1 Gradient, Divergence, Curl, and Laplacian

1.1 Cartesian Coordinates (x, y, z)

$$\nabla V = \frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z \tag{1}$$

$$\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \tag{2}$$

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \tag{3}$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \tag{4}$$

1.2 Cylindrical Coordinates (ρ, ϕ, z)

$$\nabla V = \frac{\partial V}{\partial \rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\hat{a}_\phi + \frac{\partial V}{\partial z}\hat{a}_z \tag{5}$$

$$\nabla \cdot \vec{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \tag{6}$$

$$\nabla \times \vec{A} = \frac{1}{\rho}\begin{vmatrix} \hat{a}_\rho & \rho\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix} \tag{7}$$

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} \tag{8}$$

1.3 Spherical Coordinates (r, θ, ϕ)

$$\nabla V = \frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{a}_\phi \tag{9}$$

$$\nabla \cdot \vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi} \tag{10}$$

$$\nabla \times \vec{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{a}_r & r\hat{a}_\theta & r\sin\theta\hat{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix} \tag{11}$$

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} \tag{12}$$

2 Useful Vector Identities

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B}) \tag{13}$$

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B}) \tag{14}$$

$$(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = (\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D}) - (\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C}) \tag{15}$$

$$\nabla(\psi V) = \psi\nabla V + V\nabla\psi \tag{16}$$

$$\nabla\cdot(\psi\vec{A}) = \nabla\psi\cdot\vec{A} + \psi\nabla\cdot\vec{A} \tag{17}$$

$$\nabla\times(\psi\vec{A}) = \nabla\psi\times\vec{A} + \psi\nabla\times\vec{A} \tag{18}$$

$$\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B}) \tag{19}$$

$$\nabla\times(\vec{A}\times\vec{B}) = \vec{A}(\nabla\cdot\vec{B}) - \vec{B}(\nabla\cdot\vec{A}) + (\vec{B}\cdot\nabla)\vec{A} - (\vec{A}\cdot\nabla)\vec{B} \tag{20}$$

$$\nabla(\vec{A}\cdot\vec{B}) = (\vec{A}\cdot\nabla)\vec{B} + (\vec{B}\cdot\nabla)\vec{A} + \vec{A}\times(\nabla\times\vec{B}) + \vec{B}\times(\nabla\times\vec{A}) \tag{21}$$

$$\nabla\times\nabla V = 0 \tag{22}$$

$$\nabla\cdot(\nabla\times\vec{A}) = 0 \tag{23}$$

$$\nabla\times\nabla\times\vec{A} = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A} \tag{24}$$

$$\iiint_V \nabla\cdot\vec{A}\,dv = \oiint_S \vec{A}\cdot\vec{ds} \tag{25}$$

$$\iint_S \nabla\times\vec{A}\cdot\vec{ds} = \oint_C \vec{A}\cdot\vec{dl} \tag{26}$$

$$\oiint_S \vec{A}\times\vec{ds} = -\iiint_V \nabla\times\vec{A}\,dv \tag{27}$$

$$\oiint_S \psi\vec{ds} = \iiint_V \nabla\psi\,dv \tag{28}$$

$$\oint_C \psi\,\vec{dl} = -\iint_S \nabla\psi\times\vec{ds} \tag{29}$$

$$\oiint_S \Phi\nabla\Psi\cdot\vec{ds} = \iiint_V \left(\nabla\Phi\cdot\nabla\Psi + \Phi\nabla^2\Psi\right)dv \tag{30}$$

$$\iiint_V \left(\Phi\nabla^2\Psi - \Psi\nabla^2\Phi\right)dv = \oiint_S (\Phi\nabla\Psi - \Psi\nabla\Phi)\cdot\vec{ds} \tag{31}$$

$$\iiint_V (\nabla\times\vec{A}\cdot\nabla\times\vec{B} - \vec{A}\cdot\nabla\times\nabla\times\vec{B})dv = \oiint_S (\vec{A}\times\nabla\times\vec{B})\cdot\vec{ds} \tag{32}$$

$$\iiint_V (\vec{B}\cdot\nabla\times\nabla\times\vec{A} - \vec{A}\cdot\nabla\times\nabla\times\vec{B})dv = \oiint_S (\vec{A}\times\nabla\times\vec{B} - \vec{B}\times\nabla\times\vec{A})\cdot\vec{ds} \tag{33}$$

$$\nabla\frac{1}{|\vec{r}-\vec{r}'|} = -\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} \qquad \nabla\cdot\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} = 4\pi\delta(\vec{r}-\vec{r}') \tag{34}$$

$$\frac{d\vec{A}}{dt} = \frac{\partial\vec{A}}{\partial t} + \vec{A}\nabla\cdot\vec{v} - \nabla\times(\vec{A}\times\vec{v}) \qquad \vec{v} = \frac{d\vec{r}}{dt} = \text{velocity} \tag{35}$$

$$\nabla^2(\psi V) = \psi\nabla^2 V + V\nabla^2\psi + 2\nabla V\cdot\nabla\psi \tag{36}$$

The Taylor series expansion of a function $\psi(\vec{r})$ is:

$$\psi(\vec{r}+\vec{h}) = \psi(\vec{r}) + (\vec{h}\cdot\nabla)\psi(\vec{r}) + \frac{1}{2!}(\vec{h}\cdot\nabla)^2\psi(\vec{r}) + \ldots + \frac{1}{n!}(\vec{h}\cdot\nabla)^n\psi(\vec{r}) + \ldots \tag{37}$$

$$f(x+h) = f(x) + h\frac{d}{dx}f(x) + \frac{h^2}{2!}\frac{d^2}{dx^2}f(x) + \ldots + \frac{h^n}{n!}\frac{d^n}{dx^n}f(x) + \ldots \tag{38}$$