Jordan-Hölder theorem for modules

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Here R denotes a ring with unity.

Definition 1. An R-module M is said to be *simple* if M does not have any proper nonzero submodule.

Definition 2. An R-module M is said to be of finite length if it satisfies the following equivalent conditions:
(1) M is both noetherian and artinian.
(2) There exists a series

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

of submodules of M such that for every i, the quotient module M_i/M_{i-1} is a simple R-module.

The above series is called a Jordan-Hölder series for M. The number n is called the *length* of this series and the quotient submodules M_i/M_{i-1}, $i = 1, 2, \cdots n$, are called the *quotient factors* of this series.

Theorem 3. *(Jordan-Hölder) Let M be an R-module of finite length and let*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_{n-1} \subset M_n = M, \quad (1)$$

$$0 = N_0 \subset N_1 \subset \cdots \subset N_{m-1} \subset N_m = M \quad (2)$$

be two Jordan-Hölder series for M. Then we have $m = n$ and the quotient factors of these series are the same.

Proof. We prove the result by induction on k, where k is the length of a Jordan-Hölder series of M of minimum length. Without loss of generality suppose that the series (1) is a series of M with minimum length. In particular we have $m \geq n$. If $n = 1$ then M is a simple module and the length of every other Jordan-Hölder series of M is also 1 and the only quotient factor is M and the result is proved.

Now suppose that $n > 1$. Consider two submodules M_{n-1} and N_{m-1} and put $K = M_{n-1} \cap N_{m-1}$.

There are two possibilities:
(i) $M_{n-1} = N_{m-1}$.
(ii) $M_{n-1} \neq N_{m-1}$.

In the first case we have $K = M_{n-1} = N_{m-1}$ and consider two Jordan-Hölder series:

$$0 = M_0 \subset M_1 \subset \cdots \subset M_{n-1} = K,$$

$$0 = N_0 \subset N_1 \subset \cdots \subset N_{m-1} = K.$$

The above series shows that K has a Jordan-Holder series of length $\leq n-1$, so the induction hypothesis implies that $n-1 = m-1$ and the quotient factors of above series are the same. Consequently the Jordan-Hölder series in (1) and (2) have the same length and the same quotient factors.

In the second case, we have $K \subsetneq M_{n-1}$ and $K \subsetneq N_{m-1}$. As $M_{n-1} \neq N_{m-1}$ and M_{n-1} and N_{m-1} are maximal in M we obtain $M_{n-1} + N_{m-1} = M$. Consequently we have:

$$M_{n-1}/K = M_{n-1}/(M_{n-1} \cap N_{m-1}) \simeq (M_{n-1} + N_{m-1})/N_{m-1} = M/N_{m-1}.$$

So

$$M_{n-1}/K \simeq M/N_{m-1}, \tag{3}$$

similarly we have

$$N_{m-1}/K \simeq M/M_{n-1}. \tag{4}$$

In particular two quotient modules M_{n-1}/K and N_{m-1}/K are simple modules. As M is both artinian and notherian, K is as well. In particular K has a Jordan-Hölder series as follows:

$$0 = K_0 \subset K_1 \subset \cdots \subset K_r = K.$$

We therefore obtain two new Jordan-Hölder series for M:

$$0 = K_0 \subset K_1 \subset \cdots \subset K_r = K \subset M_{n-1} \subset M_n = M \tag{5}$$

$$0 = K_0 \subset K_1 \subset \cdots \subset K_r = K \subset N_{m-1} \subset N_m = M \tag{6}$$

By (1), M_{n-1} has a Jordan-Hölder series of length $\leq n-1$ so we can apply the induction hypothesis for M_{n-1}, so all Jordan-Hölder series of M_{n-1} are of the same length. By (5), M_{n-1} has a Jordan-Hölder series of length $r+1$ and by (1), M_{n-1} has a Jordan-Hölder series of length $n-1$ so we have $r+1 = n-1$ and two Jordan-Hölder series

$$0 = K_0 \subset K_1 \subset \cdots \subset K_r = K \subset M_{n-1}$$

and

$$0 = M_0 \subset M_1 \subset \cdots \subset M_{n-1}$$

have the same quotient factors. Hence the length and the quotient factors of two series (1) and (5) are the same. Also by (6), N_{m-1} has a series of length $r+1 = n-1$. By induction the length and the quotient factors of the below Jordan-Hölder series of N_{m-1} are the same:

$$0 = K_0 \subset K_1 \subset \cdots \subset K_r = K \subset N_{m-1}$$

and

$$0 = N_0 \subset N_1 \subset \cdots \subset N_{m-1}.$$

Consequently the length and the quotient factors of two series (2) and (6) are the same. By (3) and (4), the length and the quotient factors of two series (5) and (6) are the same. So the lengths and the quotient factors of two series (1) and (2) are the same. □

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