Chapter 15

15.1 The chain represented by

$$P = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

is irreducible and aperiodic.

The second chain is also irreducible and aperiodic.

The third chain has two aperiodic closed sets $\{e_1, e_2\}$ and $\{e_3, e_4\}$ and a transient state e_5.

15.2 Note that both the row sums and column sums are unity in this case. Hence P represents a doubly stochastic matrix here, and

$$P^n = \frac{1}{m+1} \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix}$$

$$\lim_{n\to\infty} P\{\mathbf{x}_n = e_k\} = \frac{1}{m+1}, \qquad k = 0, 1, 2, \cdots m.$$

15.3 This is the "success runs" problem discussed in Example 15-11 and 15-23. From Example 15-23, we get

$$u_{i+1} = p_{i,i+1} u_i = \frac{1}{i+1} u_i = \frac{u_o}{(i+1)!}$$

so that from (15-206)

$$\sum_{k=1}^{\infty} u_k = u_0 \sum_{k=1}^{\infty} \frac{1}{k!} = e \cdot u_0 = 1$$

gives $u_0 = 1/e$ and the steady state probabilities are given by

$$u_k = \frac{1/e}{k!}, \qquad k = 1, 2, \cdots$$

15.4 If the zeroth generation has size m, then the overall process may be considered as the sum of m independent and identically distributed branching processes $\mathbf{x}_n^{(k)}$, $k = 1, 2, \cdots m$, each corresponding to unity size at the zeroth generation. Hence if π_0 represents the probability of extinction for any one of these individual processes, then the overall probability of extinction is given by

$$\begin{aligned}
&\lim_{n\to\infty} P[\mathbf{x}_n = 0 | \mathbf{x}_0 = m] = \\
&= P[\{\mathbf{x}_n^{(1)} = 0 | \mathbf{x}_0^{(1)} = 1\} \bigcap \{\mathbf{x}_n^{(2)} = 0 | \mathbf{x}_0^{(2)} = 1\} \bigcap \cdots \{\mathbf{x}_n^{(m)} = 0 | \mathbf{x}_0^{(m)} = 1\}] \\
&= \textstyle\prod_{k=1}^m P[\mathbf{x}_n^{(k)} = 0 | \mathbf{x}_0^{(k)} = 1] \\
&= \pi_0^m
\end{aligned}$$

15.5 From (15-288)-(15-289),

$$P(z) = p_0 + p_1 z + p_2 z^2, \quad \text{since } p_k = 0, \quad k \geq 3.$$

Also $p_0 + p_1 + p_2 = 1$, and from (15-307) the extinction probability is given by sloving the equation

$$P(z) = z.$$

Notice that

$$\begin{aligned}
P(z) - z &= p_0 - (1 - p_1)z + p_2 z^2 \\
&= p_0 - (p_0 + p_2)z + p_2 z^2 \\
&= (z - 1)(p_2 z - p_0)
\end{aligned}$$

and hence the two roots of the equation $P(z) = z$ are given by

$$z_1 = 1, \qquad z_2 = \frac{p_0}{p_2}.$$

Thus if $p_2 < p_0$, then $z_2 > 1$ and hence the smallest positive root of $P(z) = z$ is 1, and it represents the probability of extinction. It follows that such a tribe which does not produce offspring in abundence is bound to extinct.

15.6 Define the branching process $\{\mathbf{x}_n\}$

$$\mathbf{x}_{n+1} = \sum_{k=1}^{\mathbf{x}_n} \mathbf{y}_k$$

where $\mathbf{y}_k$ are i.i.d random variables with common moment generating function $P(z)$ so that (see (15-287)-(15-289))

$$P'(1) = E\{\mathbf{y}_k\} = \mu.$$

Thus

$$\begin{aligned} E\{\mathbf{x}_{n+1}|\mathbf{x}_n\} &= E\{\textstyle\sum_{k=1}^{\mathbf{x}_n} \mathbf{y}_k|\mathbf{x}_n = m\} \\ &= E\{\textstyle\sum_{k=1}^{m} \mathbf{y}_k|\mathbf{x}_n = m\} \\ &= E\{\textstyle\sum_{k=1}^{m} \mathbf{y}_k\} = mE\{\mathbf{y}_k\} = \mathbf{x}_n\,\mu \end{aligned}$$

Similarly

$$\begin{aligned} E\{\mathbf{x}_{n+2}|\mathbf{x}_n\} &= E\{E\{\mathbf{x}_{n+2}|\mathbf{x}_{n+1}, \mathbf{x}_n\}\} \\ &= E\{E\{\mathbf{x}_{n+2}|\mathbf{x}_{n+1}\}|\mathbf{x}_n\} \\ &= E\{\mu\mathbf{x}_{n+1}|\mathbf{x}_n\} = \mu^2\,\mathbf{x}_n \end{aligned}$$

and in general we obtain

$$E\{\mathbf{x}_{n+r}|\mathbf{x}_n\} = \mu^r\,\mathbf{x}_n. \tag{i}$$

Also from (15-310)-(15-311)

$$E\{\mathbf{x}_n\} = \mu^n. \tag{ii}$$

Define

$$\mathbf{w}_n = \frac{\mathbf{x}_n}{\mu^n}. \tag{iii}$$

This gives

$$E\{\mathbf{w}_n\} = 1.$$

Dividing both sider of (i) with μ^{n+r} we get

$$E\{\frac{\mathbf{x}_{n+r}}{\mu^{n+r}}|\mathbf{x}_n = x\} = \mu^r \cdot \frac{\mathbf{x}_n}{\mu^{n+r}} = \frac{\mathbf{x}_n}{\mu^n} = \mathbf{w}_n$$

or

$$E\{\mathbf{w}_{n+r}|\mathbf{w}_n = \frac{x}{\mu^n} \triangleq w\} = \mathbf{w}_n$$

which gives

$$E\{\mathbf{w}_{n+r}|\mathbf{w}_n\} = \mathbf{w}_n,$$

the desired result.

15.7

$$\mathbf{s}_n = \mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_n$$

where $\mathbf{x}_n$ are i.i.d. random variables. We have

$$\mathbf{s}_{n+1} = \mathbf{s}_n + \mathbf{x}_{n+1}$$

so that

$$E\{\mathbf{s}_{n+1}|\mathbf{s}_n\} = E\{\mathbf{s}_n + \mathbf{x}_{n+1}|\mathbf{s}_n\} = \mathbf{s}_n + E\{\mathbf{x}_{n+1}\} = \mathbf{s}_n.$$

Hence $\{\mathbf{s}_n\}$ represents a Martingale.

15.8 (a) From Bayes' theorem

$$\begin{aligned} P\{\mathbf{x_n} = j|\mathbf{x_{n+1}} = i\} &= \frac{P\{\mathbf{x}_{n+1} = i|\mathbf{x}_n = j\}\, P\{\mathbf{x}_n = j\}}{P\{\mathbf{x}_{n+1} = i\}} \\ &= \frac{q_j\, p_{ji}}{q_i} = p^*_{ij}, \end{aligned} \tag{i}$$

where we have assumed the chain to be in steady state.

(b) Notice that time-reversibility is equivalent to

$$p^*_{ij} = p_{ij}$$

and using (i) this gives

$$p^*_{ij} = \frac{q_j\, p_{ji}}{q_i} = p_{ij} \tag{ii}$$

or, for a time-reversible chain we get

$$q_j\, p_{ji} = q_i\, p_{ij}. \tag{iii}$$

Thus using (ii) we obtain by direct substitution

$$\begin{aligned} p_{ij}\, p_{jk}\, p_{ki} &= \left(\frac{q_j}{q_i}\, p_{ji}\right) \left(\frac{q_k}{q_j}\, p_{kj}\right) \left(\frac{q_i}{q_k}\, p_{ik}\right) \\ &= p_{ik}\, p_{kj}\, p_{ji}, \end{aligned}$$

the desired result.

15.9 (a) It is given that $A = A^T$, $(a_{ij} = a_{ji})$ and $a_{ij} > 0$. Define the i^{th} row sum

$$r_i = \sum_k a_{ik} > 0, \qquad i = 1, 2, \cdots$$

and let

$$p_{ij} = \frac{a_{ij}}{\sum_k a_{ik}} = \frac{a_{ij}}{r_i}.$$

Then

$$\begin{aligned} p_{ji} &= \frac{a_{ji}}{\sum\limits_m a_{jm}} = \frac{a_{ji}}{r_j} = \frac{a_{ij}}{r_j} \\ &= \frac{r_i}{r_j}\, \frac{a_{ij}}{r_i} = \frac{r_i}{r_j}\, p_{ij} \end{aligned} \tag{i}$$

or

$$r_i\, p_{ij} = r_j\, p_{ji}.$$

Hence

$$\sum_i r_i\, p_{ij} = \sum_i r_j\, p_{ji} = r_j \sum_i p_{ji} = r_j, \tag{ii}$$

since

$$\sum_i p_{ji} = \frac{\sum_i a_{ji}}{r_j} = \frac{r_j}{r_j} = 1.$$

Notice that (ii) satisfies the steady state probability distribution equation (15-167) with

$$q_i = c\, r_i, \qquad i = 1, 2, \cdots$$

where c is given by

$$c \sum_i r_i = \sum_i q_i = 1 \Longrightarrow c = \frac{1}{\sum_i r_i} = \frac{1}{\sum_i \sum_j a_{ij}}.$$

Thus

$$q_i = \frac{r_i}{\sum_i r_i} = \frac{\sum_j a_{ij}}{\sum_i \sum_j a_{ij}} > 0 \qquad \text{(iii)}$$

represents the stationary probability distribution of the chain.

With (iii) in (i) we get

$$p_{ji} = \frac{q_i}{q_j}\, p_{ij}$$

or

$$p_{ij} = \frac{q_j\, p_{ji}}{q_i} = p^*_{ij}$$

and hence the chain is time-reversible.

15.10 (a) $M = (m_{ij})$ is given by

$$M = (I - W)^{-1}$$

or

$$(I - W)M = I$$
$$M = I + WM$$

which gives

$$\begin{aligned} m_{ij} &= \delta_{ij} + \textstyle\sum_k w_{ik}\, m_{kj}, \quad e_i, e_j \in T \\ &= \delta_{ij} + \textstyle\sum_k p_{ik}\, m_{kj}, \quad e_i, e_j \in T \end{aligned}$$

(b) The general case is solved in pages 743-744. From page 744, with $N = 6$ (2 absorbing states; 5 transcient states), and with $r = p/q$ we obtain

$$m_{ij} = \begin{cases} \dfrac{(r^j - 1)(r^{6-i} - 1)}{(p-q)(r^6 - 1)}, & j \le i \\[2ex] \dfrac{(r^i - 1)(r^{6-i} - r^{j-i})}{(p-q)(r^6 - 1)}, & j \ge i. \end{cases}$$

15.11 If a stochastic matrix $A = (a_{ij}), a_{ij} > 0$ corresponds to the two-step transition matrix of a Markov chain, then there must exist another stochastic matrix P such that

$$A = P^2, \qquad P = (p_{ij})$$

where

$$p_{ij} > 0, \quad \sum_j p_{ij} = 1,$$

and this may not be always possible. For example in a two state chain, let

$$P = \begin{pmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{pmatrix}$$

so that

$$A = P^2 = \begin{pmatrix} \alpha^2 + (1-\alpha)(1-\beta) & (\alpha+\beta)(1-\alpha) \\ (\alpha+\beta)(1-\beta) & \beta^2 + (1-\alpha)(1-\beta) \end{pmatrix}.$$

This gives the sum of this its diagonal entries to be

$$\begin{aligned} a_{11} + a_{22} &= \alpha^2 + 2(1-\alpha)(1-\beta) + \beta^2 \\ &= (\alpha+\beta)^2 - 2(\alpha+\beta) + 2 \\ &= 1 + (\alpha+\beta-1)^2 \ \geq 1. \end{aligned} \tag{i}$$

Hence condition (i) necessary. Since $0 < \alpha < 1$, $0 < \beta < 1$, we also get $1 < a_{11} + a_{22} \leq 2$. Futher, the condition (i) is also sufficient in the 2×2 case, since $a_{11} + a_{22} > 1$, gives

$$(\alpha + \beta - 1)^2 = a_{11} + a_{22} - 1 > 0$$

and hence

$$\alpha + \beta = 1 \pm \sqrt{a_{11} + a_{22} - 1}$$

and this equation may be solved for all admissible set of values $0 < \alpha < 1$ and $0 < \beta < 1$.

15.12 In this case the chain is irreducible and aperiodic and there are no absorption states. The steady state distribution $\{u_k\}$ satisfies (15-167),and hence we get

$$u_k = \sum_j u_j \, p_{jk} = \sum_{j=0}^{N} u_j \binom{N}{k} p_j^k \, q_j^{N-k}.$$

Then if $\alpha > 0$ and $\beta > 0$ then "fixation to pure genes" does not occur.

15.13 The transition probabilities in all these cases are given by (page 765) (15A-7) for specific values of $A(z) = B(z)$ as shown in Examples 15A-1, 15A-2 and 15A-3. The eigenvalues in general satisfy the equation

$$\sum_j p_{ij}\, x_j^{(k)} = \lambda_k\, x_i^{(k)}, \quad k = 0, 1, 2, \cdots N$$

and trivially $\sum_j p_{ij} = 1$ for all i implies $\lambda_0 = 1$ is an eigenvalue in all cases.

However to determine the remaining eigenvalues we can exploit the relation in (15A-7). From there the corresponding conditional moment generating function in (15-291) is given by

$$G(s) = \sum_{j=0}^{N} p_{ij}\, s^j \tag{i}$$

where from (15A-7)

$$\begin{aligned} p_{ij} &= \frac{\{A^i(z)\}_j\,\{B^{N-i}(z)\}_{N-j}}{\{A^i(z)\,B^{N-i}(z)\}_N} \\ &= \frac{\text{coefficient of } s^j\, z^N \text{ in } \{A^i(sz)\,B^{N-i}(z)\}}{\{A^i(z)\,B^{N-i}(z)\}_N} \end{aligned} \tag{ii}$$

Substituting (ii) in (i) we get the compact expression

$$G(s) = \frac{\{A^i(sz)\,B^{N-i}(z)\}_N}{\{A^i(z)\,B^{N-i}(z)\}_N}. \tag{iii}$$

Differentiating $G(s)$ with respect to s we obtain

$$\begin{aligned} G'(s) &= \sum_{j=0}^{N} P_{ij}\, j\, s^{j-1} \\ &= \frac{\{iA^{i-1}(sz)\,A'(sz)z\,B^{N-i}(z)\}_N}{\{A^i(z)\,B^{N-i}(z)\}_N} \\ &= i \cdot \frac{\{A^{i-1}(sz)\,A'(sz)\,B^{N-i}(z)\}_{N-1}}{\{A^i(z)\,B^{N-i}(z)\}_N}. \end{aligned} \tag{iv}$$

Letting $s = 1$ in the above expression we get

$$G'(1) = \sum_{j=0}^{N} p_{ij}\, j = i\, \frac{\{A^{i-1}(z)\, A'(z)\, B^{N-i}(z)\}_{N-1}}{\{A^{i}(z)\, B^{N-i}(z)\}_{N}}. \tag{v}$$

In the special case when $A(z) = B(z)$, Eq.(v) reduces to

$$\sum_{j=0}^{N} p_{ij}\, j = \lambda_1\, i \tag{vi}$$

where

$$\lambda_1 = \frac{\{A^{N-1}(z)\, A'(z)\}_{N-1}}{\{A^{N}(z)\}_{N}}. \tag{vii}$$

Notice that (vi) can be written as

$$Px_1 = \lambda_1\, x_1, \quad x_1 = [0, 1, 2, \cdots N]^T$$

and by direct computation with $A(z) = B(z) = (q + pz)^2$ (Example 15A-1) we obtain

$$\begin{aligned} \lambda_1 &= \frac{\{(q+pz)^{2(N-1)}\, 2p(q+pz)\}_N}{\{(q+pz)^{2N}\}_N} \\ &= \frac{2p\{(q+pz)^{2N-1}\}_{N-1}}{\{(q+pz)^{2N}\}_N} = \frac{2p\binom{2N}{N-1} q^N\, p^{N-1}}{\binom{2N}{N} q^N\, p^N} = 1. \end{aligned}$$

Thus $\sum_{j=0}^{N} p_{ij}\, j = i$ and from (15-224) these chains represent Martingales. (Similarly for Examples 15A-2 and 15A-3 as well).

To determine the remaining eigenvalues we differentiate $G'(s)$ once more. This gives

$$\begin{aligned} G''(s) &= \sum_{j=0}^{N} p_{ij}\, j(j-1)\, s^{j-2} \\ &= \frac{\{i(i-1)A^{i-2}\,(sz)[A'(sz)]^2\, z\, B^{N-i}(z) + iA^{i-1}(sz)\, A''(sz) z\, B^{N-i}(z)\}_{N-1}}{\{A^i(z)\, B^{N-i}(z)\}_N}\;. \\ &= \frac{\{i\, A^{i-2}(sz)\, B^{N-i}(z)[(i-1)\,(A'(sz))^2 + A(sz)\, A''(sz)]\}_{N-2}}{\{A^i(z)\, B^{N-i}(z)\}_N}. \end{aligned}$$

With $s=1$, and $A(z)=B(z)$, the above expression simplifies to

$$\sum_{j=0}^{N} p_{ij}\, j(j-1) = \lambda_2\, i(i-1) + i\mu_2 \tag{viii}$$

where

$$\lambda_2 = \frac{\{A^{N-2}(z)\,[A'(z)]^2\}_{N-2}}{\{A^N(z)\}_N}$$

and

$$\mu_2 = \frac{\{A^{N-1}(z)\,A''(z)\}_{N-2}}{\{A^N(z)\}_N}.$$

Eq. $(viii)$ can be rewritten as

$$\sum_{j=0}^{N} p_{ij}\, j^2 = \lambda_2\, i^2 + (\text{polynomial in } i \text{ of degree } \le 1)$$

and in general repeating this procedure it follows that (show this)

$$\sum_{j=0}^{N} p_{ij}\, j^k = \lambda_k\, i^k + (\text{polynomial in } i \text{ of degree } \le k-1) \tag{ix}$$

where

$$\lambda_k = \frac{\{A^{N-k}(z)\,[A'(z)]^k\}_{N-k}}{\{A^N(z)\}_N}, \quad k = 1, 2, \cdots N. \tag{x}$$

Equations $(viii)$–(x) motivate to consider the identities

$$P\, q_k = \lambda_k\, q_k \tag{xi}$$

where q_k are polynomials in i of degree $\le k$, and by proper choice of constants they can be chosen in that form. It follows that λ_k, $k = 1, 2, \cdots N$ given by (ix) represent the desired eigenvalues.

(a) The transition probabilities in this case follow from Example 15A-1 (page 765-766) with $A(z) = B(z) = (q+pz)^2$. Thus using (ix) we obtain the desired eigenvalues to be

$$\begin{aligned}
\lambda_k &= \frac{\{(q+pz)^{2(N-k)}[2p(q+pz)]^k\}_{N-k}}{\{(q+pz)^{2N}\}_N} \\
&= 2^k\,p^k\,\frac{\{(q+pz)^{2N-k}\}_{N-k}}{\{(q+pz)^{2N}\}_N\}} \\
&= 2^k\,\frac{\binom{2N-k}{N-k}}{\binom{2N}{N}}, \quad k=1,2,\cdots N.
\end{aligned}$$

(b) The transition probabilities in this case follows from Example 15A-2 (page 766) with

$$A(z) = B(z) = e^{\lambda(z-1)}$$

and hence

$$\begin{aligned}
\lambda_k &= \frac{\{e^{\lambda(N-k)(z-1)}\,\lambda^k\,e^{\lambda k(z-1)}\}_{N-k}}{\{e^{\lambda N(z-1)}\}_N} \\
&= \frac{\lambda^k\,\{e^{\lambda N z}\}_{N-k}}{\{e^{\lambda N z}\}_N} = \frac{\lambda^k\,(\lambda N)^{N-k}/(N-k)!}{(\lambda N)^N/N!} \\
&= \frac{N!}{(N-k)!\,N^k} = \left(1-\frac{1}{N}\right)\left(1-\frac{2}{N}\right)\cdots\left(1-\frac{k-1}{N}\right), \quad k=1,2,\cdots N
\end{aligned}$$

(c) The transition probabilities in this case follow from Example 15A-3 (page 766-767) with

$$A(z) = B(z) = \frac{q}{1-pz}.$$

Thus

$$\begin{aligned}
\lambda_k &= p^k\,\frac{\{1/(1-pz)^{N+k}\}_{N-k}}{\{1/(1-pz)^N\}_N} \\
&= (-1)^k\,\frac{\binom{-(N+k)}{N-k}}{\binom{-N}{N}} = \frac{\binom{2N-1}{N-k}}{\binom{2N-1}{N}}, \quad r=2,3,\cdots N
\end{aligned}$$

15.14 From (15-240), the mean time to absorption vector is given by

$$m = (I - W)^{-1}\, E, \quad E = [1, 1, \cdots 1]^T,$$

where

$$W_{ik} = p_{jk}, \quad j, k = 1, 2, \cdots N - 1,$$

with p_{jk} as given in (15-30) and (15-31) respectively.

15.15 The mean time to absorption satisfies (15-240). From there

$$\begin{aligned} m_i \; &= 1 + \sum_{k \in T} p_{ik}\, m_k = 1 + p_{i,i+1}\, m_{i+1} + p_{i,i-1}\, m_{i-1} \\ &= 1 + p\, m_{i+1} + q\, m_{i-1}, \end{aligned}$$

or

$$m_k = 1 + p\, m_{k+1} + q\, m_{k-1}.$$

This gives

$$p\,(m_{k+1} - m_k) = q\,(m_k - m_{k-1}) - 1$$

Let

$$M_{k+1} = m_{k+1} - m_k$$

so that the above iteration gives

$$\begin{aligned} M_{k+1} \; &= \frac{q}{p}\, M_k - \frac{1}{p} \\ &= \left(\frac{q}{p}\right)^k M_1 - \frac{1}{p}\left[1 + \frac{q}{p} + \left(\frac{q}{p}\right)^2 + \cdots + \left(\frac{q}{p}\right)^{k-1}\right] \\ &= \begin{cases} \left(\frac{q}{p}\right)^k M_1 - \dfrac{1}{p-q}\left\{1 - (\frac{q}{p})^k\right\}, & p \neq q \\ M_1 - \dfrac{k}{p}, & p = q \end{cases} \end{aligned}$$

This gives

$$
\begin{aligned}
m_i &= \sum_{k=0}^{i-1} M_{k+1} \\
&= \begin{cases} \left(M_1 + \frac{1}{p-q}\right) \sum\limits_{k=0}^{i-1} \left(\frac{q}{p}\right)^k - \frac{i}{p-q}, & p \neq q \\ iM_1 - \frac{i(i-1)}{2p}, & p = q \end{cases} \\
&= \begin{cases} \left(M_1 + \frac{1}{p-q}\right) \frac{1-(q/p)^i}{1-q/p} - \frac{i}{p-q}, & p \neq q \\ iM_1 - \frac{i(i-1)}{2p}, & p = q \end{cases}
\end{aligned}
$$

where we have used $m_o = 0$. Similarly $m_{a+b} = 0$ gives

$$M_1 + \frac{1}{p-q} = \frac{a+b}{p-q} \cdot \frac{1-q/p}{1-(q/p)^{a+b}}.$$

Thus

$$m_i = \begin{cases} \frac{a+b}{p-q} \cdot \frac{1-(q/p)^i}{1-(q/p)^{a+b}} - \frac{i}{p-q}, & p \neq q \\ i(a+b-i), & p = q \end{cases}$$

which gives for $i = a$

$$
\begin{aligned}
m_a &= \begin{cases} \frac{a+b}{p-q} \cdot \frac{1-(q/p)^a}{1-(q/p)^{a+b}} - \frac{a}{p-q}, & p \neq q \\ ab, & p = q \end{cases} \\
&= \begin{cases} \frac{b}{2p-1} - \frac{a+b}{2p-1} \cdot \frac{1-(p/q)^b}{1-(p/q)^{a+b}}, & p \neq q \\ ab, & p = q \end{cases}
\end{aligned}
$$

by writing

$$\frac{1-(q/p)^a}{1-(q/p)^{a+b}} = 1 - \frac{(q/p)^a - (q/p)^{a+b}}{1-(q/p)^{a+b}} = 1 - \frac{1-(p/q)^b}{1-(p/q)^{a+b}}$$

(see also problem 3-10).