Deep learning

Sum Product Networks

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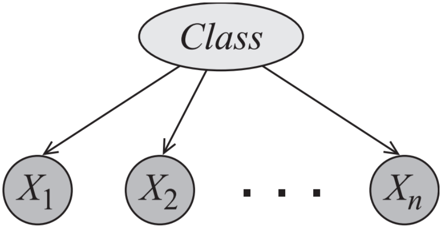
Introduction

1. We assume $x_1, x_2, \ldots, x_m$ are IID random variables are sampled from an unknown distribution $\mathcal{D}$, where each x_i is k-dimensional vector.
2. We require to specify a high-dimensional distribution $p(x_1, \ldots, x_k)$ on the data and possibly some latent variables.
3. The specific form of p will depend on some parameters w.
4. The basic operations will be to
   - **Structure learning:** Specifying the parametric/non-parametric form of $p(x_1, \ldots, x_k)$.
   - **Parameter learning:** Adjusting $p(x_1, \ldots, x_k)$ to the data.
   - **Inference:** Computing marginals and modes of $p(x_1, \ldots, x_k)$.
5. Working with fully flexible joint distributions is intractable!

1. How the form of density function is specified?
2. We specify the form of density function in such a way that parameter learning and inference become easier.
3. For example, we can consider the following conditional form.

$$p(x_1, \ldots, x_k) = p(x_1|x_2)p(x_1|x_3)\ldots p(x_1|x_k)$$

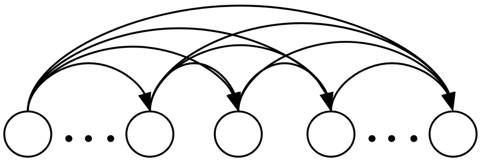
4. Consider the Naive Bayes classifier. We have $p(class, x) = p(class)\prod_{j=1}^{n} p(x_j \mid class)$



1. Or consider the following forms

$$p(x_1, \ldots, x_k) = p(x_k|x_{k-1})p(x_{k-1}|x_{k-2}) \ldots p(x_2|x_1)$$

$$p(x_1, \ldots, x_k) = \prod_{i=1}^{k} p(x_i|x_1, x_2, \ldots, x_{i-1})$$



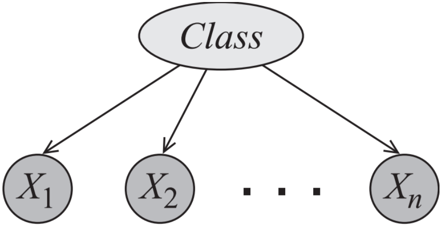
2. We must work with structured or compact distributions.
3. For example, distributions in which the random variables interact directly with only very few others in simple ways (why?).
4. One solution is to use probabilistic graphical models.

1. **Simple queries:** computing posterior marginal $p(x_1|E = e)$
2. **Conjunctive queries:** Computing $p(x_1, x_2|E = e)$
3. How do you answer the following query?

$$p(x_1) = \sum_{x_2} \sum_{x_3} p(x_1, x_2, x_3)$$

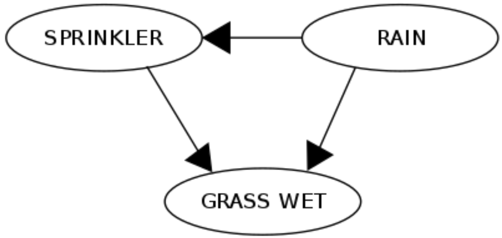
4. How do you answer the query $p(x_1)$ when density function has the following form?

$$p(x_1, \ldots, x_k) = p(x_1|x_2)p(x_1|x_3)\ldots p(x_1|x_k)$$



1. A simple Bayesian network

	SPRINKLER	
RAIN	T	F
F	0.4	0.6
T	0.01	0.99



RAIN	
T	F
0.2	0.8

		GRASS WET	
SPRINKLER	RAIN	T	F
F	F	0.0	1.0
F	T	0.8	0.2
T	F	0.9	0.1
T	T	0.99	0.01

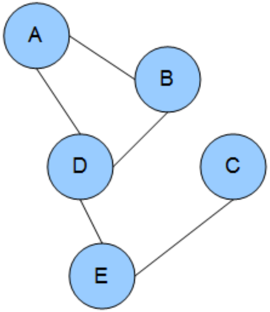
$$p(G, S, R) = p(G|S, R)p(S|R)p(R)$$

1. How calculate $p(x_1, \ldots, x_k)$ using Bayesian networks?
2. If a Bayesian network can be factorized, then we can write

$$p(x_1, \ldots, x_k) = \prod_{v \in V} p(x_v | pa(v))$$

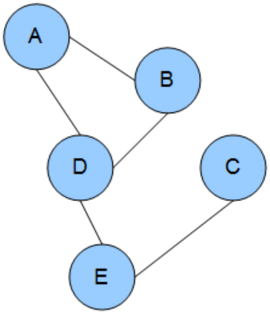
   where $pa(v)$ is the set of parents of v.
3. Cooper proved that exact inference in Bayesian networks is NP-hard.

1. A Markov network is a set of random variables having a Markov property described by an undirected graph.



2. Each edge represents dependency.
   - A depends on B and D.
   - B depends on A and D.
   - D depends on A, B, and E.
   - E depends on D and C.
   - C depends on E.

1. Consider the following network.



2. We'll assume p is a general undirected model of the following form

$$p(x_1, \ldots, x_n; w) = \frac{\bar{p}(x_1, \ldots, x_n; w)}{Z(w)} = \frac{1}{Z(w)} \prod_k \phi_k(x\{k\}; w),$$

where the ϕ_k are the factors and $Z(w)$ is the normalization constant and $x\{k\}$ is a subset of variables .

3. How do yo compute $Z(w)$?

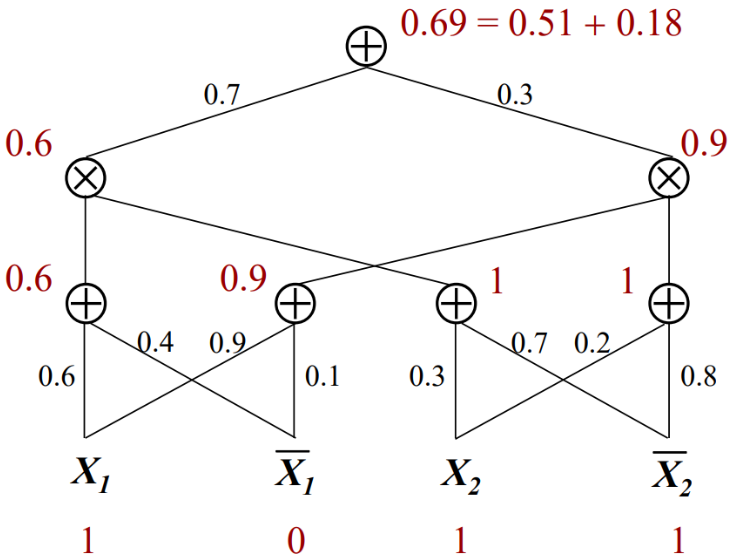
Limitations of Graphical Models

1. Graphical models are limited in some aspects
   - Many compact distributions cannot be represented as a GM.
   - The cost of exact inference in GM is exponential in the worst case (using approximate techniques).
   - Because learning requires inference, learning GM will be difficult .
   - Some distributions require GM with many layers of hidden variables to be compactly encoded.
2. An alternative are sum product networks (Poon and Domingos 2011).
   - New deep model with many layers of hidden variables.
   - Exact inference is tractable (linear in the size of the model).

Sum Product Networks

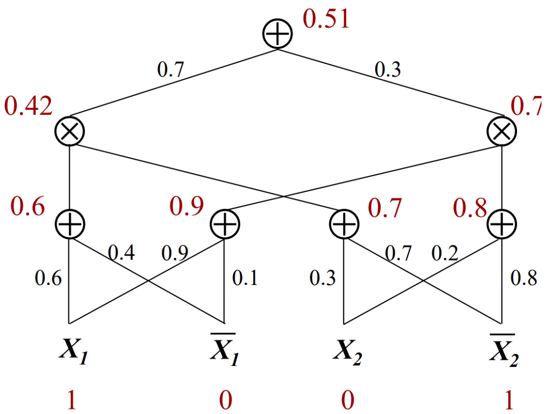
1. A SPN is rooted DAG whose leaves are $x_1, \dots, x_n$ and $\bar{x}_1, \dots, \bar{x}_n$ with internal sum and product nodes, where each edge (i, j) emanating from sum node i has a weight $w_{ij} \geq 0$.
2. The value of a product node is the product of the value of its children.
3. The value of a sum node i is $\sum_{j \in Ch(i)} w_{ij} v_j$, where $Ch(j)$ are the children of node i and v_j is the value of node j
4. The value of a SPN is the value of the root after a bottom up evaluation.
5. Layers of sum and product nodes usually alternate.

1. An example of SPN

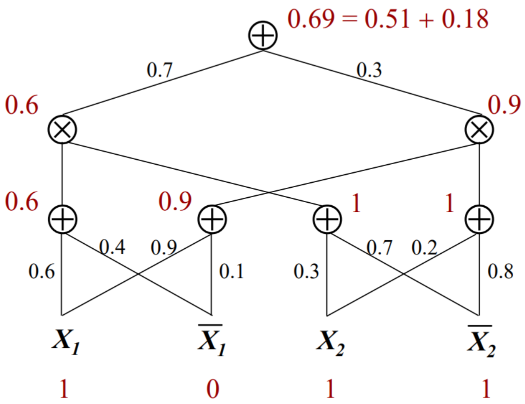


2. What is the output of the above network?

SPN represents a joint distribution over a set of random variables. What is value of $p(x_1 = 1, x_2 = 0)$?



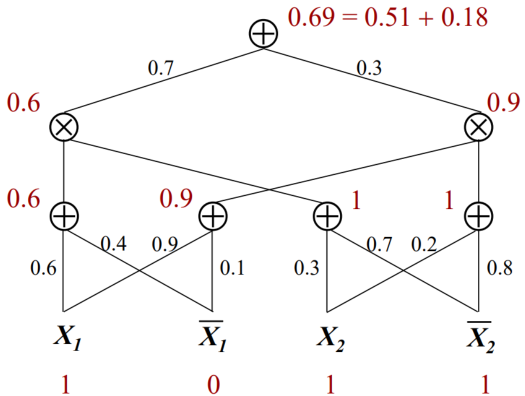
SPN represents a joint distribution over a set of random variables. What is value of $p(x_1 = 1)$?



SPN Semantics

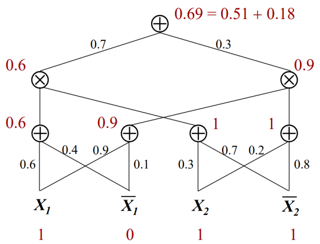
A valid SPN encodes a hierarchical mixture distribution.

- Sum nodes: hidden variables (mixture)
- Product nodes: factorization (independence)



Valid SPN

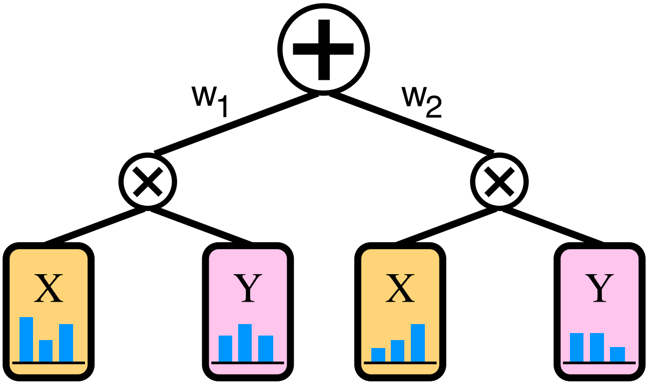
- The scope of a node is the set of variables that appear in the sub-SPN rooted at the node
- An SPN is decomposable iff no variable appears in more than one child of a product node.
- An SPN is complete when each sum node has children with identical scopes.
- An SPN is consistent iff no variable appears negated in one child of a product node and non-negated in another.
- A consistent and complete SPN is a valid SPN. An SPN is valid if it always correctly computes the probability of evidence.



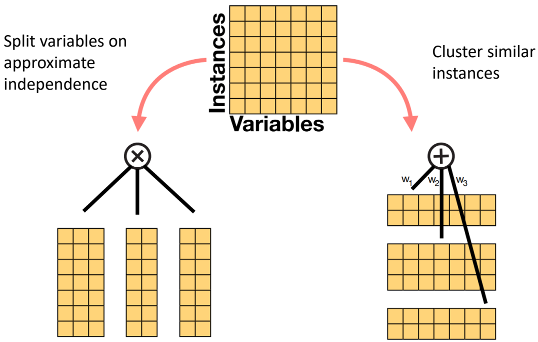
Building and using an SPN

1. We must specify the structure of SPN (structure Estimation or structure learning).
2. We must find the parameters of SPN (parameter learning).
3. We must answer queries (inference).

1. What is SPN for univariate distribution?
2. → A univariate distribution is an SPN
3. What is SPN for product of disjoint random variables?
4. → A product of SPNs over disjoint variables is an SPN.
5. What is SPN for a mixture model?
6. → A weighted sum of SPNs over the same variables is an SPN.



1. In a structure learning, one alternates between
    - Data Clustering: sum nodes
    - Variable partitioning: product nodes



2. Some others use SVD decomposition (Adel, Balduzzi, and Ghodsi 2015).

SPN learning

1. Initialize the SPN using a dense valid SPN.
2. Learn the SPN weights using gradient descent or EM.
3. Add some penalty to the weights so that they tend to be zero.
4. Prune edges with zero weights at convergence.

Algorithm 1 LearnSPN

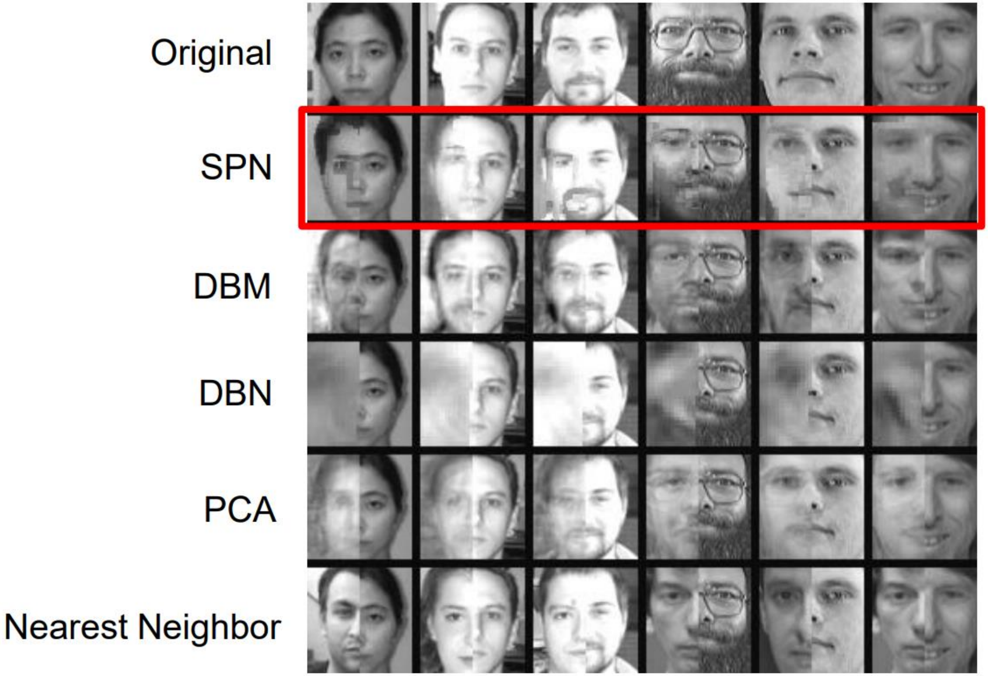
```
Input: Set D of instances over variables X.
Output: An SPN with learned structure and parameters.
S ← GenerateDenseSPN(X)
InitializeWeights(S)
repeat
    for all d ∈ D do
        UpdateWeights(S, Inference(S, d))
    end for
until convergence
S ← PruneZeroWeights(S)
return S
```

Applications

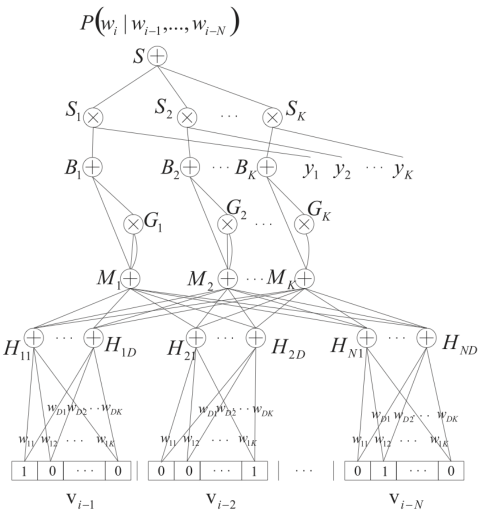
Image completion

1. Main evaluation: Caltech-101
   - 101 categories, e.g., faces, cars, elephants
   - Each category: 30 – 800 images
2. Each category: Last third for test
3. Test images: Unseen objects

Image completion



1. Fixed structure SPN encoding the conditional probability $p(w_i|w_{i-1}\dots,w_{i-N})$ as an Nth order language model (Cheng et al. 2014).



1. Perplexity scores (PPL) of different language models

Model	Individual PPL	+KN5
TrainingSetFrequency	528.4	
KN5 [3]	141.2	
Log-bilinear model [4]	144.5	115.2
Feedforward neural network [5]	140.2	116.7
Syntactical neural network [8]	131.3	110.0
RNN [6]	124.7	105.7
LDA-augmented RNN [9]	113.7	98.3
SPN-3	**104.2**	**82.0**
SPN-4	**107.6**	**82.4**
SPN-4'	**100.0**	**80.6**

Other applications

1. Image completion
2. Image classification
3. Activity recognition
4. Click-through logs
5. Nucleic acid sequences
6. Collaborative filtering

Advantages of SPNs

1. Unlike graphical models, SPNs are tractable over high treewidth models.
2. SPNs are deep architectures with full probabilistic semantics
3. SPNs can incorporate features into an expressive model without requiring approximate inference.

Reading

Readings

1. Read the survey paper (Paris, Sanchez-Cauce, and Diez 2020).

- Adel, Tameem, David Balduzzi, and Ali Ghodsi (2015). "Learning the Structure of Sum-Product Networks via an SVD-based Algorithm". In: *Proceedings of the Thirty-First Conference on Uncertainty in Artificial Intelligence*, pp. 32–41.
- Cheng, Wei-Chen et al. (2014). "Language modeling with sum-product networks". In: *Proceedings of the 15th Annual Conference of the International Speech Communication Association*, pp. 2098–2102.
- Paris, Iago, Raquel Sanchez-Cauce, and Francisco Javier Diez (2020). "Sum-product networks: A survey". In: *CoRR* abs/2004.01167. arXiv: 2004.01167.
- Poon, Hoifung and Pedro M. Domingos (2011). "Sum-Product Networks: A New Deep Architecture". In: *Proceedings of the Twenty-Seventh Conference on Uncertainty in Artificial Intelligence*, pp. 337–346.

Questions?