1 Useful Integrals

$$\int \frac{1}{(x^2+a^2)^{\frac{1}{2}}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right) = \sinh^{-1}\left(\frac{x}{a}\right) \tag{1}$$

$$\int \frac{x}{(x^2+a^2)^{\frac{1}{2}}}\,dx = \sqrt{x^2+a^2} \tag{2}$$

$$\int \frac{x^2}{(x^2+a^2)^{\frac{1}{2}}}\,dx = \frac{1}{2}\,x\sqrt{x^2+a^2} - \frac{1}{2}\,a^2\ln\left(x+\sqrt{x^2+a^2}\right) \tag{3}$$

$$\int \frac{1}{x\,(x^2+a^2)^{\frac{1}{2}}}\,dx = -\frac{1}{a}\tanh^{-1}\left(\frac{a}{\sqrt{x^2+a^2}}\right) \tag{4}$$

$$\int \frac{1}{x^2\,(x^2+a^2)^{\frac{1}{2}}}\,dx = -\frac{\sqrt{x^2+a^2}}{a^2x} \tag{5}$$

$$\int \frac{1}{(x^2+a^2)^{\frac{3}{2}}}\,dx = \frac{x}{a^2\sqrt{x^2+a^2}} \tag{6}$$

$$\int \frac{x}{(x^2+a^2)^{\frac{3}{2}}}\,dx = \frac{-1}{\sqrt{x^2+a^2}} \tag{7}$$

$$\int \frac{x^2}{(x^2+a^2)^{\frac{3}{2}}}\,dx = \frac{-x}{\sqrt{x^2+a^2}} + \ln\left(x+\sqrt{x^2+a^2}\right) \tag{8}$$

$$\int \frac{x^3}{(x^2+a^2)^{\frac{3}{2}}}\,dx = \frac{2a^2+x^2}{\sqrt{x^2+a^2}} \tag{9}$$

$$\int \frac{1}{x\,(x^2+a^2)^{\frac{3}{2}}}\,dx = \frac{1}{a^2\sqrt{x^2+a^2}} - \frac{1}{a^3}\tanh^{-1}\left(\frac{a}{\sqrt{x^2+a^2}}\right) \tag{10}$$

$$\int \frac{1}{x^2\,(x^2+a^2)^{\frac{3}{2}}}\,dx = -\frac{a^2+2x^2}{x\,a^4\sqrt{x^2+a^2}} \tag{11}$$

$$\int \frac{1}{(x^2+a^2)\sqrt{x^2+b^2}}\,dx = \frac{1}{a\sqrt{b^2-a^2}}\tan^{-1}\left[\frac{x\sqrt{b^2-a^2}}{a\sqrt{x^2+b^2}}\right] \tag{12}$$

$$\int \frac{1}{(x^2+a^2)}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) \tag{13}$$

$$\int_0^{2\pi} \frac{1}{a+b\cos\alpha}\,d\alpha = \frac{2\pi}{\sqrt{a^2-b^2}} \qquad a>b \tag{14}$$

$$\int_0^{2\pi} \frac{\sin^2\alpha}{a+b\cos\alpha}\,d\alpha = \frac{2\pi}{b^2}\left(a-\sqrt{a^2-b^2}\right) \qquad a>b \tag{15}$$

$$\int_0^{2\pi} \frac{\sin\alpha}{a+b\cos\alpha}\,d\alpha = 0 \qquad a>b \tag{16}$$

$$\int_0^{\pi} \frac{\sin\alpha}{a+b\cos\alpha}\,d\alpha = \frac{1}{b}\ln\frac{a+b}{a-b} \qquad a>b \tag{17}$$

$$\int_0^{2\pi} \frac{\cos\alpha}{a+b\cos\alpha}\,d\alpha = \frac{2\pi}{b}\left(1-\frac{a}{\sqrt{a^2-b^2}}\right) \qquad a>b \tag{18}$$

$$\int \frac{1}{(R^2+Z^2-2ZRx)^{\frac{3}{2}}}\,dx = \frac{1}{ZR\sqrt{R^2+Z^2-2ZRx}} \tag{19}$$

$$\int \frac{Z-Rx}{(R^2+Z^2-2ZRx)^{\frac{3}{2}}}\,dx = \frac{Zx-R}{Z^2\sqrt{R^2+Z^2-2ZRx}} \tag{20}$$

$$\int \frac{1}{(R^2+Z^2-2ZRx)^{\frac{1}{2}}}\,dx = -\frac{\sqrt{R^2+Z^2-2ZRx}}{ZR} \tag{21}$$

$$\int \frac{x}{(R^2+Z^2-2ZRx)^{\frac{1}{2}}}\,dx = -\frac{\sqrt{R^2+Z^2-2ZRx}\,\left(R^2+Z^2+RZx\right)}{3Z^2R^2} \tag{22}$$

$$\int \frac{Z-Rx}{(R^2+Z^2-2ZRx)^{\frac{1}{2}}}\,dx = \frac{(R^2-2Z^2+ZRx)\sqrt{R^2+Z^2-2ZRx}}{3Z^2R} \tag{23}$$

2 Gradient, Divergence, Curl, and Laplacian

2.1 Cartesian Coordinates (x, y, z)

$$\nabla V = \frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z} \tag{24}$$

$$\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \tag{25}$$

$$\nabla \times \vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{z} \tag{26}$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \tag{27}$$

2.2 Cylindrical Coordinates (ρ, ϕ, z)

$$\nabla V = \frac{\partial V}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\hat{\phi} + \frac{\partial V}{\partial z}\hat{z} \tag{28}$$

$$\nabla \cdot \vec{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \tag{29}$$

$$\nabla \times \vec{A} = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{\rho} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right)\hat{\phi} + \left(\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\phi) - \frac{1}{\rho}\frac{\partial A_\rho}{\partial \phi}\right)\hat{z} \tag{30}$$

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} \tag{31}$$

2.3 Spherical Coordinates (r, θ, ϕ)

$$\nabla V = \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{\phi} \tag{32}$$

$$\nabla \cdot \vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi} \tag{33}$$

$$\nabla \times \vec{A} = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial \theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial \phi}\right)\hat{r} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{\partial}{\partial r}(rA_\phi)\right)\hat{\theta} + \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right)\hat{\phi} \tag{34}$$

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} \tag{35}$$

3 Fourier series

If m and n are integer numbers:

$$f(x) = \frac{A_0}{2} + \sum_{m=1}^{\infty} \left(A_m \cos \frac{m\pi x}{L} + B_m \sin \frac{m\pi x}{L} \right) \qquad 0 \le x \le L \tag{36}$$

$$A_m = \frac{2}{L} \int_0^L f(x) \cos \frac{m\pi x}{L}\, dx \qquad m \ge 0 \tag{37}$$

$$B_m = \frac{2}{L} \int_0^L f(x) \sin \frac{m\pi x}{L}\, dx \qquad m \ge 1 \tag{38}$$

$$\int_0^L \sin \frac{m\pi x}{L}\, \sin \frac{n\pi x}{L}\, dx = \begin{cases} 0, & m \ne n \\ \frac{L}{2}, & m = n \end{cases} \tag{39}$$

$$\int_0^L \cos \frac{m\pi x}{L}\, \cos \frac{n\pi x}{L}\, dx = \begin{cases} 0, & m \ne n \\ \frac{L}{2}, & m = n \end{cases} \tag{40}$$

$$\int_0^L \sin \frac{m\pi x}{L}\, \cos \frac{n\pi x}{L}\, dx = 0 \qquad \forall\ m, n \tag{41}$$

$$\int_0^L \sin \frac{m\pi x}{L}\, dx = \frac{L}{m\pi} (1 - \cos m\pi) \tag{42}$$

$$\int_0^L \cos \frac{m\pi x}{L}\, dx = 0 \qquad m \ge 1 \tag{43}$$

$$\int_0^L x \sin \frac{m\pi x}{L}\, dx = -\frac{L^2}{m\pi} \cos m\pi \tag{44}$$

$$\int_0^L x \cos \frac{m\pi x}{L}\, dx = -\frac{L^2}{m^2\pi^2} (1 - \cos m\pi) \tag{45}$$

$$\int_0^L \left(1 - \frac{x}{L}\right) \sin \frac{m\pi x}{L}\, dx = \frac{L}{m\pi} \tag{46}$$

$$\int_0^L \left(1 - \frac{x}{L}\right) \cos \frac{m\pi x}{L}\, dx = \frac{L}{m^2\pi^2} (1 - \cos m\pi) \tag{47}$$

4 Useful Vector Identities

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B}) \tag{48}$$

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B}) \tag{49}$$

$$(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = (\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D}) - (\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C}) \tag{50}$$

$$\nabla(\psi V) = \psi\nabla V + V\nabla\psi \tag{51}$$

$$\nabla\cdot(\psi\vec{A}) = \nabla\psi\cdot\vec{A} + \psi\nabla\cdot\vec{A} \tag{52}$$

$$\nabla\times(\psi\vec{A}) = \nabla\psi\times\vec{A} + \psi\nabla\times\vec{A} \tag{53}$$

$$\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B}) \tag{54}$$

$$\nabla\times(\vec{A}\times\vec{B}) = \vec{A}(\nabla\cdot\vec{B}) - \vec{B}(\nabla\cdot\vec{A}) + (\vec{B}\cdot\nabla)\vec{A} - (\vec{A}\cdot\nabla)\vec{B} \tag{55}$$

$$\nabla(\vec{A}\cdot\vec{B}) = (\vec{A}\cdot\nabla)\vec{B} + (\vec{B}\cdot\nabla)\vec{A} + \vec{A}\times(\nabla\times\vec{B}) + \vec{B}\times(\nabla\times\vec{A}) \tag{56}$$

$$\nabla\times\nabla V = 0 \tag{57}$$

$$\nabla\cdot(\nabla\times\vec{A}) = 0 \tag{58}$$

$$\nabla\times\nabla\times\vec{A} = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A} \tag{59}$$

$$\iiint_V \nabla\cdot\vec{A}\,dv = \oiint_S \vec{A}\cdot\vec{ds} \tag{60}$$

$$\iint_S \nabla\times\vec{A}\cdot\vec{ds} = \oint_C \vec{A}\cdot\vec{dl} \tag{61}$$

$$\oiint_S \vec{A}\times\vec{ds} = -\iiint_V \nabla\times\vec{A}\,dv \tag{62}$$

$$\oiint_S \psi\vec{ds} = \iiint_V \nabla\psi\,dv \tag{63}$$

$$\oint_C \psi\,\vec{dl} = -\iint_S \nabla\psi\times\vec{ds} \tag{64}$$

$$\oiint_S \Phi\nabla\Psi\cdot\vec{ds} = \iiint_V \left(\nabla\Phi\cdot\nabla\Psi + \Phi\nabla^2\Psi\right)dv \tag{65}$$

$$\iiint_V \left(\Phi\nabla^2\Psi - \Psi\nabla^2\Phi\right)dv = \oiint_S (\Phi\nabla\Psi - \Psi\nabla\Phi)\cdot\vec{ds} \tag{66}$$

$$\iiint_V (\nabla\times\vec{A}\cdot\nabla\times\vec{B} - \vec{A}\cdot\nabla\times\nabla\times\vec{B})dv = \oiint_S (\vec{A}\times\nabla\times\vec{B})\cdot\vec{ds} \tag{67}$$

$$\iiint_V (\vec{B}\cdot\nabla\times\nabla\times\vec{A} - \vec{A}\cdot\nabla\times\nabla\times\vec{B})dv = \oiint_S (\vec{A}\times\nabla\times\vec{B} - \vec{B}\times\nabla\times\vec{A})\cdot\vec{ds} \tag{68}$$

$$\nabla\frac{1}{|\vec{r}-\vec{r}'|} = -\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} \qquad \nabla\cdot\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} = 4\pi\delta(\vec{r}-\vec{r}') \tag{69}$$

$$\frac{d\vec{A}}{dt} = \frac{\partial\vec{A}}{\partial t} + \vec{A}\nabla\cdot\vec{v} - \nabla\times(\vec{A}\times\vec{v}) \qquad \vec{v} = \frac{d\vec{r}}{dt} = \text{velocity} \tag{70}$$

$$\nabla^2(\psi V) = \psi\nabla^2 V + V\nabla^2\psi + 2\nabla V\cdot\nabla\psi \tag{71}$$