1 Useful Integrals

$$\int_0^{2\pi} e^{jx\cos(\phi-\alpha)} e^{jn\phi} d\phi = 2\pi J_n(x) j^n e^{jn\alpha}$$

$$\int_0^{\pi} \cos(n\phi) e^{jx\cos\phi} d\phi = \pi J_n(x) j^n$$

$$\int_0^{\pi} \cos\left[x\sin(\phi) - n\phi\right]\, d\phi = \pi J_n(x)$$

$$\int_0^{\frac{\pi}{2}} \cos\left[x\sin(\alpha)\right]\, d\alpha = \int_0^{\frac{\pi}{2}} \cos\left[x\cos(\alpha)\right]\, d\alpha = \frac{\pi}{2} J_0(x)$$

$$\int_0^{\pi} \cos\left(x\sin(\alpha)\right)\, d\alpha = \int_0^{\pi} \cos\left(x\cos(\alpha)\right)\, d\alpha = \pi J_0(x)$$

$$\int_0^{\frac{\pi}{2}} \sin\left[x\sin(\alpha)\right] \sin\alpha \, d\alpha = \frac{\pi}{2} J_1(x)$$

$$\int_0^{\frac{\pi}{2}} \cos\left[x\sin(\alpha)\right] \cos^2\alpha \, d\alpha = \frac{\pi}{2x} J_1(x)$$

$$\int_0^{\frac{\pi}{2}} \cos\left[x\sin(\alpha)\right] \cos 2\alpha \, d\alpha = \frac{\pi}{2} J_2(x)$$

$$\int_0^{\frac{\pi}{2}} \cos\left[x\sin(\alpha)\right] \cos(2n\alpha)\, d\alpha = (-1)^n \int_0^{\frac{\pi}{2}} \cos\left[x\cos(\alpha)\right] \cos(2n\alpha)\, d\alpha = \frac{\pi}{2} J_{2n}(x)$$

$$\int_{-\frac{l}{2}}^{\frac{l}{2}} e^{jkx} \cos\alpha x \, dx = l \frac{K\sin K\cos A - A\cos K\sin A}{K^2 - A^2} \qquad A = \frac{\alpha l}{2} \quad K = \frac{kl}{2}$$

$$\int_{-\frac{l}{2}}^{\frac{l}{2}} e^{jkx} dx = l \frac{\sin K}{K}$$

$$\int_{-\frac{l}{2}}^{\frac{l}{2}} e^{jkx} \sin\alpha x \, dx = -jl \frac{K\cos K\sin A - A\sin K\cos A}{K^2 - A^2}$$

$$\int_{x_0-\frac{w}{2}}^{x_0+\frac{w}{2}} \sin\alpha x \, dx = w \frac{\sin(W)\sin(2X_0)}{W} \qquad W = \frac{\alpha w}{2} \quad X_0 = \frac{\alpha x_0}{2}$$

$$\int_{x_0-\frac{w}{2}}^{x_0+\frac{w}{2}} \cos\alpha x \, dx = w \frac{\sin(W)\cos(2X_0)}{W}$$

$$\int_0^{\frac{\pi}{2}} \sin^{2m} x dx = \frac{1}{2} \int_0^{\pi} \sin^{2m} x dx = \frac{(2m-1)!!}{(2m)!!} \frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}} \cos^{2m} x dx = \frac{1}{2} \int_0^{\pi} \cos^{2m} x dx = \frac{(2m-1)!!}{(2m)!!} \frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}} \sin^{2m+1} x dx = \frac{1}{2} \int_0^{\pi} \sin^{2m+1} x dx = \frac{(2m)!!}{(2m+1)!!}$$

$$\int_0^{\frac{\pi}{2}} \cos^{2m+1} x dx = \frac{(2m)!!}{(2m+1)!!}$$

$$\int_0^{\pi} \frac{\cos\left[\alpha\cos\theta\right] - \cos\alpha}{\sin\theta} d\theta = \mathrm{Si}(2\alpha)\sin\alpha - \mathrm{Cin}(2\alpha)\cos\alpha$$

2 Sine and Cosine Integrals

2.1 Definitions

$$\begin{aligned}
\mathrm{Si}(x) &= \int_0^x \frac{\sin t}{t} dt \\
\mathrm{Cin}(x) &= \int_0^x \frac{1-\cos t}{t} dt \\
\mathrm{Ci}(x) &= \gamma + \ln(x) - \mathrm{Cin}(x) \qquad \gamma = 0.5772156649\cdots \\
\mathrm{Si}(ax) &= \int_0^x \frac{\sin(at)}{t} dt \\
\mathrm{Cin}(ax) &= \int_0^x \frac{1-\cos(at)}{t} dt
\end{aligned}$$

$$\begin{aligned}
\mathrm{Si}(-x) &= -\mathrm{Si}(x) \\
\mathrm{Cin}(-x) &= \mathrm{Cin}(x) \\
\mathrm{Ci}(-x) &= \mathrm{Ci}(x) - j\pi \\
\int_0^x \frac{1+\cos(\pi t)}{1+t} dt &= \mathrm{Cin}(\pi + \pi x) - \mathrm{Cin}(\pi) \\
\int_0^x \frac{1+\cos(\pi t)}{1-t} dt &= \mathrm{Cin}(\pi) - \mathrm{Cin}(\pi - \pi x) \\
\int_0^x \frac{\sin(\pi t)}{1+t} dt &= \mathrm{Si}(\pi) - \mathrm{Si}(\pi + \pi x) \\
\int_0^x \frac{\sin(\pi t)}{1-t} dt &= \mathrm{Si}(\pi) - \mathrm{Si}(\pi - \pi x)
\end{aligned}$$

2.2 Asymptotic Expansions for Small Argument

$$\begin{aligned}
\mathrm{Si}(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!} = x - \frac{x^3}{18} + \frac{x^5}{600} - \frac{x^7}{35280} + \cdots \\
\mathrm{Cin}(x) &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n}}{(2n)(2n)!} = \frac{x^2}{4} - \frac{x^4}{96} + \frac{x^6}{4320} - \ldots
\end{aligned}$$

2.3 Asymptotic Expansions for Large Argument

$$\begin{aligned}
\mathrm{Si}(x) &= \frac{\pi}{2} - \frac{\cos x}{x}\left(1 - \frac{2}{x^2} + \frac{24}{x^4} - \frac{720}{x^6} + \ldots\right) - \frac{\sin x}{x^2}\left(1 - \frac{6}{x^2} + \frac{120}{x^4} - \frac{5040}{x^6} + \ldots\right) \\
\mathrm{Ci}(x) &= \frac{\sin x}{x}\left(1 - \frac{2}{x^2} + \frac{24}{x^4} - \frac{720}{x^6} + \ldots\right) - \frac{\cos x}{x^2}\left(1 - \frac{6}{x^2} + \frac{120}{x^4} - \frac{5040}{x^6} + \ldots\right)
\end{aligned}$$

If $R = \sqrt{a^2 + (z-d)^2}$ then

$$\begin{aligned}
\int_0^h \frac{e^{-jkR}}{R} e^{\mp jkz} dz &= \pm e^{\mp jkd} \left\{ \mathrm{Ci}(u_1) - \mathrm{Ci}(u_0) - j\left[\mathrm{Si}(u_1) - \mathrm{Si}(u_0)\right] \right\} \\
&= \pm e^{\mp jkd} \left\{ \ln \frac{u_1}{u_0} + \mathrm{Cin}(u_0) - \mathrm{Cin}(u_1) + j\left[\mathrm{Si}(u_0) - \mathrm{Si}(u_1)\right] \right\}
\end{aligned}$$

$$u_0 = k\left(\sqrt{a^2 + d^2} \mp d\right) \qquad u_1 = k\left[\sqrt{a^2 + (h-d)^2} \mp (d-h)\right]$$