Chapter 16

16.1 Use (16-132) with $r = 1$. This gives

$$p_n = \begin{cases} \dfrac{\rho^n}{n!}\, p_0, & n \le 1 \\ \rho^n\, p_0, & 1 < n \le m \end{cases}$$

$$= \rho^n\, p_0, \qquad 0 \le n \le m$$

Thus

$$\sum_{n=0}^{m} p_n = p_0 \sum_{n=0}^{m} \rho^n = p_0\, \frac{(1-\rho^{m+1})}{1-\rho} = 1$$

$$\Longrightarrow \quad p_0 = \frac{1-\rho}{1-\rho^{m+1}}$$

and hence

$$p_n = \frac{1-\rho}{1-\rho^{n+1}}\, \rho^n, \qquad 0 \le n \le m, \quad \rho \ne 1$$

and $lim\, \rho \to 1$, we get

$$p_n = \frac{1}{m+1}, \qquad \rho = 1.$$

16.2 (a) Let $n_1(t) = X + Y$, where X and Y represent the two queues. Then

$$\begin{aligned} p_n &= P\{n_1(t) = n\} = P\{X + Y = n\} \\ &= \sum_{k=0}^{n} P\{X = k\}\, P\{Y = n-k\} \\ &= \sum_{k=0}^{n} (1-\rho)\rho^k\, (1-\rho)\rho^{n-k} \\ &= (n+1)(1-\rho)^2 \rho^n, \qquad n = 0, 1, 2, \cdots \end{aligned} \tag{i}$$

where $\rho = \lambda/\mu$.

(b) When the two queues are merged, the new input rate $\lambda' = \lambda + \lambda = 2\lambda$. Thus from (16-102)

$$p_n = \begin{cases} \dfrac{(\lambda'/\mu)^n}{n!} p_0 = \dfrac{(2\rho)^n}{n!}\, p_0, & n < 2 \\ \dfrac{2^2}{2!}\,(\dfrac{\lambda'}{2\mu})^n\, p_0 = 2\rho^n p_0, & n \geq 2. \end{cases}$$

Hence

$$\begin{aligned} \sum_{k-0}^{\infty} p_k &= p_0(1 + 2\rho + 2\sum_{k=2}^{\infty} \rho^k) \\ &= p_0(1 + 2\rho + \frac{2\rho^2}{1-\rho}) \\ &= \frac{p_0}{1-\rho}((1+2\rho)\,(1-\rho) + 2\rho^2) \\ &= \frac{p_0}{1-\rho}\,(1+\rho) = 1 \end{aligned}$$

$$\Longrightarrow p_0 = \frac{1-\rho}{1+\rho}, \qquad (\rho = \lambda/\mu). \tag{ii}$$

Thus

$$p_n = \begin{cases} 2\,(1-\rho)\,\rho^n/(1+\rho), & n \leq 1 \\ (1-\rho)/(1+\rho), & n = 0 \end{cases} \tag{iii}$$

(c) For an $M/M/1$ queue the average number of items waiting is given by (use (16-106) with $r = 1$)

$$E\{X\} = L_1' = \sum_{n=2}^{\infty} (n-1)\, p_n$$

where p_n is an in (16-88). Thus

$$\begin{aligned} L_1' &= \sum_{n=2}^{\infty} (n-1)(1-\rho)\,\rho^n \\ &= (1-\rho)\,\rho^2 \sum_{n=2}^{\infty} (n-1)\,\rho^{n-2} \\ &= (1-\rho)\,\rho^2 \sum_{k=1}^{\infty} k\,\rho^{k-1} \\ &= (1-\rho)\,\rho^2 \frac{1}{(1-\rho)^2} = \frac{\rho^2}{(1-\rho)}. \end{aligned} \tag{iv}$$

Since $n_1(t) = X + Y$ we have

$$\begin{aligned} L_1 &= E\{n_1(t)\} = E\{X\} + E\{Y\} \\ &= 2L_1' = \frac{2\rho^2}{1-\rho} \end{aligned} \tag{v}$$

For L_2 we can use (16-106)-(16-107) with $r = 2$. Using (iii), this gives

$$\begin{aligned} L_2 &= p_r \frac{\rho}{(1-\rho)^2} \\ &= 2\,\frac{(1-\rho)\,\rho^2}{1+\rho}\,\frac{\rho}{(1-\rho)^2} = \frac{2\,\rho^3}{1-\rho^2} \\ &= \frac{2\,\rho^2}{1-\rho}\left(\frac{\rho}{1+\rho}\right) < L_1 \end{aligned} \tag{vi}$$

From (vi), a single queue configuration is more efficient then two separate queues.

16.3 The only non-zero probabilities of this process are

$$\lambda_{0,0} = -\lambda_0 = -m\lambda, \quad \lambda_{0,1} = \mu$$

$$\lambda_{i,i+1} = (m-i)\,\lambda, \quad \lambda_{i,i-1} = i\mu$$

$$\lambda_{i,i} = [(m-i)\,\lambda + i\mu], \quad i = 1, 2, \cdots, m-1$$

$$\lambda_{m,m} = -\lambda_{m,m-1} = -m\mu.$$

Substituting these into (16-63) text, we get

$$m\,\lambda\,p_0 = \mu\,p_1 \tag{i}$$

$$[(m-i)\lambda + i\mu]\,p_i = (m-i+1)\,p_{i-1} + (i+1)\,\mu\,p_{i+1}, \quad i = 1, 2, \cdots, m-1 \tag{ii}$$

and

$$m\,\mu\,p_m = \lambda\,p_{m-1}. \tag{iii}$$

Solving (i)-(iii) we get

$$p_i = \binom{m}{i}\left(\frac{\lambda}{\lambda+\mu}\right)^i \left(\frac{\mu}{\lambda+\mu}\right)^{m-i}, \quad i = 0, 1, 2, \cdots, m$$

16.4 (a) In this case

$$p_n = \begin{cases} \dfrac{\lambda}{\mu_1}\,\dfrac{\lambda}{\mu_1}\,\cdots\,\dfrac{\lambda}{\mu_1} = \left(\dfrac{\lambda}{\mu_1}\right)^n p_0, & n < m \\ \dfrac{\lambda}{\mu_1}\,\dfrac{\lambda}{\mu_1}\,\cdots\,\dfrac{\lambda}{\mu_1}\,\dfrac{\lambda}{\mu_2}\,\cdots\,\dfrac{\lambda}{\mu_2}\,p_0, & n \geq m \end{cases}$$

$$= \begin{cases} \rho_1^n\,p_0, & n < m \\ \rho_1^{m-1}\,\rho_2^{n-m+1}\,p_0, & n \geq m, \end{cases}$$

where

$$\begin{aligned} \sum_{n=0}^{\infty} p_n &= p_0\left[\sum_{k=0}^{m-1}\rho_1^k + \rho_1^{m-1}\rho_2\sum_{n=0}^{\infty}\rho_2^n\right] \\ &= p_0\left[\frac{1-\rho_1^m}{1-\rho_1} + \frac{\rho_2\rho_1^{m-1}}{1-\rho_2}\right] = 1 \end{aligned}$$

gives

$$p_0 = \left(\frac{1-\rho_1^m}{1-\rho_1} + \frac{\rho_2 \rho_1^{m-1}}{1-\rho_2} \right)^{-1}.$$

(b)

$$\begin{aligned}
L &= \sum_{n=0}^{\infty} n\, p_n \\
&= p_0 \left[\sum_{n=0}^{m-1} n\, \rho_1^n + \sum_{n=m}^{\infty} n\, \rho_1^{m-1}\, \rho_2^{n-m+1} \right] \\
&= p_0 \left[\rho_1 \sum_{n=0}^{m-1} n\, \rho_1^{n-1} + \rho_1 \, \left(\frac{\rho_1}{\rho_2} \right)^{m-2} \sum_{n=m}^{\infty} n\, \rho_2^{n-1} \right] \\
&= p_0 \left[\rho_1 \frac{d}{d\rho_1} \, \left(\sum_{n=0}^{m-1} \rho_1^n \right) + \rho_1 \, \left(\frac{\rho_1}{\rho_2} \right)^{m-2} \frac{d}{d\rho_2} \sum_{n=m}^{\infty} \rho_2^n \right] \\
&= p_0 \left[\rho_1 \, \frac{d}{d\rho_1} \, \left(\frac{1-\rho_1^m}{1-\rho} \right) + \rho_1 \, \left(\frac{\rho_1}{\rho_2} \right)^{m-2} \frac{d}{d\rho} \, \left(\frac{\rho^m}{1-\rho} \right) \right] \\
&= p_0 \left[\frac{\rho_1 [1 + (m-1)\rho_1^m - m\rho_1^{m-1}]}{(1-\rho_1)^2} + \frac{\rho_2\, \rho_1^{m-1} + [m - (m-1)\rho_2]}{(1-\rho_2)^2} \right].
\end{aligned}$$

16.5 In this case

$$\lambda_i = \begin{cases} \lambda, & j < r \\ p\lambda, & j \geq r \end{cases} \qquad \mu_i = \begin{cases} j\mu, & j < r \\ r\mu, & j \geq r. \end{cases}$$

Using (16-73)-(16-74), this gives

$$p_n = \begin{cases} \dfrac{(\lambda/\mu)^n}{n!}\, p_0, & n < r \\ \dfrac{(\lambda/\mu)^r}{r!}\, (p\lambda/r\mu)^{n-r}, & n \geq r. \end{cases}$$

16.6

$$
\begin{aligned}
P\{w > t\} &= \sum_{n=r}^{m-1} p_n \, P(w > t|n) \\
&= \sum_{n=r}^{m-1} p_n \, (1 - F_w(t|n)) = \sum p_r \left(\frac{\lambda}{r\mu}\right)^{n-r} \; (1 - F_w(t|n))
\end{aligned}
$$

$$
f_w(t|n) = e^{-\gamma\mu t} \, \frac{(\gamma\mu)^{n-r+1} \, t^{n-r}}{(n-r)!)} \quad (\textit{see } 16.116)
$$

and

$$
F_w(t|n) = 1 - \sum_{k=0}^{n-r} \frac{(\gamma\mu t)^k}{k!} \, e^{-\gamma\mu t} \quad (\textit{see } 4.)
$$

so that

$$
1 - F_w(t|n) = \sum_{k=0}^{n-r} \frac{(\gamma\mu t)^k}{k!} \, e^{-\gamma\mu t}
$$

$$
\begin{aligned}
P\{w > t\} &= \sum_{n=r}^{m-1} p_r \left(\frac{\lambda}{\gamma\mu}\right)^{n-r} \sum_{k=0}^{n-r} \frac{(\gamma\mu t)^k}{k!} \, e^{-\gamma\mu t} \\
&= \sum_{i=0}^{m-r-1} p_r \, \rho^i \sum_{k=0}^{i} \frac{(\gamma\mu t)^k}{k!} \, e^{-\gamma\mu t}, \quad n - r = i \\
&= p_r \, e^{-\gamma\mu t} \sum_{k=0}^{m-r-1} \rho^k \sum_{i=0}^{k} \frac{(\gamma\mu t)^i}{i!} \\
&= \sum_{k=0}^{m-r-1} \sum_{i=0}^{k} = \sum_{i=0}^{m-r-1} \sum_{k=i}^{m-r-1}
\end{aligned}
$$

$$
\begin{aligned}
P\{w > t\} &= p_r \, e^{-\gamma\mu t} \sum_{i=0}^{m-r-1} \frac{(\gamma\mu t)^i}{i!} \sum_{k=i}^{m-r-1} \rho^k \\
&= \frac{p_r}{1-\rho} \, e^{-\gamma\mu t} \sum_{i=0}^{m-r-1} \frac{(\gamma\mu t)^i}{i!} \, (\rho^i - \rho^{m-r}), \quad \rho = \lambda/\gamma\mu.
\end{aligned}
$$

Note that $m \to \infty \Longrightarrow M/M/r/m \Longrightarrow M/M/r$ and

$$\begin{aligned} P(w > t) &= \frac{p_r}{1-\rho}\, e^{-\gamma\mu t} \sum_{i=0}^{\infty} \frac{(\gamma\mu\rho t)^i}{i!} \\ &= \frac{p_r}{1-\rho}\, e^{-\gamma\mu(1-\rho)t} \quad t > 0. \end{aligned}$$

and it agrees with (16.119)

16.7 (a) Use the hints

(b)

$$-\sum_{n=1}^{\infty} (\lambda+\mu)\, p_n\, z^n + \frac{\mu}{z} \sum_{n=1}^{\infty} p_{n+1}\, z^{n+1} + \lambda \sum_{n=1}^{\infty} \sum_{k=1}^{n} p_{n-k}\, c_k\, z^n = 0$$

$$-(\rho+1)\,(P(z)-p_0) + \frac{\mu}{z}\,(P(z)-p_0-p_1 z) + \lambda \sum_{k=1}^{\infty} c_k z^k \sum_{m=0} p_m\, z^m = 0$$

which gives

$$P(z)[1-z-\rho z\,(1-C(z))] = p_0(1-z)$$

or

$$P(z) = \frac{p_0(1-z)}{1-z-\rho z\,(1-C(z))}.$$

$$1 = P(1) = \frac{-p_0}{-1-\rho+\rho_z\, C'(z)+\rho C(z)} = \frac{-p_0}{-1+\rho C'(1)}$$
$$\Longrightarrow p_0 = 1-\rho_0, \quad \rho_0 = \rho C'(1).$$

Let

$$D(z) = \frac{1-C(z)}{1-z}.$$

Then

$$P(z) = \frac{1-\rho_L}{1-\rho z D(z)}.$$

(c) This gives

$$P'(z) = \frac{(1-\rho_c)}{(1-\rho z D(z))^2}\, (\rho D(z) + \rho z D'(z))$$

$$
\begin{aligned}
L = P'(1) \quad &= \frac{(1-\rho_c)}{(1-\rho_c)^2}\,\rho\,(D(1)+D'(1)) \\
&= \frac{1}{(1-\rho_c)}\;(C'(1)+D'(1))
\end{aligned}
$$

$$C'(1) = E(x)$$

$$D(z) = \frac{1-C(z)}{1-z}$$

$$
\begin{aligned}
D'(z) \quad &= \frac{(1-z)\,(-C'(z))-(1-C(z))\,(-1)}{(1-z)^2} \\
&= \frac{1-C(z)-(1-z)C'(z)}{(1-z)^2}
\end{aligned}
$$

By L-Hopital's Rule

$$
\begin{aligned}
D'(1) \quad &= \lim_{z\to 1} \frac{-C'(z)-(-1)C'(z)-(1-z)C''(z)}{-2(1-z)} \\
&= \lim_{z\to 1} = 1/2C''(z) = \frac{C''(z)}{2} \\
&= 1/2 \sum k(k-1)\,C_k = \frac{E(X^2)-E(X)}{2}
\end{aligned}
$$

$$L = \frac{\rho\,(E(X)+E(X^2))}{2\,(1-\rho E(X))}.$$

(d)

$$C(z)z^m \quad E(X) = m$$

$$P(z) = \frac{1-\rho}{1-\rho\sum_{k=1}^{m} z^k}$$

$$D(z) = \frac{1-z^m}{1-z} = \sum_{k=0}^{m-1} z^k$$

$$E(X) = m, \quad E(X^2) = m^2$$

$$L = \frac{\rho(m+m^2)}{2(1-\rho m)}$$

(e)

$$C(z) = \frac{qz}{1-Pz}$$

$$P(z) = \frac{1-\rho_0}{1-\rho z D(z)}, \quad C(z) = \frac{qz}{1-pz}$$

$$D(z) = \frac{1-C(z)}{1-z} = \frac{1-\frac{qz}{1-P(z)}}{1-z} = \frac{1-Pz-(1-P)_z}{(1-z)(1-Pz)} = \frac{1-z}{(1-z)(1-Pz)} = \frac{1}{1-Pz}$$

$$P(z) = \frac{(1-\rho_0)(1-pz)}{1-pz-\rho z} = \frac{(1-\rho_0)(1-pz)}{1-(p+\rho)z}$$

$$C'(1) = \frac{(1-pz)q - qz(-p)}{(1-Pz)^2} = \frac{q}{q^2} = \frac{1}{q}$$

$$D(z) = \frac{1-C(z)}{1-z}$$

$$D(1) = C'(1)$$

$$L = P'(1) = \frac{1-\rho_c}{(1-\rho_c)^2}\ (\rho \cdot C'(1) + \rho \cdot D'(1))$$

$$D'(z) = \frac{-(1-z)C'(z) - (1-C(z))\,(\rho-1)}{(1-z)^2} = \frac{1-C(z)-(1-z)C'(1)}{(1-z)^2}$$

$$\begin{aligned}\lim\nolimits_{z\to 1} D'(z) \quad &= \lim\nolimits_{z\to 1} \frac{-C'(z)-(-1)C'(z)-(1-z)C''(z)}{2(1-z)} \\ &= \frac{-(1-z)C''(z)}{-2(1-z)} = \frac{\rho''(z)}{2}\end{aligned}$$

$$D'(1) = \frac{C''(1)}{2}$$

$$L = \frac{1}{(1-\rho_c)}\left(\rho E(X) + \frac{\rho\,(E(X^2)-E(X))}{2}\right) = \frac{\rho E(X) + \rho E(X^2)}{2(1-\rho_c)}.$$

16.8 (a) Use the hints.
(b)

$$-\sum_{n=1}^{\infty}(\lambda+\mu)\,p_n\,z^n + \frac{\mu}{z^n}\sum_{n=1}^{\infty} p_{n+m}\,z^{n+m} + \lambda z\sum_{n=1}^{\infty} p_{n-1}\,z^{n-1} = 0$$

or

$$-(1+\rho)\,(P(z)-p_0)+\frac{1}{z^m}\left(P(z)-\sum_{k=0}^{m}p_k\,z^k\right)+\rho z P(z)=0$$

which gives

$$P(z)\left[\rho\,z^{m+1}-(\rho+1)\,z^m+1\right]=\sum_{k=0}^{m}p_k\,z^k-p_0\,(1+\rho)\,z^m$$

or

$$P(z)=\frac{\sum_{k=0}^{m}p_k\,z^k-p_0\,(1+\rho)\,z^m}{\rho\,z^{m+1}-(\rho+1)\,z^m+1}=\frac{N(z)}{M(z)}. \tag{i}$$

(c) Consider the denominator polynomial $M(z)$ in (i) given by

$$M(z)=\rho\,z^{m+1}-(1+\rho)\,z^m+1=f(z)+g(z)$$

where

$$f(z)=-(1+\rho)\,z^m,$$
$$g(z)=1+\rho\,z^{m+1}.$$

Notice that $|f(z)|>|g(z)|$ in a circle defined by $|z|=1+\varepsilon$, $\varepsilon>0$. Hence by Rouche's Theorem $f(z)$ and $f(z)+g(z)$ have the same number of zeros inside the unit circle ($|z|=1+\varepsilon$). But $f(z)$ has m zeros inside the unit circle. Hence $f(z)+g(z)=M(z)$ also has m zeros inside the unit circle. Hence

$$M(z)=M_1(z)\,(z-z_0) \tag{ii}$$

where $|z_0|>1$ and $M_1(z)$ is a polynomial of degree m whose zeros are all *inside* or on the unit circle. But the moment generating function $P(z)$ is analytic inside and on the unit circle. Hence all the m zeros of $M(z)$ that are inside or on the unit circle must cancel out with the zeros of the numerator polynomial of $P(z)$. Hence

$$N(z)=M_1(z)\,a. \tag{iii}$$

Using (ii) and (iii) in (i) we get

$$P(z) = \frac{N(z)}{M(z)} = \frac{a}{z - z_0}.$$

But $P(1) = 1$ gives $a = 1 - z_0$
or

$$\begin{aligned} P(z) &= \frac{z_0 - 1}{z_0 - z} \\ &= \left(1 - \frac{1}{z_0}\right) \sum_{n=0}^{\infty} (z/z_0)^n \end{aligned}$$

$$\Longrightarrow p_n = \left(1 - \frac{1}{z_0}\right) \left(\frac{1}{z_0}\right)^n = (1 - r)\, r^n, \quad n \geq 0 \qquad \text{(iv)}$$

where $r = 1/z_0$.

(d) Average system size

$$L = \sum_{n=0}^{\infty} n\, p_n = \frac{r}{1 - r}.$$

16.9 (a) Use the hints in the previous problem.
(b)

$$\begin{aligned} -\sum_{n=m}^{\infty} (\lambda + \mu)\, p_n\, z^n + \mu \sum_{n=m}^{\infty} p_{n+m}\, z^n + \lambda \sum_{n=m}^{\infty} p_{n-1}\, z^n \\ -(1+\rho)\left(P(z) - \sum_{k=0}^{m-1} p_k\, z^k\right) + \frac{1}{z^m}\left(P(z) - \sum_{k=0}^{2m-1} p_k\, z^k\right) \\ +\rho\, z \left(P(z) - \sum_{k=0}^{m-2} p_k\, z^k\right) = 0. \end{aligned}$$

After some simplifications we get

$$P(z)\left[\rho\, z^{m+1} - (\rho + 1)\, z^m + 1\right] = (1 - z^m) \sum_{k=0}^{m-1} p_k\, z^k$$

or

$$P(z) = \frac{(1 - z^m) \displaystyle\sum_{k=0}^{m-1} p_k\, z^k}{\rho\, z^{m+1} - (\rho + 1)\, z^m + 1} = \frac{(z_0 - 1) \displaystyle\sum_{k=0}^{m-1} z^k}{m\,(z_0 - z)}$$

where we have made use of Rouche's theerem and $P(z) \equiv 1$ as in problem 16-8.
(c)

$$P(z) = \sum_{n=0}^{\infty} p_n \, z^n = \frac{1-r}{m} \, \frac{\sum_{k=0}^{m-1} z^k}{1 - rz}$$

gives

$$p_n = \begin{cases} (1 + r + \cdots + r^k) \, p_0, & k \le m-1 \\ r^{n-m+1} \, (1 + r + \cdots + r^{m-1}) \, p_0, & k \ge m \end{cases}$$

where

$$p_0 = \frac{1-r}{m}, \qquad r = \frac{1}{z_0}.$$

Finally

$$L = \sum_{n=0}^{\infty} n \, p_n = P_n^{'}(1).$$

But

$$P^{'}(z) = \left(\frac{1-r}{m}\right) \frac{\sum_{k=1}^{m-1} k \, z^{k-1} \, (1 - rz) - \sum_{k=0}^{m-1} z^k \, (-r)}{(1 - rz)^2}$$

so that

$$\begin{aligned} L &= P^{'}(1) = \frac{1-r}{m} \, \frac{m - 1 + r}{(1-r)^2} = \frac{m - (1-r)}{m \, (1-r)} \\ &= \frac{1}{1-r} - \frac{1}{m}. \end{aligned}$$

16.10 Proceeding as in (16-212),

$$\begin{aligned} \psi_A(u) &= \int_0^{\infty} e^{-u\tau} dA(\tau) \\ &= \left(\frac{\lambda \, m}{u + \lambda \, m \, z}\right)^m . \end{aligned}$$

This gives

$$\begin{aligned} B(z) &= \psi_A(\psi(1-z)) \\ &= \left(\frac{\lambda\, m}{\mu\,(1-z)+\lambda\, m}\right)^m \\ &= \left(\frac{1}{1+\frac{1}{\rho}(1-z)}\right)^m \\ &= \left(\frac{\rho}{(1+\rho)-z}\right)^m, \qquad \rho = \frac{\lambda}{m\,\mu}. \end{aligned} \tag{i}$$

Thus the equation $B(z) = z$ for π_0 reduce to

$$\left(\frac{\rho}{(1+\rho)-z}\right)^m = z$$

or

$$\frac{\rho}{(1+\rho)-z} = z^{1/m},$$

which is the same as

$$\rho\, z^{-1/m} = (1+\rho) - z \tag{ii}$$

Let $x = z^{-1/m}$. Sustituting this into (ii) we get

$$\rho\, x = (1+\rho) - x^{-m}$$

or

$$\rho\, x^{m+1} - (1+\rho)\, x^m + 1 = 0 \tag{iii}$$

16.11 From Example 16.7, Eq.(16-214), the characteristic equation for $Q(z)$ is given by $(\rho = \lambda/m\,\mu)$

$$1 - z[1+\rho\,(1-z)]^m = 0$$

which is equivalent to

$$1 + \rho\,(1 - z) = z^{-1/m}. \tag{i}$$

Let $x = z^{1/m}$ in this case, so that (i) reduces to

$$[(1 + \rho) - \rho\, x^m]\, x = 1$$

or the characteristic equation satisfies

$$\rho\, x^{m+1} - (1 + \rho)\, x + 1 = 0. \tag{ii}$$

16.12 Here the service time distribution is given by

$$\frac{dB(t)}{dt} = \sum_{i=1}^{k} d_i\, \delta(t - T_i)$$

and this Laplace transform equals

$$\Phi_s(s) = \sum_{i=1}^{k} d_i\, e^{-s\,T_i} \tag{i}$$

substituting (i) into (15.219), we get

$$\begin{aligned} A(z) &= \Phi_s\,(\lambda\,(1 - z)) \\ &= \sum_{i=1}^{k} d_i\, e^{-\lambda\,T_i\,(1-z)} \\ &= \sum_{i=1}^{k} d_i\, e^{-\lambda\,T_i}\, e^{\lambda\,T_i\,z} \\ &= \sum_{i=1}^{k} d_i\, e^{-\lambda\,T_i} \sum_{j=0}^{\infty} \tfrac{(\lambda\,T_i)^j\, z^j}{j!} = \sum_{j=0}^{\infty} a_j\, z^j. \end{aligned}$$

Hence

$$a_j = \sum_{i=1}^{k} d_i\, e^{-\lambda\,T_i}\, \frac{(\lambda\,T_i)^j}{j!}, \qquad j = 0, 1, 2, \cdots. \tag{i}$$

To get an explicit formula for the steady state probabilities $\{q_n\}$, we can make use of the analysis in (16.194)-(16.204) for an $M/G/1$ queue. From (16.203)-(16.204), let

$$c_0 = 1 - a_0, \qquad c_n = 1 - \sum_{k=0}^{n} a_k, \qquad n \geq 1$$

and let $\{c_k^{(m)}\}$ represent the $m-$fold convolution of the sequence $\{c_k\}$ with itself. Then the steady-state probabilities are given by (16.203) as

$$q_n = (1-\rho) \sum_{m=0}^{\infty} \sum_{k=0}^{n} a_k \, c_{n-k}^{(m)}.$$

(b) *State-Dependent Service Distribution*
Let $B_i(t)$ represent the service-time distribution for those customers entering the system, where the most recent departure left i customers in the queue. In that case, (15.218) modifies to

$$a_{k,i} = P\{A_k | B_i\}$$

where

$$A_k = \text{"}k \text{ customers arrive during a service time"}$$

and

$$B_i = \text{"}i \text{ customers in the system at the most recent departure."}$$

This gives

$$\begin{aligned} a_{k,i} &= \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^k}{k!} \, dB_i(t) \\ &= \begin{cases} \displaystyle\int_0^\infty e^{-\lambda t} \frac{(\lambda t)^k}{k!} \mu_1 \, e^{-\mu_1 t} \, dt = \frac{\mu_1 \lambda^k}{(\lambda + \mu_1)^{k+1}}, & i = 0 \\ \displaystyle\int_0^\infty e^{-\lambda t} \frac{(\lambda t)^k}{k!} \mu_2 \, e^{-\mu_2 t} \, dt = \frac{\mu_2 \lambda^k}{(\lambda + \mu_2)^{k+1}}, & i \geq 1 \end{cases} \end{aligned} \tag{i}$$

This gives

$$A_i(z) = \sum_{k=0}^{\infty} a_{k,i}\, z^k = \begin{cases} \dfrac{1}{1+\rho_1(1-z)}, & i = 0 \\ \dfrac{1}{1+\rho_2(1-z)}, & i \geq 1 \end{cases} \tag{ii}$$

where $\rho_1 = \lambda/\mu_1$, $\rho_2 = \lambda/\mu_2$. Proceeding as in Example 15.24, the steady state probabilities satisfy [(15.210) gets modified]

$$q_j = q_0\, a_{j,0} + \sum_{i=1}^{j+1} q_i\, a_{j-i+1,i} \tag{iii}$$

and (see(15.212))

$$\begin{aligned} Q(z) &= \sum_{j=0}^{\infty} q_j\, z^j \\ &= q_0 \sum_{j=0}^{\infty} a_{j,0}\, z^j + \sum_{j=0}^{\infty} q_i\, a_{j-i+1,i} \\ &= q_0\, A_0(z) + \sum_{i=1}^{\infty} q_i\, z^i \sum_{m=0}^{\infty} a_{m,i}\, z^m\, z^{-1} \\ &= q_0\, A_0(z) + (Q(z) - q_0)\, A_1(z)/z \end{aligned} \tag{iv}$$

where (see (ii))

$$A_0(z) = \frac{1}{1+\rho_1(1-z)} \tag{v}$$

and

$$A_1(z) = \frac{1}{1+\rho_2(1-z)}. \tag{vi}$$

From (iv)

$$Q(z) = \frac{q_0\,(z\, A_0(z) - A_1(z))}{z - A_1(z)}. \tag{vii}$$

Since

$$
\begin{aligned}
Q(1) &= 1 = \frac{q_0 \left[A_0'(1) + A_0(1) - A_1'(1) \right]}{1 - A_1'(1)} \\
&= \frac{q_0 \, (1 + \rho_1 - \rho_2)}{1 - \rho_2}
\end{aligned}
$$

we obtain

$$
q_0 = \frac{1 - \rho_2}{1 + \rho_1 - \rho_2}. \tag{viii}
$$

Substituting (viii) into (vii) we can rewrite $Q(z)$ as

$$
\begin{aligned}
Q(z) &= (1 - \rho_2) \, \frac{(1 - z) \, A_1(z)}{A_1(z) - z} \cdot \frac{1}{1 + \rho_1 - \rho_2} \, \frac{1 - z \, A_0(z)/A_1(z)}{1 - z} \\
&= \left(\frac{1 - \rho_2}{1 - \rho_2 \, z} \right) \frac{1}{1 + \rho_1 - \rho_2} \, \frac{1 - \frac{\rho_2}{1 + \rho_1} \, z}{1 - \frac{\rho_1}{1 + \rho_1} \, z} \\
&= Q_1(z) \, Q_2(z)
\end{aligned} \tag{ix}
$$

where

$$
Q_1(z) = \frac{1 - \rho_2}{1 - \rho_2 \, z} = (1 - \rho_2) \sum_{k=0}^{\infty} \rho_2^k \, z^k
$$

and

$$
Q_2(z) = \frac{1}{1 + \rho_1 - \rho_2} \, \left(1 - \frac{\rho_2}{1 + \rho_1} \, z \right) \sum_{i=0}^{\infty} \left(\frac{\rho_1}{1 + \rho_1} \right)^i \, z^i.
$$

Finally substituting. $Q_1(z)$ and $Q_2(z)$ into (ix) we obtain

$$
q_n = q_0 \left[\sum_{i=0}^{n} \left(\frac{\rho_1}{1 + \rho_1} \right)^{n-i} \, \rho_2^i - \sum_{i=0}^{n-1} \rho_2^{i+1} \, \frac{\rho_1^{n-i-1}}{(1 + \rho_1)^{n-i}} \right]. \quad n = 1, 2, \cdots
$$

with q_0 as in (viii).

16.13 From (16-209), the Laplace transform of the waiting time distribution is given by

$$\begin{aligned}\Psi_w(s) &= \frac{1-\rho}{1-\lambda\left(\frac{1-\Phi_s(s)}{s}\right)} \\ &= \frac{1-\rho}{1-\rho\,\mu\left(\frac{1-\Phi_s(s)}{s}\right)}.\end{aligned} \tag{i}$$

Let

$$\begin{aligned}F_r(t) &= \mu\int_0^t [1-B(\tau)]d\tau \\ &= \mu\left[t-\int_0^t B(\tau)d\tau\right].\end{aligned} \tag{ii}$$

represent the residual service time distribution. Then its Laplace transform is given by

$$\begin{aligned}\Phi_F(s) &= L\left\{F_r(t)\right\} = \mu\left(\frac{1}{s}-\frac{\Phi_s(s)}{s}\right) \\ &= \mu\left(\frac{1-\Phi_s(s)}{s}\right).\end{aligned} \tag{iii}$$

Substituting (iii) into (i) we get

$$\Psi_w(s) = \frac{1-\rho}{1-\rho\,\Phi_F(s)} = (1-\rho)\sum_{n=0}^{\infty}[\rho\,\Phi_F(s)]^n, \quad |\Phi_F(s)|<1. \tag{iv}$$

Taking inverse transform of (iv) we get

$$F_w(t) = (1-\rho)\sum_{n=0}^{\infty}\rho^n\,F_r^{(n)}(t),$$

where $F_r^{(n)}(t)$ is the n^{th} convolution of $F_r(t)$ with itself.

16.14 Let ρ in (16.198) that represents the average number of customers that arrive during any service period be greater than one. Notice that

$$\rho = A^{'}(1) > 1$$

where

$$A(z) = \sum_{k=0}^{\infty} a_k\, z^k$$

From Theorem 15.9 on Extinction probability (pages 759-760) it follows that if $\rho = A^{'}(1) > 1$, the eqution

$$A(z) = z \tag{i}$$

has a unique positive root $\pi_0 < 1$. On the other hand, the transient state probabilities $\{\sigma_i\}$ satisfy the equation (15.236). By direct substitution with $x_i = \pi_0^i$ we get

$$\sum_{j=1}^{\infty} p_{ij}\, x_j = \sum_{j=1}^{\infty} a_{j-i+1}\, \pi_0^j \tag{ii}$$

where we have made use of $p_{ij} = a_{j-i+1}$, $i \geq 1$ in (15.33) for an $M/G/1$ queue. Using $k = j - i + 1$ in (ii), it reduces to

$$\sum_{k=2-i}^{\infty} a_k\, \pi_0^{k+i-1} = \pi_0^{i-1} \sum_{k=0}^{\infty} a_k\, \pi_0^k$$

$$= \pi_0^{i-1}\, \pi_0 = \pi_0^i = x_i \tag{iii}$$

since π_0 satisfies (i). Thus if $\rho > 1$, the $M/G/1$ system is transient with probabilities $\sigma_i = \pi_0^i$.

16.15 (a) The transition probability matrix here is the truncated version of (15.34) given by

$$P = \begin{pmatrix} a_0 & a_1 & a_2 & \cdot & \cdot & \cdot & \cdot & a_{m-2} & 1 - \sum_{k=0}^{m-2} a_k \\ a_0 & a_1 & a_2 & \cdot & \cdot & \cdot & \cdot & a_{m-2} & 1 - \sum_{k=0}^{m-2} a_k \\ 0 & a_0 & a_1 & \cdot & \cdot & \cdot & \cdot & a_{m-3} & 1 - \sum_{k=0}^{m-3} a_k \\ \vdots & \vdots & \vdots & & & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & a_0 & a_1 & 1-(a_0+a_1) \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & a_0 & 1-a_0 \end{pmatrix} \tag{i}$$

and it corresponds to the upper left hand block matrix in (15.34) followed by an m^{th} column that makes each row sum equal to unity.

(b) By direct sybstitution of (i) into (15-167), the steady state probabilities $\{q_j^*\}_{j=0}^{m-1}$ satisfy

$$q_j^* = q_0^* \, a_j + \sum_{i=1}^{j+1} q_i^* \, a_{j-i+1}, \qquad j = 0, 1, 2, \cdots, m-2 \tag{ii}$$

and the normalization condition gives

$$q_{m-1}^* = 1 - \sum_{i=0}^{m-2} q_i^*. \tag{iii}$$

Notice that (ii) in the same as the first $m-1$ equations in (15-210) for an $M/G/1$ queue. Hence the desired solution $\{q_j^*\}_{j=0}^{m-1}$ must satisfy the first $m-1$ equations in (15-210) as well. Since the unique solution set to (15.210) is given by $\{q_j\}_{j=0}^{\infty}$ in (16.203), it follows that the desired probabilities satisfy

$$q_j^* = c \, q_j, \qquad j = 0, 1, 2, \cdots, m-1 \tag{iv}$$

where $\{q_j\}_{j=0}^{m-1}$ are as in (16.203) for an $M/G/1$ queue. From (iii) we also get the normalization constant c to be

$$c = \frac{1}{\sum_{i=0}^{m-1} q_i}. \tag{v}$$

16.16 (a) The event $\{X(t) = k\}$ can occur in several mutually exclusive ways, *viz.*, in the interval $(0, t)$, n customers arrive and k of them continue their service beyond t. Let A_n = "n arrivals in $(0, t)$", and $B_{k,n}$ ="exactly k services among the n arrivals continue beyond t", then by the theorem of total probability

$$P\{X(t) = k\} = \sum_{n=k}^{\infty} P\{A_n \cap B_{k,n}\} = \sum_{n=k}^{\infty} P\{B_{k,n}|A_n\}P(A_n).$$

But $P(A_n) = e^{-\lambda t}(\lambda t)^n/n!$, and to evaluate $P\{B_{k,n}|A_n\}$, we argue as follows: From (9.28), under the condition that there are n arrivals in $(0, t)$, the joint distribution of the arrival instants agrees with the joint distribution of n independent random variables arranged in increasing order and distributed uniformly in $(0, t)$. Hence the probability that a service time S does not terminate by t, given that its starting time $\mathbf{x}$ has a uniform distribution in $(0, t)$ is given by

$$p_t = \int_0^t P(S > t - x|\mathbf{x} = x) f_{\mathbf{x}}(x) dx$$

$$= \int_0^t [1 - B(t - x)] \frac{1}{t} dx = \frac{1}{t} \int_0^t (1 - B(\tau)) \, d\tau = \frac{\alpha(t)}{t}$$

Thus $B_{k,n}$ given A_n has a Binomial distribution, so that

$$P\{B_{k,n}|A_n\} = \binom{n}{k} p_t^k (1 - p_t)^{n-k}, \quad k = 0, 1, 2, \cdots n,$$

and

$$
\begin{aligned}
P\{X(t)=k\} &= \sum_{n=k}^{\infty} e^{-\lambda t}\,\frac{(\lambda t)^n}{n!}\binom{n}{k}\left(\frac{\alpha(t)}{t}\right)^k\,\left(\frac{1}{t}\int_0^t B(\tau)d\tau\right)^{n-k} \\
&= e^{-\lambda t}\,\frac{[\lambda\alpha(t)]^k}{k!}\sum_{n=k}^{\infty}\frac{\left(\lambda t\frac{1}{t}\int_0^t B(\tau)d\tau\right)^{n-k}}{(n-k)!} \\
&= \frac{[\lambda\alpha(t)]^k}{k!}\,e^{-\lambda\left[t-\int_0^t B(\tau)d\tau\right]} \\
&= \frac{[\lambda\alpha(t)]^k}{k!}\,e^{-\lambda\int_0^t[1-B(\tau)]d\tau} \\
&= \frac{[\lambda\alpha(t)]^k}{k!}\,e^{-\lambda\,\alpha(t)}, \qquad k=0,1,2,\cdots
\end{aligned}
\tag{i}
$$

(b)

$$
\begin{aligned}
\lim_{t\to\infty}\alpha(t) &= \int_0^{\infty}[1-B(\tau)]d\tau \\
&= E\{\mathbf{s}\}
\end{aligned}
\tag{ii}
$$

where we have made use of (5-52)-(5-53). Using (ii) in (i), we obtain

$$
\lim_{t\to\infty} P\{x(t)=k\} = e^{-\rho}\,\frac{\rho^k}{k!} \tag{iii}
$$

where $\rho=\lambda\,E\{\mathbf{s}\}$.