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Ranjbar-Motlagh, Alireza (BR-FMG)
Rigidity of spheres in Riemannian manifolds and a non-embedding theorem. (English summary)
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Let $f\colon M\to\overline{M}$ be an isometric immersion between Riemannian manifolds. The aim of this paper is to find the minimum possible conditions on M and $\overline{M}$ (in terms of curvatures and external diameter) such that the image of f is contained in a geodesic sphere. The author proves the following pinching theorem: Let $f\colon M^n\to\overline{M}^{n+k}$ be an isometric C^2 immersion. Denote the l-mean curvature vector of $f(M)$ in $\overline{M}$ by H_l. Suppose that the image of f is contained in $B(p,R)\setminus C_p$, where $B(p,R)$ denotes the closed ball in $\overline{M}$ centered at p with radius R and C_p is the cut locus of p, and there is a point $x_0\in M$ such that $f(x_0)\in\partial B(p,R)$. Suppose that the Hessian of the distance function on $\overline{M}$, $r(y)=r_p(y)$, is bounded from below by $m(r)>0$ on the tangent bundle of $\partial B(p,r)$. Moreover, suppose that for all $x\in M$ and any unit normal vector field $\eta(f(x))$ on M, we have $|\langle H_l(f(x)),\eta(f(x))\rangle|\le m(r(f(x)))|\langle H_{l-1}(f(x)),\eta(f(x))\rangle|$, where $l\ge 1$ is an integer (define $|\langle H_0(f(x)),\eta(f(x))\rangle|:=1$). If M is connected, then the image of f is contained in the geodesic sphere $\partial B(p,R)$.

The result generalizes previous results by F. Fontenele and S. L. Silva [Arch. Math. (Basel) **73** (1999), no. 6, 474–480; MR1725184 (2000j:53065)], S. Markvorsen [Math. Z. **183** (1983), no. 3, 407–411; MR0706398 (85d:53029)] and the reviewer [Geom. Dedicata **68** (1997), no. 1, 73–78; MR1485385 (2000a:53103)]. In addition, the author proves a non-embedding result that extends well-known results of S. Chern and N. H. Kuiper [Ann. of Math. (2) **56** (1952), 422–430; MR0050962 (14,408e)] and J. D. Moore [Duke Math. J. **42** (1975), 191–193; MR0377776 (51 #13945)]. The proofs of the results are based on the maximum principle.

Reviewed by *Theodoros Vlachos*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.