# Lower Bound on Scattered Power from Antennas

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***Abstract*—A lower bound on the total power scattered from a lossless and matched receiving antenna, for a given incident wave, is presented. The vector spherical wave expansion is used to describe the scattering and power absorption mechanisms. Scattering is the main reason behind antenna absorption. This poses a fundamental limit on minimization of antenna scattering. In other words, if we are to extract any power from the incident wave, the receiving antenna cannot be completely invisible, and scattering is an inevitable consequence of the power capturing process. To demonstrate the phenomena, a linear array antenna is optimized such that its scattered power is increasingly reduced and the total amount of scattered power for various incident angles is compared to our predicted lower bound.**

***Index Terms*— Antenna theory, Hilbert space, scattering from antennas.**

## I. Introduction

ABSORPTION efficiency of receiving antennas, which is defined as the ratio of the absorption cross section to the sum of the absorption and total scattering cross sections [1], is an important parameter in radar and, more recently, in sensing applications. Performance of antennas in noninvasive sensing devices such as near-field scanning optical microscope strongly depends on their scattering properties. Using plasmonic cover for cloaking the sensors in order to prevent them from disturbing the sampled fields has been proposed recently [2].

Scattering and reradiation from antennas has been the subject of numerous papers for many years. Canonical minimum-scattering antennas were introduced in [3], where the authors conjectured that, in general, the scattered power must be greater than or equal to the absorbed power in a receiving antenna. They proved this statement only for a particular class of conjugate-matched antennas. Recently, a common mistake, which is to assume that the internal power dissipated in the equivalent circuit of a receiving antenna is equal to the scattered power, has been debated [4], [5]. This mistake would lead to the wrong conclusion that a conjugately matched antenna should scatter power as much as it absorbs. However, Green had already shown that one could realize a matched antenna in which the absorbed power exceeded the scattered power [6]. More recently, it has been claimed that the ratio of the scattered power to the absorbed power can be as small as zero or, equivalently, the absorption efficiency can approach 100% as closely as desired [1]. A new dispersion relation for the extinction cross section was used in [7] to derive an upper bound on the gain-bandwidth product of antennas. In [8], a fundamental limit on the maximum absorbed power in an arbitrary antenna was derived. It was shown that, in optimal receiving conditions, all outward-traveling spherical waves must vanish, and the absorption efficiency reaches 50%. The minimum power scattered from a receiving antenna *as a function of the load impedance* was presented in [9], where the scattering properties of the antenna were assumed to be completely known. It must be noted that the scattered fields in [9] are a mixture of not only the structural scattering, but also the reradiation from the mismatched load.

The primary focus of this letter is to reexamine the process of power absorption in receiving antennas based on their scattering characteristics and, more importantly, to derive a lower bound on the total power scattered from a lossless and matched antenna for which only the transmitting radiation pattern is given. To this end, the widely used scattering matrix formulation based on vector spherical wave expansion is employed [3], [10], [11]. In this formulation, due to the completeness of spherical modes in free space, the fields exterior to the antenna structure are expanded in terms of vector spherical wave functions.

In Section II, the scattering matrix description of a receiving antenna is introduced and the physical role of the scattered fields in the absorption process is explained. In Section III, a lower bound on the total power scattered by a lossless and matched antenna with a prescribed complex radiation pattern is derived. The validity of the results is examined in Section IV by minimizing the scattered power from a four-element array of uniformly spaced half-wave dipoles. To this end, the lossless and reciprocal coupling network that connects the antenna elements to the matched load is optimized.

## II. Scattering Matrix Description

A complete network description of a receiving antenna can be obtained by the spherical mode decomposition of electromagnetic fields outside of the smallest sphere that encloses the antenna structure [3], [11]. In this formulation, every spherical mode in free-space constitutes a hypothetical *port*. All accessible waveguide ports are denoted by the index $\alpha$, and all orthonormal free-space modes are designated by the index $\beta$. Analogous to incident and reflected waves in waveguide terminology, modes occur in pairs of incoming and outgoing spherical waves in free space; all modes propagating toward the antenna reference plane are indicated by the vector "$\mathbf{a}$," and the outgoing modes are denoted by the vector "$\mathbf{b}$." The generalized scattering matrix $\mathbf{S}$ establishes the linear relationship between these two vectors, namely, $\mathbf{b} = \mathbf{S}\mathbf{a}$. As shown in Fig. 1, the incident plane wave contains both incoming and outgoing spherical modes. Since the plane-wave function is finite at the origin, both incoming and outgoing spherical waves must be equally excited

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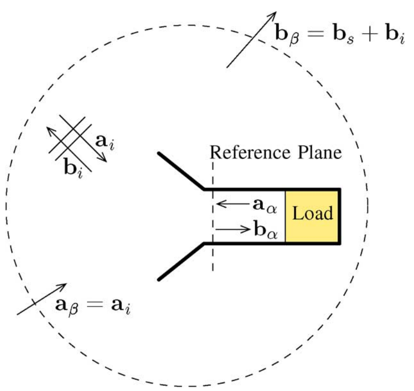


Fig. 1. Receiving antenna under illumination of a plane wave.

$(\mathbf{a}_i = \mathbf{b}_i)$. Equation $\mathbf{b} = \mathbf{S}\mathbf{a}$ can be expressed in a partitioned matrix form [3]

$$\begin{pmatrix} \mathbf{b}_\alpha \\ \mathbf{b}_\beta \end{pmatrix} = \begin{pmatrix} \mathbf{S}_{\alpha\alpha} & \mathbf{S}_{\alpha\beta} \\ \mathbf{S}_{\beta\alpha} & \mathbf{S}_{\beta\beta} \end{pmatrix} \begin{pmatrix} \mathbf{a}_\alpha \\ \mathbf{a}_\beta \end{pmatrix}. \tag{1}$$

For simplicity, we assume that the antenna has one single mode and matched input port. By this assumption, $\mathbf{S}_{\beta\alpha}$ will be a column vector whose components uniquely determine the complex radiation pattern of the antenna in transmitting mode. For reciprocal antennas, $\mathbf{S}_{\beta\alpha} = \mathbf{S}_{\alpha\beta}^T$, where the superscript $T$ denotes the *transpose*. As shown in Fig. 1, $\mathbf{b}_\beta$ contains the expansion coefficients of the outgoing spherical modes including those of the scattered field $(\mathbf{b}_s)$ and the outgoing modes appeared in the modal expansion of the incident plane wave $(\mathbf{b}_i)$. The scattering properties of the antenna are completely determined by $\mathbf{S}_{\beta\beta}$. In the absence of antenna, the incoming and outgoing modes are equally excited, thus, no power will be delivered to an arbitrary point in free space. The presence of the antenna disturbs the outgoing modes because of the scattered fields. A destructive interference reduces the power carried by the outgoing modes into the exterior region and thus disturbs the power balance between the incoming and outgoing waves. The difference is interpreted as the power extracted from the incident wave by the antenna. Taking this physical process into consideration, it can be inferred that there must be a lower bound on the amount of scattered power that is imposed by the energy conservation relation.

## III. Lower Bound on the Total Scattered Power

Consider a lossless reciprocal antenna located at origin and connected to a matched load through a single-mode transmission line or waveguide. Therefore, $\mathbf{a}_\alpha = 0$ and the antenna reradiation, i.e., the radiation with the transmitting pattern due to the reflected wave from the load, disappears. Moreover, it is reasonable to assume that the antenna is matched to the connecting line and, consequently, $\mathbf{S}_{\alpha\alpha}$ is zero [3]. With these assumptions, only the structural (also known as residual) scattering from the antenna is preserved. It is further assumed that the free-space modes as well as the waveguide modes are normalized and, since the antenna is lossless, the scattering matrix is unitary. Therefore, we can write [3]

$$\mathbf{S}_{\beta\alpha}^\dagger \mathbf{S}_{\beta\alpha} = \mathbf{I}_{\alpha\alpha} \tag{2}$$

$$\mathbf{S}_{\beta\alpha}^\dagger \mathbf{S}_{\beta\beta} = \mathbf{0} \tag{3}$$

$$\mathbf{S}_{\alpha\beta}^\dagger \mathbf{S}_{\alpha\beta} + \mathbf{S}_{\beta\beta}^\dagger \mathbf{S}_{\beta\beta} = \mathbf{I}_{\beta\beta}. \tag{4}$$

The superscript $\dagger$ denotes *the conjugate of the transpose*. After multiplying both sides of (3) from the right by $\mathbf{a}_\beta$ and using $\mathbf{a}_\alpha = 0$, we obtain the following orthogonality relation:

$$\mathbf{S}_{\beta\alpha}^\dagger \mathbf{b}_\beta = \mathbf{b}_\beta^\dagger \mathbf{S}_{\beta\alpha} = 0. \tag{5}$$

Since the spherical modes are orthonormal, the absorbed power $P_a$ and the scattered power $P_s$ can be expressed as

$$P_a = |\mathbf{b}_\alpha|^2 \tag{6}$$

$$P_s = |\mathbf{b}_s|^2 = |\mathbf{a}_\beta - \mathbf{b}_\beta|^2. \tag{7}$$

As solutions of the free-space wave equation, the spherical modes form a closed linear manifold. Therefore, we can introduce a Hilbert space [12]. The following definitions are adopted here, which are compatible with the nature of power equations:

$$\langle \chi_1, \chi_2 \rangle = \chi_2^\dagger \chi_1 \qquad \chi_1, \chi_2 \in \mathcal{H} \tag{8}$$

$$\|\chi\|^2 = |\chi|^2 = \langle \chi, \chi \rangle. \tag{9}$$

The physical quantities involved can now be expressed in terms of the inner product and norm in the above Hilbert space. Although an infinite set of spherical modes is excited in the scattering process, only a limited number of them are significant, and the antenna mainly interacts with those finite numbers of incoming modes. Hence, we could only consider a finite-dimensional space spanned by these *effective* modes.

According to (7), the scattered power is equal to the square of the Euclidean distance between $\mathbf{a}_\beta$ and $\mathbf{b}_\beta$. The absorbed power can also be expressed in terms of an inner product

$$P_a = \|\mathbf{b}_\alpha\|^2 = \|\mathbf{S}_{\alpha\beta}\mathbf{a}_\beta\|^2 = \left|\left\langle \mathbf{S}_{\beta\alpha}, \mathbf{a}_\beta^* \right\rangle\right|^2 \tag{10}$$

because $\mathbf{S}_{\alpha\beta}^T = \mathbf{S}_{\beta\alpha}$. The superscript $^*$ denotes *the complex conjugate*. Conservation of energy can be expressed in the following form:

$$\|\mathbf{b}_\beta\|^2 + \|\mathbf{b}_\alpha\|^2 = \|\mathbf{a}_\beta\|^2 \tag{11}$$

since $\mathbf{a}_\alpha = 0$. Substituting $\|\mathbf{b}_\alpha\|^2$ from (10) yields

$$\|\mathbf{b}_\beta\|^2 = \|\mathbf{a}_\beta^*\|^2 - \left|\left\langle \mathbf{S}_{\beta\alpha}, \mathbf{a}_\beta^* \right\rangle\right|^2. \tag{12}$$

We can decompose our Hilbert space into a direct sum of two complimentary orthogonal subspaces as follows [12]:

$$\mathcal{A} = \operatorname{span}\{\mathbf{S}_{\beta\alpha}\} \qquad \mathcal{H} = \mathcal{A} \oplus \mathcal{A}^\perp \tag{13}$$

From (2) and (12), one can immediately conclude that $\|\mathbf{b}_\beta\|$ is equal to the projection of $\mathbf{a}_\beta^*$ in the subspace $\mathcal{A}^\perp$. This is expressed by

$$\|\mathbf{b}_\beta\| = \|\operatorname{Prj}_{\mathcal{A}^\perp}(\mathbf{a}_\beta^*)\|. \tag{14}$$

Based on the triangular inequality in Hilbert spaces [12], the scattered power can be minimized with the constraints given in (5) and (14). Equation (5) could also be written in terms of an inner product

$$\mathbf{S}_{\beta\alpha}^\dagger \mathbf{b}_\beta = \langle \mathbf{b}_\beta, \mathbf{S}_{\beta\alpha} \rangle = 0.$$

The scattered power will be minimum if $\mathbf{b}_\beta$ lies in the direction of the projection of $\mathbf{a}_\beta$ in $\mathcal{A}^\perp$ subspace [12]. Mathematically, it means $\mathbf{b}_\beta$ must be parallel with $\mathbf{c}_\beta \triangleq \mathbf{a}_\beta - \langle \mathbf{a}_\beta, \mathbf{S}_{\beta\alpha}\rangle \mathbf{S}_{\beta\alpha}$. This relationship can be written as $\mathbf{c}_\beta = \mathbf{\Psi}_{\beta\beta}\mathbf{a}_\beta$ in which $\mathbf{\Psi}_{\beta\beta}$ is known as the *complementary projection operator* defined by

$$\mathbf{\Psi}_{\beta\beta} = \mathbf{I}_{\beta\beta} - \mathbf{S}_{\beta\alpha}\mathbf{S}^\dagger_{\beta\alpha}. \tag{15}$$

The lower bound on the scattered power can now be evaluated by considering the minimum possible geometrical distance between $\mathbf{a}_\beta$ and $\mathbf{b}_\beta$

$$\begin{aligned} P_{s,\min} &= |\mathbf{a}_\beta - \mathbf{b}_\beta|^2_{\min} \\ &= |\langle \mathbf{S}_{\beta\alpha}, \mathbf{a}_\beta\rangle|^2 + \left| |\operatorname{Prj}_{\mathcal{A}^\perp}(\mathbf{a}^*_\beta)| - |\operatorname{Prj}_{\mathcal{A}^\perp}(\mathbf{a}_\beta)|\right|^2. \end{aligned} \tag{16}$$

After some algebraic manipulations, (16) can be expressed in a more convenient form

$$\begin{aligned} P_{s,\min} = &\mathbf{a}^\dagger_\beta \left(\mathbf{S}_{\beta\alpha}\mathbf{S}^\dagger_{\beta\alpha}\right)\mathbf{a}_\beta \\ &+ \left| \left(\mathbf{a}^\dagger_\beta \mathbf{\Psi}_{\beta\beta}\mathbf{a}_\beta\right)^{1/2} - \left(\mathbf{a}^\dagger_\beta \mathbf{\Psi}^*_{\beta\beta}\mathbf{a}_\beta\right)^{1/2}\right|^2. \end{aligned} \tag{17}$$

This expression gives the lower bound on the scattered power explicitly in terms of the coefficients of the spherical mode decomposition of the radiation pattern in transmit. Its practical application for concrete radiation patterns is straightforward, as will be illustrated in Section IV.

If the antenna complex pattern (neglecting a separable phase factor) is symmetric with respect to the origin, it can receive equal amounts of power from two plane-waves propagating in opposite directions with conjugate polarization vectors. It can be easily shown that the two plane waves described above have conjugate excitation coefficients in their spherical mode expansions, and therefore $|\mathbf{b}_\alpha| = |\mathbf{S}_{\alpha\beta}\mathbf{a}_\beta|$ must be equal to $|\mathbf{S}_{\alpha\beta}\mathbf{a}^*_\beta| = |\mathbf{S}^*_{\alpha\beta}\mathbf{a}_\beta|$. Consequently, for this class of antennas, the first term on the right-hand side of (16) represents the absorbed power according to (6). In other words, (16) requires the minimum scattered power to be *greater than or equal to* the absorbed power, which means the absorption efficiency cannot exceed 50%. This result was previously derived by Kahn and Kurss in an alternative way [3].

Note that in our formulations it is assumed that the antenna is finite in size so that it can be enclosed by a hypothetical sphere. The same analysis can be applied to two-dimensional problems by cylindrical mode expansion and considering the absorbed and scattered powers per unit length. As a matter of fact, any two- or three-dimensional localized absorber cannot violate the absolute lower bound and become completely invisible. Some authors have tried to present examples of antennas with no scattered power. For example, as shown in [13], one can design an *infinite periodic array* of loaded dipoles with no scattered or reflected power, but such antennas are not realizable in the real world. In fact, for such cases, the result of (17) is zero.

## IV. Example

It was shown that there is a lower bound on the scattered power from a matched antenna for a given incident wave. In this section, we try to minimize the antenna scattering to obtain an antenna whose scattered power closely matches the lower bound. This optimization is performed for a particular incident plane wave. Finally, the antenna scattered power for other incident angles is compared to minimum achievable values given by (17).

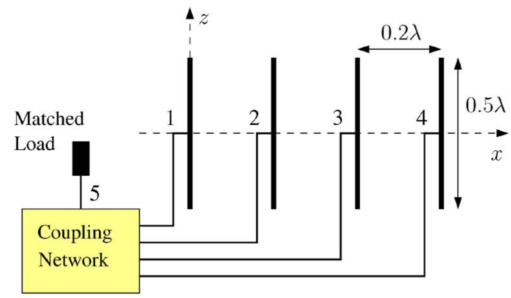


Fig. 2. Four-element equispaced half-wave dipole array. The incident plane wave is linearly polarized along the $z$-axis and is traveling along the $x$-axis.

A four-element half-wave dipole array with a lossless reciprocal coupling network that connects the antenna elements to a conjugate-matched load is considered. As shown in Fig. 2, the element spacing is $0.2\lambda$. Note that the 5-port coupling network is considered as part of the antenna structure.

In order to minimize the scattering, the matching network is optimized by the Differentiated Meta Particle Swarm Optimization algorithm with democratic implementation in evolution of particles [14]. Since the coupling network is lossless and reciprocal, it can be represented by a purely imaginary and symmetric impedance matrix whose elements are the optimization variables, i.e., the optimization algorithm adjusts 15 independent variables in the impedance matrix. It is assumed that a unit amplitude plane wave propagating in $-\hat{x}$ direction illuminates the antenna and is vertically polarized. The scattering problem is numerically analyzed by the method of moments with sinusoidal basis functions. With port numbers shown in Fig. 2, the following impedance matrix was obtained after the optimization:

$$Z_{\text{opt}} = j\begin{pmatrix} 314.8 & 135.1 & -13.2 & -111.0 & 123.3 \\ 135.1 & 604.9 & 666.1 & 181.3 & -7.4 \\ -13.2 & 666.1 & 284.3 & -514.2 & -480.7 \\ -111.0 & 181.3 & -514.2 & 971.3 & 716.2 \\ 123.3 & -7.4 & -480.7 & 716.2 & 788.3 \end{pmatrix}\Omega.$$

Radiation patterns of the optimized antenna in two principal planes are shown in Fig. 3. It must be noted that the antenna gain in the direction of incident wave is considerably reduced because for this direction the scattering from the antenna was minimized. This greatly reduces the capability of the antenna to absorb power, which is translated into a very low gain in the transmitting mode.

The total scattering cross section of the antenna for different incident angles is plotted in Fig. 4. $\sigma_\mathrm{s}$ is the total scattered power divided by the power density of the incident wave. A unit amplitude plane wave with linear polarization in $\hat{\theta}$-direction was assumed. Dashed lines represent our theoretical prediction of the minimum possible scattering cross section based on (17). Modal coefficients of the transmitting antenna that constitute $\mathbf{S}_{\beta\alpha}$ are approximately determined by numerical integration of

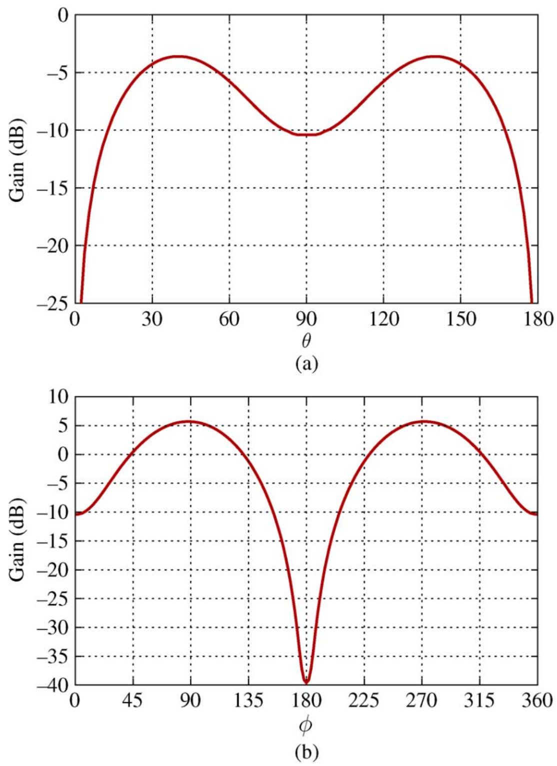


Fig. 3. Antenna gain after minimization of the scattered power. (a) E-plane. (b) H-plane.

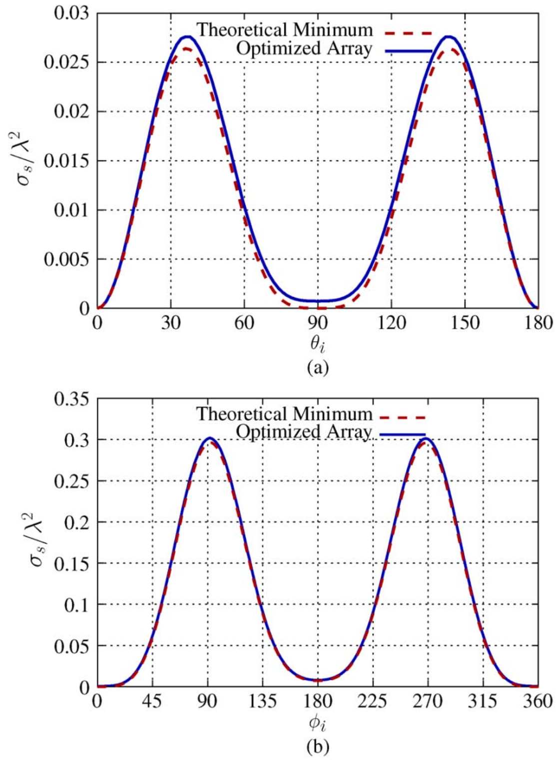


Fig. 4. Normalized total scattering cross section versus the angle of incidence. Solid lines represent the total scattering cross section obtained from method of moments (MoM) analysis, and dashed lines show the lower bound from (17). (a) E-plane or $xoz$-plane. (b) H-plane or $xoy$-plane.

the simulated far-field pattern multiplied by the complex conjugate of normalized spherical modes when the antenna is transmitting. In numerical calculations, the infinite modal series must be truncated. As discussed in [15] the number of spherical harmonics with independent radial behavior that is required to represent the antenna field is in the order of $ka$, where $a$ is the largest dimension of the antenna structure and $k$ is the free-space wavenumber. Spherical wave expansion of plane waves with arbitrary polarization can be found in various textbooks and papers [11].

## V. Conclusion

Spherical modal analysis was employed to investigate the physical process of power absorption in antennas. It was concluded that if the absorbed power is not zero, the energy conservation relation imposes a fundamental limit on the minimum achievable scattering from the antenna. A direct consequence of this limit is that the absorption efficiency of receiving antennas cannot approach 100% as closely as desired unless there is no power reception at all. Furthermore, a lower bound on the amount of scattered power emitted from a receiving antenna for which only the transmitting pattern is given was derived for the first time.

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