Useful Approximations for the Directivity and Beamwidth of Large Scanning Dolph-Chebyshev Arrays

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Abstract—Approximate but accurate formulas are presented in a closed form that makes possible an easy examination and computation of directivity and beamwidth for large scanning Dolph-Chebyshev arrays. Array length, number of elements, spacing of the elements, sidelobe level, and angle of scan are parameters. Element spacings less than $\lambda/2$ are included. Comparisons between exact and approximate theory are presented in a graphical form that illustrates the lower limits of array size for which the approximations are valid.

The maximum directivity for Chebyshev arrays is in principle limited, regardless of array size. Values of maximum directivity, and the particular array designs required to achieve them, are given for several fixed array lengths. The directivity-beamwidth product is evaluated over a wide range of sidelobe levels and array lengths, and the region over which this product is essentially constant is specified. In the interests of achieving joint minimization of beamwidth and maximization of directivity, the particular Chebyshev design such that the directivity-beamwidth ratio is maximized is determined.

INTRODUCTION

THE Dolph-Chebyshev array is an optimally designed linear array of uniformly spaced isotropic antenna elements. This design, among all linear arrays of isotropic elements equal in number and spacing, is optimum in the following sense. The beamwidth of its radiation pattern is minimum for a specified sidelobe level; alternatively, the sidelobe level of its radiation pattern is minimum for a specified beamwidth. Ever since Dolph [1] introduced this design, numerous attempts [2] have been made to simplify the calculation of the array excitation coefficients needed to produce its radiation pattern. This author [3] has recently obtained a rather simple approximate representation by means of which these calculations can be simplified for spacings that include even the superdirective region. One of the purposes of the present paper is to introduce approximate representations for directivity and beamwidth for large Dolph-Chebyshev arrays that may be used for scanning purposes. (It should be recalled [4], however, that the particular design that is optimum at broadside will not generally be so when its pattern is electronically scanned to some other direction.)

These approximation formulas are then used to examine, for example, extremal properties of the directivity as a function of the sidelobe level, interelement spacing, and array length. The results are shown to be consistent with Elliott's observation [5] that the directivity of a Dolph pattern of given sidelobe level does not increase indefinitely with array length, but instead asymptotically approaches a limit that is 3 dB greater than the value of the sidelobe level.

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DIRECTIVITY

For symmetric arrays of $2N+1$ elements, broadside operation yields the following expression for directivity

$$D = \frac{\mu R^2}{\int_0^{\mu} T_N^2(a\cos\psi + b)\,d\psi}, \tag{1}$$

where R is the sidelobe voltage ratio, μ is $2\pi d/\lambda$, d being the interelement spacing, ψ is $\mu \sin\theta$, θ being the direction of radiation measured from broadside, and $T_N(z)$ is the Chebyshev polynomial in z of degree N. For spacings $d \geq \lambda/2$, DuHamel [6] has shown that

$$a = \frac{z_0 + 1}{2}, \qquad b = \frac{z_0 - 1}{2}, \tag{2}$$

where $T_N(z_0) = R$. Solving for z_0 yields

$$z_0 = \cosh\left(\frac{1}{N}\cosh^{-1} R\right), \tag{3}$$

since $R > 1$. It is clear that $z_0 > 1$ and that $a + b = z_0$.

We now consider the quantity $(a\cos\psi + b)$, which is illustrated in Fig. 1. Let ψ_0 be that value of ψ such that $(a\cos\psi + b) = 1$. For large N and moderate R it is clear that z_0 will be only slightly larger than 1, and hence that ψ_0 will be rather small.

For spacings satisfying $\pi \leq \mu \leq 2\pi - \psi_0$, where we note that $\mu = \pi$ corresponds to $d = \lambda/2$ and $\mu = 2\pi$ corresponds to $d = \lambda$, (1) becomes

$$D = \frac{\mu R^2}{\int_0^{\psi_0} \cosh^2[N\cosh^{-1}(a\cos\psi + b)]\,d\psi + \int_{\psi_0}^{\mu} \cos^2[N\cos^{-1}(a\cos\psi + b)]\,d\psi}. \tag{4}$$

As recently reported [7], for large N it can be shown that

$$\int_0^{\psi_0} \cosh^2[N\cosh^{-1}(a\cos\psi + b)]\,d\psi \sim \frac{\pi\psi_0}{4} I_1(2N\psi_0), \tag{5}$$

where I_1 is the modified Bessel function of the first kind, of

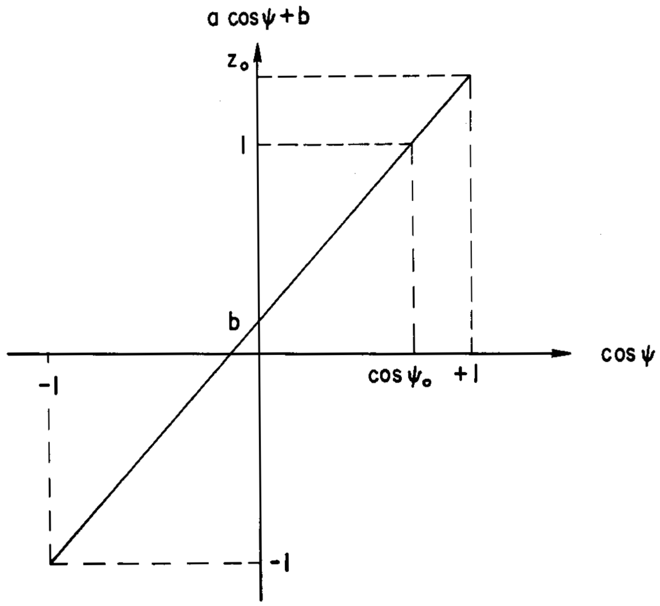


Fig. 1. Plot of the quantity ($a \cos \psi + b$) versus $\cos \psi$.

order 1, and

$$\int_{\psi_0}^{\mu} \cos^2 [N \cos^{-1} (a \cos \psi + b)]\, d\psi \sim \frac{1}{2} (\mu - \psi_0).$$

For reasonably large arrays (say, $N > 10$) the directivity can then be approximately represented by

$$D \sim \frac{4\mu R^2}{2(\mu - \psi_0) + \pi \psi_0 I_1(2N\psi_0)}. \tag{6}$$

From the foregoing, it is clear that as N increases without limit, a approaches unity, and b approaches zero, and hence ψ_0 approaches zero. Again, for large N, it has been shown [7] that

$$z_0 \sim \cosh \psi_0.$$

Reference to (3) indicates then that

$$N\psi_0 \sim \cosh^{-1} R. \tag{7}$$

So for a fixed R, the second term in the denominator of (6) tends to vanish with increasing N, showing that for any spacing in the assumed range the directivity of a Chebyshev array has the limiting value of $2R^2$, a result that is in precise agreement with the previously mentioned result of Elliott [5]. This gain limitation (not apparent in uniformly illuminated arrays, for example) is of no serious practical consequence, even for rather large arrays.

For interelement spacings such that $2\pi - \psi_0 \leq \mu \leq 2\pi$, it follows quite similarly (see Fig. 1) that

$$D \sim \frac{4\mu R^2}{4(\pi - \psi_0) + \pi \psi_0 I_1(2N\psi_0) + 4 \displaystyle\int_{2\pi-\psi_0}^{\mu} \cosh^2 [N \cosh^{-1} (a \cos \psi + b)]\, d\psi}. \tag{8}$$

In (8), as in (6), for fixed R (or equivalently $N\psi_0$) we again observe the property that the maximum directivity $2R^2$ is approached as we let N increase without limit. We have thus seen that this property quite generally exists for all spacings between half wavelength and full wavelength.

Applying asymptotic techniques similar to those leading to (6), we ultimately obtain for (8) the result

$$D \sim \frac{4\mu R^2}{4(\pi - \psi_0) + \pi \psi_0 [I_1(2N\psi_0) + I_1(2N\psi_0 \sqrt{1 - \beta^2}) \sqrt{1 - \beta^2}\, Q(\beta)]}, \tag{9}$$

where $\beta = (2\pi - \mu)/\psi_0$ and $Q(\beta)$, proportional to an incomplete error function, has the following rational function approximation [8]

$$Q(\beta) = \frac{a_1}{x} + \frac{a_2}{x^2} + \frac{a_3}{x^3},$$

for which

$$x = 1 + p \left[\frac{2N\psi_0}{\sqrt{1 - \beta^2}} \right]^{\frac{1}{2}} \beta$$

and

$$a_1 = 0.34802$$
$$a_2 = -0.09588$$
$$a_3 = 0.74786$$
$$p = 0.33267.$$

Referring again to Fig. 1 we can see how approximate representations of directivity for spacings greater than full wavelength are obtained in terms of results already presented; however, such spacings are not ordinarily of great interest, since they introduce grating lobes.

For the so-called "superdirective" region of element spacings less than half wavelength (to be specific, we shall consider the region $\pi \geq \mu \geq \psi_0$), it has been shown [6] that the quantities a and b of (1) are

$$a = \frac{z_0 + 1}{1 - \cos\mu}; \qquad b = -\frac{z_0 \cos\mu + 1}{1 - \cos\mu}. \tag{10}$$

Let ψ_1 be that value of ψ such that

$$a \cos\psi + b = -1.$$

Use of (10) demonstrates that $\psi_1 = \mu$. The formulation of directivity given by (6) can, therefore, be extended, with a slight modification, to include the superdirective region. Thus,

$$D \sim \frac{4\mu R^2}{2(\mu - \psi_0) + \pi\psi_0 I_1\left(2N\psi_0/\sin\frac{\mu}{2}\right)}. \tag{11}$$

As for the optimum normalized spacing μ, for broadside operation other studies [9] have shown that directivity increases somewhat linearly with spacing, and then after reaching a peak at just short of full-wavelength spacing, rather abruptly decreases (as grating lobes begin to appear) to a value at full wavelength that is equal to that at half wavelength. Close examination of (6) and (8) indicates that this peak usually occurs within the μ-region for which (6) is valid, and very near the end of that region if N is reasonably large.

The formalism for directivity as represented by (6), (9), and (11) can be simplified further. First of all, for the region for which (6) is valid, (7) holds, and thereby

$$D \sim \frac{2R^2}{1 + \frac{\pi}{2\mu N} I_1(2\cosh^{-1} R)\cosh^{-1} R}. \tag{6a}$$

For the region $2\pi - \psi_0 \leq \mu \leq 2\pi$, for which (9) is applicable, use of (7) yields

$$D \sim \frac{2R^2}{\frac{2\pi}{\mu}\left\{1 + \frac{1}{4N}\left[I_1(2\cosh^{-1} R) + I_1(2\sqrt{1-\beta^2}\cosh^{-1} R)\sqrt{1-\beta^2}\,Q(\beta)\right]\cosh^{-1} R\right\}}. \tag{9a}$$

For the superdirective region, it has been shown [7] that

$$\frac{N\psi_0}{\sin\frac{\mu}{2}} \approx \cosh^{-1} R,$$

holding for arbitrarily small values of μ. (It is important to note here that in this region ψ_0 is a function of μ.) It is then true that (11) can be approximated by

$$D \sim \frac{2R^2}{1 + \frac{\pi}{2\mu N} I_1(2\cosh^{-1} R)\cosh^{-1} R \sin\frac{\mu}{2}}. \tag{11a}$$

It is sometimes useful to have the directivity expressed in terms of the length L of the array, here defined to be $2Nd$. Equations (6a) and (11a) become, respectively,

$$D \sim \frac{2R^2}{1 + \frac{\lambda}{2L} I_1(2\cosh^{-1} R)\cosh^{-1} R} \tag{12}$$

and

$$D \sim \frac{2R^2}{1 + \frac{\lambda}{2L} I_1(2\cosh^{-1} R)\cosh^{-1} R \sin\frac{\mu}{2}}. \tag{13}$$

From (12), we see that the directivity is independent of element spacing for $\pi \leq \mu \leq 2\pi - \psi_0$. The variation of D with R for several fixed array lengths is shown in Fig. 2.

A most significant deduction to be drawn from (13) is that the maximum attainable directivity is $2R^2$, even in the superdirective region. For a given array length L and sidelobe ratio R, this maximum value is approached as the spacing decreases and the number of elements increases; so in this limited sense the Chebyshev array can be made superdirective relative to the directivity of an array of identical length and sidelobe level, but larger spacing.

Calculations of directivity performed on the basis of the approximate results in (6a), (9a), and (11a) are compared in Fig. 3 with exact results based on a formulation by Brown and Sharp [10]. For all practical purposes, one incurs no significant loss in accuracy when using the approximate results. Because of the asymptotic nature of the approximations, one expects good agreement for large values of N, but the curves show that agreement is good for values as small as $N = 10$.

Sample calculations made by using the approximate expression for D in terms of the array length in (12), are presented in Fig. 4 where they are compared with the exact values [10] and values obtained from an approximation by Elliott [5]. Equation (12) has the advantage of being simpler than other such approximations while providing good computational accuracy.

Equations (12) and (13) for directivity can be simplified further to yield

$$D \sim \frac{2R^2}{1 + \frac{\lambda}{L} R^2 \sqrt{\frac{\ln 2R}{\pi}}}, \tag{14}$$

and, for the superdirective region,

$$D \sim \frac{2R^2}{1 + \frac{\lambda}{L} R^2 \sqrt{\frac{\ln 2R}{\pi}} \sin\frac{\mu}{2}}. \tag{15}$$

Calculations based on (12) and (14) were compared for $R = 100$ and $R = 10$. The results using (14) are almost identical to those of Elliott shown in Fig. 4.

The directivity as expressed by (14) is somewhat like

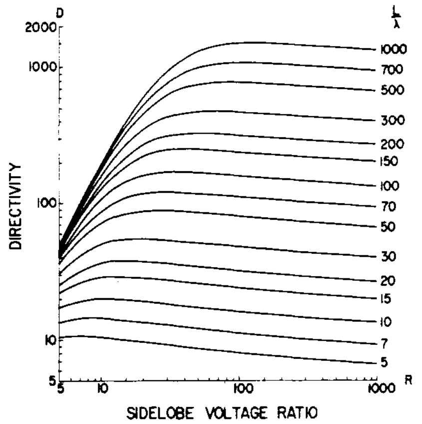


Fig. 2. The variation of directivity with sidelobe ratio for fixed array length.

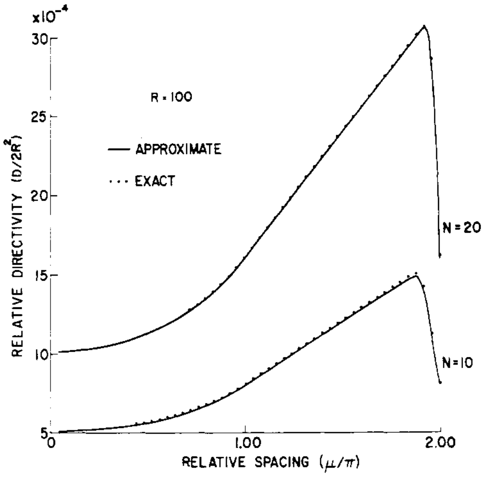
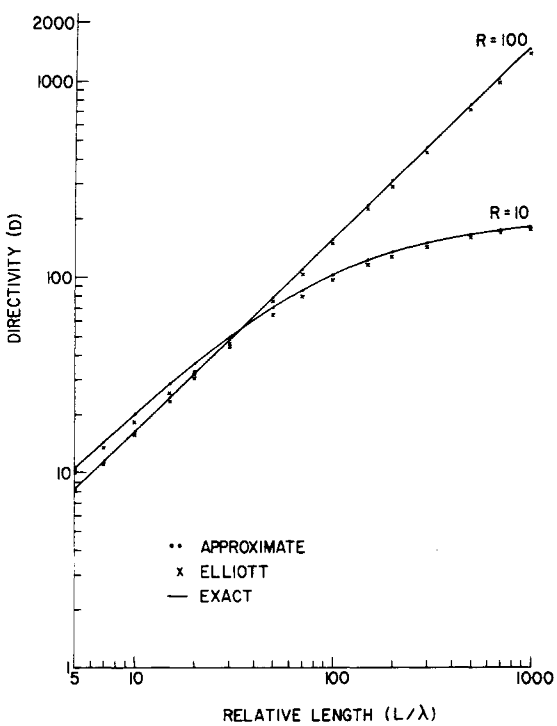


Fig. 3. Comparison between approximate and exact values of directivity; $R=100$.



4. Comparison between exact and approximate values of directivity. pproximate values are those derived in this paper and those of

Hansen's result [11] for the Taylor "ideal" space factor, a pattern equivalent to that of the Dolph-Chebyshev array, whose corresponding continuous aperture distribution represents an approximation to the envelope of the excitation coefficients of a large Chebyshev array.

Directivity and Scanning

We now consider the Dolph-Chebyshev array subjected to electronic scan, the directivity now being given by

$$D = \frac{2\mu R^2}{\left\{\int_0^{\mu-\alpha} + \int_0^{\mu+\alpha}\right\} T_N^2(a\cos\psi + b)\,d\psi}, \qquad (16)$$

where α is the uniform progressive phase shift between elements and is equal to $\mu \sin\theta_0$, θ_0 being the direction in which the main beam is pointing when the radiation pattern is scanned.

It is well known [5] that directivity is independent of scan angle for an interelement spacing that is half wavelength (or any integral multiple thereof). We now demonstrate that this remains true for spacings between half wavelength and full wavelength for the Chebyshev array, if we restrict our consideration to scanned arrays having only one main beam in the visible region. Under these restrictions, the range of scan is specified as follows. Additional main beams will occur when $\psi = \pm 2\pi$, that is, there will be no additional beams when

$$-\frac{2\pi}{\mu} + |\sin\theta_0| < -1 \quad \text{and/or} \quad \frac{2\pi}{\mu} + |\sin\theta_0| > 1,$$

the more restrictive of which implies that

$$\mu < \frac{2\pi}{1 + |\sin\theta_0|}$$

or that

$$\mu + |\alpha| < 2\pi.$$

After comparing (1) and (16) and considering the approximations leading to (6) for the directivity, we see from inspection of Fig. 1 that the directivity will be unchanged with scanning provided

$$\mu + |\alpha| \le 2\pi - \psi_0,$$

but that this will not generally be true for the region $2\pi - \psi_0 \le \mu + |\alpha| \le 2\pi$. As we have already seen, ψ_0 is usually a very small quantity. Thus, the independence we wanted to establish has been proven to exist for almost all spacings between half and full wavelength. This unusual behavior of the Dolph-Chebyshev design is intuitively reasonable in view of the characteristic uniformity of the sidelobe structure of the design [11].

Beamwidth

The half-power points of the radiation pattern are seen [6] to occur for such values of ψ that

$$T_N(a\cos\psi + b) = R/\sqrt{2},$$

or,

$$a \cos \psi = \cosh\left[\frac{1}{N} \cosh^{-1}\left(\frac{R}{\sqrt{2}}\right)\right] - b.$$

Although beamwidth can be determined from this closed-form expression, beamwidth for large N can also be approximated by an algebraic expression valid for all element spacings. This facilitates easy calculation of the half-power beamwidth in terms of the sidelobe level and array length.

For reasonably large values of R and N, it can easily be shown that

$$\cos \psi \approx 1 - \frac{3}{8a}\left(\frac{\ln 2}{N}\right)^2\left(1 + \frac{4}{3}\frac{\ln R}{\ln 2}\right),$$

and that the corresponding polar angle is

$$\theta \approx \frac{\ln 2}{2\mu N}\left[\frac{3}{a}\left(1 + \frac{4}{3}\frac{\ln R}{\ln 2}\right)\right]^{1/2}. \qquad (17)$$

In terms of array length L, and sidelobe level S in dB, with $S = 20 \log R$, the half-power beamwidth (BW) is given by

$$\mathrm{BW} \approx 0.18 \frac{\lambda}{L}\left[\frac{1}{a}(S + 4.52)\right]^{1/2}. \qquad (18)$$

Note that the beamwidth varies inversely with the square root of a, which is an exponential increasing function with decreasing spacing from half wavelength. For large arrays of spacing greater than or equal to half wavelength $a \approx 1$, and the half-power beamwidth becomes

$$\mathrm{BW} \approx 0.18 \frac{\lambda}{L}(S + 4.52)^{1/2}. \qquad (19)$$

The result as stated by (19) is similar to a somewhat more complicated approximation introduced earlier by Stegen [12]. It is also noteworthy that like the directivity for this case, the beamwidth is independent of the actual element spacing for sufficiently large N. Evaluations of the beamwidth according to (19) are presented in Fig. 5. In Fig. 6 these results are compared with Elliott's approximation [5] and the corresponding exact results.

Some discussion of these results is in order. The expression for BW in (19) has several distinct advantages. It is much simpler in form hence easier to evaluate than other such expressions. Fig. 6 shows that its numerical accuracy is more than sufficient for any practical purpose. Although Elliott's approximate formula contains the expected property of yielding a finite limiting value of beamwidth as S becomes arbitrarily small for an array of fixed length, (19) provides not only this information but is also extremely accurate even for rather large values of S (or R) that are of practical concern. For example, beamwidth computations for $S = 60$ dB (or $R = 1000$) compared extremely well with the exact computation even for rather small arrays, the deviations being for the most part of no practical consequence.

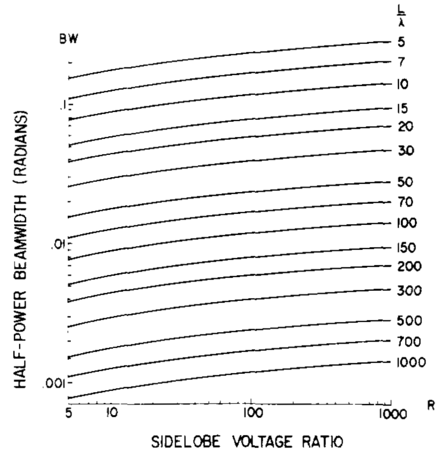


Fig. 5. Dependence of half-power beamwidth on sidelobe ratio and relative array length.

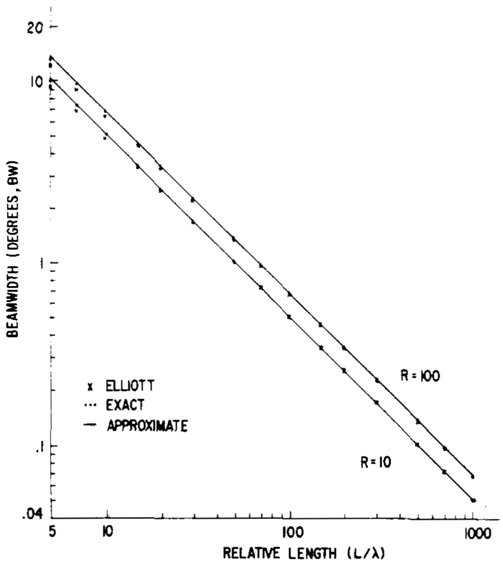


Fig. 6. Comparison between the exact values and two approximate forms for the beamwidth.

Beamwidth and Scanning

For the scanning situation with the main beam pointing in the direction θ_0,

$$\psi = \mu(\sin \theta - \sin \theta_0).$$

In terms of the series expansion for $\sin \theta$, it also follows that

$$\psi = \mu(\theta - \theta_0)\left[1 - \frac{1}{3!}(\theta^2 + \theta_0\theta + \theta_0^2) + \frac{1}{5!}(\theta^4 + \theta_0\theta^3 + \theta_0^2\theta^2 + \theta_0^3\theta + \theta_0^4) - \cdots\right].$$

If we let ψ be the value that corresponds to the half-power beamwidth, we have

$$\mathrm{BW} = 2(\theta - \theta_0)$$

and, for large N,

$$\mathrm{BW} \approx \frac{2\psi}{\mu \cos \theta_0}.$$

For spacings greater than or equal to half wavelength, this and (19) yield

$$\mathrm{BW} \approx 0.18 \frac{\lambda}{L} \sqrt{S + 4.52} \sec \theta_0. \tag{20}$$

Optimum Directivity

In Fig. 2 it is notable that the rise in directivity to a maximum is rather sharp, particularly for the larger arrays; however, for increasingly higher sidelobe ratios than that ratio at which maximum directivity occurs, the decline in directivity is rather moderate.

From (12) the condition for maximum directivity is

$$1 = \frac{\lambda}{2L} \left[I_0(2 \cosh^{-1} R) \frac{R}{\sqrt{R^2 - 1}} - I_1(2 \cosh^{-1} R) \right] \cdot \cosh^{-1} R. \tag{21}$$

Note that R_M, the solution of (21), is indeed a function of the relative length of the array (which is apparent in the graphs of Fig. 2). This information is graphically portrayed in Fig. 7. If one considers the asymptotic expansions for the various terms in (21), which ultimately yield

$$\frac{L}{\lambda} \sim \frac{\left(\frac{R_M}{2}\right)^2}{\sqrt{\pi \ln (2R_M)}}, \tag{22}$$

and notes that $\ln R_M$ is a more slowly varying quantity than R_M itself, the reason for the straight-line variation shown in Fig. 7 is quite apparent.

The maximum directivity D_M corresponding to R_M can thus be expressed as

$$D_M \sim \frac{2R_M^2}{1 + 4 \ln (2R_M)} \tag{23}$$

by using (12), (22), and asymptotic expressions for $\cosh^{-1} R$ and $I_1(2 \cosh^{-1} R)$. It is of particular interest to compare D_M with the directivity of a uniformly illuminated array of the same number of elements. The latter directivity is equal to the number of elements, for arrays whose uniform element spacings are some integral multiple of half a wavelength; this value represents the maximum directivity for all arrays of such spacing. Pritchard [13] has shown that maximum directivity for spacings between a half and a full wavelength is achieved if the illumination is nearly uniform. (This agrees with the results of Tseng and Cheng [14] for broadside operation, where perturbations of the element spacings and excitations yield only small increases from the directivity of a uniform array. But, they have also shown that rather significant increases can occur for off-broadside operation; in particular, at endfire the increase is considerable, but this can sometimes be at the expense of having the usual supergain characteristics of complicated feeding requirements and low radiation efficiency.) Furthermore, for large uniform arrays the directivity can be written in terms of the array length as $2L/\lambda$, where, throughout this paper,

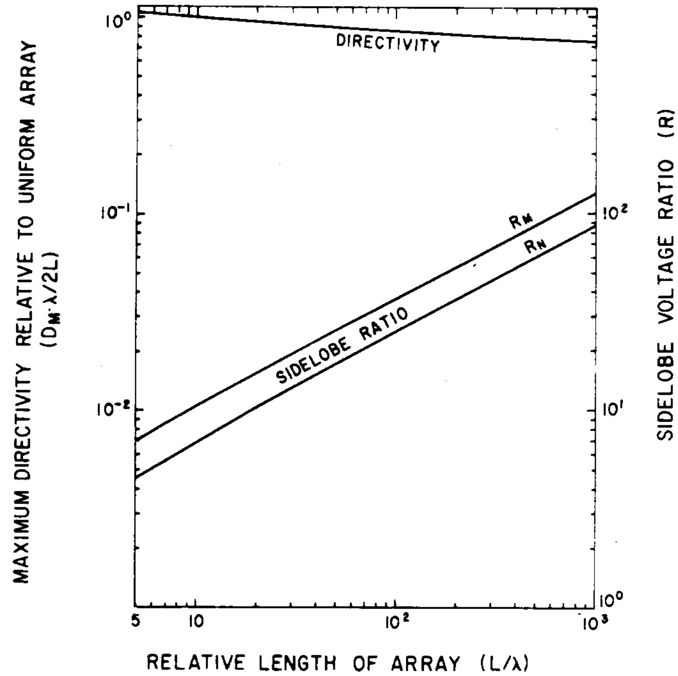


Fig. 7. The variation with relative array length of maximum directivity (D_M), the sidelobe level (R_M) at which D_M occurs, and that sidelobe level (R_N) at which maximum directivity-beamwidth ratio (D/BW) occurs.

length L is defined to be $2Nd$ for an array of $2N+1$ elements at a spacing d. By (22) and (23), this comparison takes the form

$$D_M \cdot \frac{\lambda}{2L} \sim \frac{\sqrt{\pi \ln 2R_M}}{\ln 2R_M + 0.25}. \tag{24}$$

Computations of $D_M \cdot \lambda/2L$ are presented in Fig. 7 for a wide range of array lengths. (Use of (12) to compute D_M yields results that compare within 1 percent for all array lengths considered.) In Fig. 7, D_M compares quite favorably with the maximum directivity as manifested by a uniform array over the entire range of lengths considered. The aforementioned slow variation of $\ln R_M$ with R_M in the region of interest, as applied to (24), helps to explain this fact.

Pritchard [13] called attention to the favorable comparison between D_M and the overall maximum directivity for this nonsuperdirective medium, and he also referred to an empirical formula derived by Batchelder:

$$10 \log D_M \approx 2.94 + 9.69 \log M + 9.01 \log \frac{d}{\lambda}, \tag{25}$$

where M is the number of array elements for this equal minor lobe excitation. This formula was apparently derived from data for relatively small numbers of elements (or short arrays). Consistent with this in a sense is the fairly good agreement between computations based on (25) and those based on (12) (evaluated at $R=R_M$) for small arrays; the comparison is progressively worse as the arrays increase in length. For the superdirective region, (15) will show that R_M must satisfy

$$1 = \frac{\ln 2R + 0.25}{\sqrt{\pi \ln 2R}} \; \frac{R^2 \sin \frac{\mu}{2}}{1 + \frac{\lambda}{L} R^2 \sqrt{\frac{\ln 2R}{\pi}} \sin \frac{\mu}{2}},$$

where we note that R_M is now a function of both length and interelement spacing. Corresponding to R_M is the maximum directivity D_M, given by

$$\frac{\lambda}{2L} D_M \approx \frac{\sqrt{\pi \ln 2R_M}}{(\ln 2R_M + 0.25) \sin \frac{\mu}{2}}.$$

Directivity-Beamwidth Product

Another interesting facet of many antenna designs is the constancy of the directivity-beamwidth product under certain conditions. Its advantage lies, of course, in that once one of the two quantities, directivity or beamwidth, is specified, the other is then easily obtained. In particular, it is often possible to measure beamwidth accurately, while at the same time it is rather difficult to measure directivity, especially for large arrays [15]. Very little exists in the literature concerning $D \cdot \mathrm{BW}$ for Dolph-Chebyshev designs, although Stegen [15] provided some estimates valid under certain limited conditions. Use of (12) and (19) yields the results shown in Fig. 8. They clearly indicate that the directivity-beamwidth product is, in general, not even approximately constant, particularly for large arrays of relatively high-sidelobe-level design. For Dolph-Chebyshev arrays of all lengths (or numbers of elements), the product approaches a constant with increasing R. This constant is approximately 1.91 radians or 109.5 degrees. (Use of (14) with (19) would have yielded the same results.) To get some idea of the combinations of array length L/λ and mainlobe-sidelobe ratio R such that the directivity-beamwidth product will be equal to this constant, consider the following.

In Figs. 2 and 5 the curve shapes indicate that, given any particular value of L/λ, the product will be constant for all mainlobe-sidelobe ratios larger than a value of R that is approximately R_M. On the other hand, it is clear from Fig. 8 that, given any particular value of R, the product will be constant for arrays of all lengths L/λ that are smaller than that length for which R is the particular ratio required for maximum directivity. In other words, reference to the line in Fig. 7 denoting the locus of R_M corresponding to maximum directivity D_M indicates that for all points in the $(R, L/\lambda)$ domain that are above this line the directivity-beamwidth product will be approximately equal to 109.5 degrees. This does not hold with great precision right down to the line (on which the product is approximately 102 degrees) because of the deviation of the directivity curve from linearity in the immediate vicinity of D_M, as seen in Fig. 2. But, this is equivalent to a relatively small region immediately above the R_M line.

For a Dolph-Chebyshev array that is subject to electronic scan (restricted in degree of scan only to the extent that other major lobes do not appear), it has been shown that the directivity remains unchanged for the nonsuperdirective region. On the other hand, it was shown that beamwidth is proportional to sec θ_0 (θ_0 is the beam-pointing direction measured from broadside); therefore, so also is the directivity-beamwidth product.

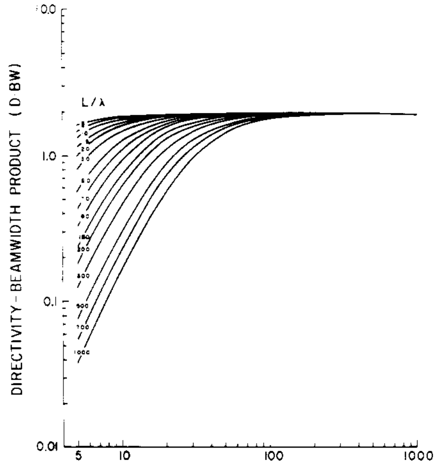


Fig. 8. Variation of directivity-beamwidth product with sidelobe level and relative array length.

Directivity-Beamwidth Ratio

We have seen that the directivity of a Dolph-Chebyshev array assumes a fairly sharp maximum value for a design corresponding to a particular sidelobe ratio R_M. On the other hand, it has been well noted that beamwidth increases with decreasing sidelobes. An obvious desire would be to simultaneously achieve maximum directivity and minimum beamwidth. However, they do not coexist for the same sidelobe design. An obvious (although admittedly arbitrary) compromise may result from maximizing the quantity D/BW considered as a function of R. Earlier, another approximate form for BW that is slightly more accurate than (19) was derived [cf. (17)]. It is

$$\mathrm{BW} \approx \frac{\ln 2}{\pi} \frac{\lambda}{L} \left(3 + 4 \frac{\ln R}{\ln 2}\right)^{\frac{1}{2}}. \tag{26}$$

Use of this and (12) leads to the following condition satisfied by R, yielding R_N corresponding to maximum D/BW:

$$1 = \frac{R}{\sqrt{R^2 - 1}} \cdot \frac{I_0(2 \cosh^{-1} R) \cosh^{-1} R}{\frac{2L}{\lambda} + I_1(2 \cosh^{-1} R) \cosh^{-1} R} + \frac{0.25}{\ln R + 0.75 \ln 2}. \tag{27}$$

Data on the ratio D/BW are plotted in Fig. 9; (12) and (26) were used in these calculations. In this figure we notice that the characteristics of the curves are very similar to those for directivity D, rising rather sharply to a maximum value, followed by a gentle decline. This is natural in view of

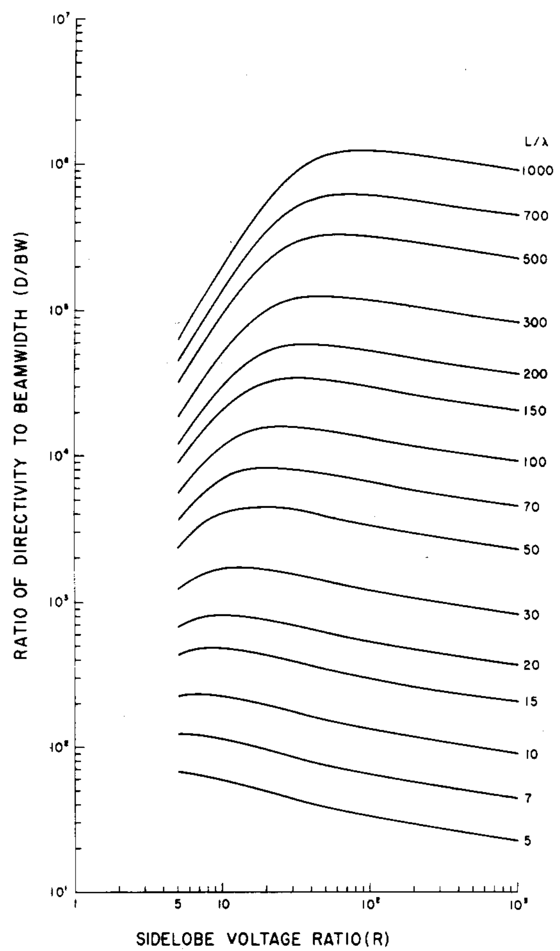


Fig. 9. Variation of directivity-beamwidth ratio with sidelobe level and relative array length.

the moderate taper of the beamwidth curves, as compared to those for directivity.

Just as was the case for R_M, we see from (27) that R_N is also a function of array length L/λ. Solutions of (27) for a wide range of array lengths were obtained and are presented in Fig. 7 for easy comparison with the R_M values already obtained. For essentially the same reasons that led (21) to be a straight line plot in Fig. 7, the curve of R_N versus L/λ is likewise essentially a straight line.

Summary

We have attempted to provide useful approximate, but nonetheless accurate formulations for the directivity and beamwidth of Dolph-Chebyshev arrays that may be subject to electronic scan. We were motivated in part by the fact that existing expressions for the directivity become computationally unwieldy as the size of the array increases beyond a moderately large number of elements. The formulas developed in this paper are as easy to use for very large arrays as for the relatively small.

Equations (6a), (9a), and (11a) will be more useful when directivity must be determined as a function of element spacing. On the other hand, (12) and (13) will be of greater use when the directivity needs to be known for arrays of given lengths. As for beamwidth, (18) together with (20), an approximate form suitable for the nonsupergain region, provide particularly simple yet accurate aids to computation whenever information as to the dependence of the beamwidth on array length, element spacing, sidelobe level, or beam direction, is desired.

It should be emphasized that the choices made for the size of some of the parameters in the comparisons between exact and approximate theory for directivity and beamwidth formulations were predicated on establishing lower limits (for array length or number of array elements) or ranges (for sidelobe level) for which the approximations would be valid. It is the nature of the development of the approximate formulas that they become more accurate for arrays that are larger or contain more elements than those chosen for the computations.

In the design of a Dolph-Chebyshev array, we have given the means whereby one can select the main beam sidelobe ratio for which directivity is maximum. We have seen that this choice is very much a function of array length; generally, the longer the array, the lower the sidelobes must be. The maximum directivity that results from this choice compares quite favorably with the directivity of a uniformly illuminated array, the comparison being less favorable the longer the array. However, for an array as long as 1000 wavelengths the difference is only about 1.28 dB; that is, the directivity of a uniform array of this length is about 33.01 dB while that of the corresponding Chebyshev array that provides maximum directivity is 31.73 dB.

The directivity-beamwidth product, a useful quantity when one of the two components is known and the other desired, or when one of the components is easily measurable while the other is not, has been calculated for large ranges of sidelobe level and array length. While it is essentially constant for certain combinations of lower sidelobes and shorter arrays, it is generally far from constant for higher sidelobes and longer arrays. Interestingly enough, it is nearly constant (102 degrees) for all conditions of sidelobe level and array length such that maximum directivity exists. This is particularly useful, since one may often wish to design for maximum directive gain.

Finally, we have considered the directivity-beamwidth ratio. Although any Dolph-Chebyshev design yields the minimum beamwidth when considered in that class of all arrays of an equal number of elements and sidelobe level, it was thought to be of interest to give some weight to the minimization of beamwidth while at the same time seeking to maximize directivity among arrays in the Dolph-Chebyshev class. This resulted in a choice of sidelobe level that is about 3.5 dB higher than that selected when maximization of directivity alone is attempted.

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Collimation of Row-and-Column Steered Phased Arrays

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Abstract—**An optically fed phased array must be provided with a means of collimation as well as with a beam steering function. The same phase shifters which are used to steer the beam can be used to collimate the beam. The use of row-and-column phase commands, while greatly simplifying the beam steering function over that required for commanding individual elements, precludes the attainment of exact collimation. A consequent phase error across the aperture results in a loss of antenna gain. For a given order of approximation to the collimation function, the minimization of the gain loss is a valid criterion for completely specifying all the parameters of the approximate collimation function. The gain loss incurred can then be determined.**

This paper develops the equations necessary to specify any order of approximation to the collimation function and the expression for the gain loss. Examples illustrate the differences between first- and second-order approximations and the effect of another parameter (f/D) on the gain loss of a typical antenna system.

Introduction

THE ADVANTAGE of the optically fed phased array is the elimination of the hardware necessary to distribute the energy over the radiating aperture. An optically fed array may be either of the reflection type or the lens type, resembling a reflector or a lens, respectively. Energy is radiated from a feed horn (or cluster of feed horns), impinges on a surface of collecting apertures, receives a phase shift at each element in the array, and is reradiated from a surface of radiating apertures, as illustrated in Fig. 1. In a reflection-type array, the collecting aperture also serves as the radiating aperture.

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In addition to steering the main beam, the element phase shifters must also serve to collimate the incident energy, that is, to convert the spherical wave incident on the array from the feed to a plane wave during transmit, and vice versa on receive. In a conventional corporately fed array, the equivalent to this function is performed by the corporate distribution network.

Collimation might be performed by the use of fixed phase shifters in each element, or by means of a lens over the entire array face. However, the former is economically undesirable inasmuch as the elements of the array would no longer be interchangeable. The latter increases the bulk and weight of the array, as well as the cost, and may limit the bandwidth over which the array itself is usable.

If each element of the array is commanded individually, the collimation may be made exact, within the limits of the accuracy and precision of the phase shifters. The beam steering function can be immensely simplified, however, if the array is commanded by rows and columns. This introduces a collimation error, but is sufficient for steering the beam to any angle in space.

The determination of the phase required at each element to steer the beam is a trivial matter. It is the purpose of this paper to consider the collimation function, and how it might be achieved in an array which is commanded by rows and columns.

By row-and-column commands it is meant that the phase shifters in a given row are commanded to a particular phase as are the phase shifters in a given column. The total phase