A New Interpretation of the Integral Equation Formulation of Cylindrical Antennas*

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Summary—**The validity of the conventional integral equation formulation of cylindrical antennas is often criticized because of an approximation of the kernel of the integral equation. If the antenna is of the form of a cylindrical tube there is an exact integral equation formulation. By means of the variational technique the integral equation for the cylindrical tube can be solved approximately. When the length of the tube is large compared to its diameter, the input impedance of the structure is found to be the same as if one had used the approximate integral equation at the very beginning of the analysis. This new procedure therefore clarifies one main ambiguity involved in the early works of the theory of cylindrical antennas based upon the approximate integral equation. It also enables us to treat the problem of relatively thick cylindrical antennas.**

Introduction

MANY ARTICLES have been written on the subject of cylindrical antennas using the integral equation technique.[1–4] Most of them are based upon Hallén's original integral equation for a thin antenna and his iteration method of solving it. Because of an approximation to the kernel of the integral equation, the result obtained has often been criticized. In the case of a transmitting antenna it is known, however, that the numerical results for the input impedance of the antenna, based upon various iteration procedures, are acceptable provided that the proper order of solution is used. The evidence is partly supported by Schelkunoff's theory of cylindrical antennas,[5] which is based upon an entirely different method, yet his result can also be obtained by a so-called $\mu\Omega$ expansion as shown by Hallén.[6]

The purpose of this paper is to examine the problem again, starting with the exact integral equation for a cylindrical shell. By means of a variational method, the integral equation can be solved approximately. No attempt is made to simplify the exact kernel of the integral equation. The exact kernel is retained in the entire analysis. Approximations are introduced only in evaluating certain definite integrals. These approximations are valid when the length of the antenna is large compared to its radius. Under this condition it is found that the expression for the input impedance of the antenna is the same as if one had used the approximate integration equation at the very beginning of the analysis. The finding demonstrates the fact that as far as the final result is concerned the use of the approximate integral equation is justified even though the equation itself may not have the unique meaning of a properly behaved integral equation. The new procedure also indicates the feasibility of treating problems involving relatively thick antennas. There seems, however, no simple way of handling some definite integrals except by numerical integration.

The Exact Integral Equation for a Cylindrical Shell and Its Approximate Form

It is known from the literature[7,8] that the boundary value problem of a perfectly conducting cylindrical shell as a center-driven antenna can be formulated exactly in terms of an integral equation containing the current distribution function $I(z)$ as the unknown function.

$$E^i(z) = \frac{j\omega\mu}{4\pi}\int_{-l}^{l} I(z')\left[1 + \frac{1}{k^2}\frac{\partial^2}{\partial z^2}\right]G(z-z', a)dz', \quad (1)$$

where $E^i(z)$ denotes the impressed field applied at the center of the antenna and distributed uniformly around the cylindrical shell, and

$$G(z-z', a) = \frac{1}{2\pi}\int_0^{2\pi}\frac{e^{-jkr}}{r}d\phi, \quad (2)$$

where

$$r = \sqrt{(z-z')^2 + \left(2a\sin\frac{\phi}{2}\right)^2}$$

l = half-length of the antenna
a = radius of the antenna
$k = 2\pi/\lambda$.

Hallén's integral equation is obtained by replacing the exact expression for $G(z-z', a)$ by an approximate expression

$$G_h(z-z', a) = \frac{e^{-jkr_a}}{r_a}, \quad (3)$$

where

$$r_a = \sqrt{(z-z')^2 + a^2}.$$

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[1] E. Hallén, "Theoretical investigations into the transmitting and receiving qualities of antennae," *Nova Acta Upsal.*, no. 4, p. 11; 1938.

[2] M. C. Gray, "A modification of Hallén's solution of the antenna problem," *Jour. Appl. Phys.*, vol. 15, p. 61; 1944.

[3] R. King and D. Middleton, "The cylindrical antenna; current and impedance," *Quar. Appl. Math.*, vol. 3, p. 302; 1946.

[4] S. A. Schelkunoff, "Advanced Antenna Theory," John Wiley & Sons, Inc., New York, N. Y., ch. 5; 1952.

[5] Schelkunoff, *ibid.*, ch. 1.

[6] E. Hallén, "Admittance Diagrams for Antennas and the Relation between Antenna Theories," Cruft Lab., Harvard Univ., Cambridge, Mass., Tech. Rep. No. 46; 1948.

[7] E. Hallén, "Properties of long antennas," *Jour. Appl. Phys.*, vol. 19, p. 1141; 1948.

[8] Schelkunoff, *op. cit.*, ch. 4.

Eq. (1) is then reduced to

$$E^i(z) = \frac{j\omega\mu}{4\pi}\int_{-l}^{l} I(z')\left(1 + \frac{1}{k^2}\frac{\partial^2}{\partial z^2}\right)G_h(z - z', a)dz'. \quad (4)$$

This approximation creates certain questions regarding the true significance of this mathematical step and its consequence. These questions have been raised and discussed by both Schelkunoff[9] and Whinnery.[10] In this investigation, we avoid this ambiguous step by considering (1) as the basic equation.

Variational Solution and the Evaluation of Integrals

To analyze (1) let us first define a normalized function $f(z)$ such that

$$\left.\begin{aligned} E^i(z) &= Vf(z) \\ \int_{-l}^{l} f(z)dz &= 1\end{aligned}\right\}, \quad (5)$$

and

where V, by definition, denotes the applied voltage to the antenna. A parametric equation can then be obtained by multiplying (1) by $I(z)$ and integrating with respect to z from $-l$ to l. Dividing the resultant equation by $[\int_{-l}^{l} f(z)I(z)dz]^2$ one obtains

$$\frac{V}{\int_{-l}^{l} f(z)I(z)dz} = \frac{j\eta}{4\pi}\cdot\frac{\int_{-l}^{l}\int_{-l}^{l} I(z)I(z')K(z-z')dzdz'}{\left[\int_{-l}^{l} f(z)I(z)dz\right]^2} \quad (6)$$

where

$$K(z - z') = k\left(1 + \frac{1}{k^2}\frac{\partial^2}{\partial z^2}\right)G(z - z', a)$$

$$\eta = (\mu/\epsilon)^{1/2} = 120\pi \text{ ohms}.$$

Denoting the quantity at the left side of (6) by Z_i, which corresponds to the input impedance of the antenna, it can be shown that the value of Z_i evaluated using the right member of (6) is stationary with respect to a small variation of $I(z)$ when the latter satisfies (1). This variational method of determining the impedance of a cylindrical antenna was first applied by Storer[11] to (4). The method is quite distinct from both Hallén's and Schelkunoff's methods yet it is simple and attractive. In calculating Z_i, using the variational formula, one major problem is the proper choice of trial function for $I(z)$. The functions which have been found adequate to use are so far limited to

$$I(z) = \sin k(l - |z|) + A_s[1 - \cos k(l - |z|)] \quad (7)$$

and

$$I(z) = \sin k(l - |z|) + A_t k(l - |z|) \cos k(l - |z|). \quad (8)$$

The constant A_s or A_t is determined according to the extreme property of Z_i such that

$$\frac{\partial Z_i}{\partial A_s} = 0 \quad \text{or} \quad \frac{\partial Z_i}{\partial A_t} = 0.$$

The first trial function was due to Storer, and the latter was suggested by this writer.[12]

The advantage of using (8) is that the impedance function will be finite for all values of l while Storer's function is only good for $l<\lambda$. In the present work, we can use the same functions as the trial functions. To perform the integration, it is convenient to change a variable and write

$$G(z - z', a) = \frac{2}{\pi}\int_0^{2a} \frac{e^{-jk\sqrt{(z-z')^2+\xi^2}}}{\sqrt{(z-z')^2+\xi^2}} \cdot \frac{d\xi}{\sqrt{(2a)^2 - \xi^2}}. \quad (9)$$

In the case of a δ-gap, (6) reduces to

$$Z_i = \frac{j\eta}{4\pi I^2(0)}\int_0^{2a}\frac{2d\xi}{\pi\sqrt{(2a)^2-\xi^2}} \cdot \int_{-l}^{l}\int_{-l}^{l} I(z)I(z')K_h(z-z')dzdz', \quad (10)$$

where

$$K_h(z - z') = k\left(1 + \frac{1}{k^2}\frac{\partial^2}{\partial z^2}\right)G_h(z - z', \xi)$$

and

$$G_h(z - z', \xi) = \frac{e^{-jk\sqrt{(z-z')^2+\xi^2}}}{\sqrt{(z-z')^2+\xi^2}}.$$

It has been shown[11,12] that when $l \gg \xi (\xi \leqq 2a)$ the double integral with respect to z and z' using the functions described by (7) or (8) can readily be evaluated. Thus, if (8) is used as the trial function, one finds that

$$j\int_{-l}^{l}\int_{-l}^{l} I(z)I(z')K_h(z-z')dzdz' = \gamma_{11}(\xi) + 2\gamma_{12}(\xi)A_t + \gamma_{22}(\xi)A_t^2. \quad (11)$$

The coefficients $\gamma_{11}(\xi)$, $\gamma_{12}(\xi)$, and $\gamma_{22}(\xi)$ consist of a *linear* function of Ω_ξ and some terms independent of ξ, for example,

9 S. A. Schelkunoff, "Concerning Hallén's integral equation for cylindrical antennas," Proc. IRE, vol. 33, pp. 872–878; December, 1945.

10 J. R. Whinnery, "The Integral Equation Solution of the Antenna Problem," U. S. Navy Elec. Res., Univ. of California, Berkeley, Calif., Rep. No. 128; 1947.

11 J. Storer, "Variational Solution to the Problem of the Symmetrical Cylindrical Antenna," Cruft Lab., Harvard Univ., Tech. Rep. No. 101; 1950.

12 C. T. Tai, "A Variational Solution to the Problem of Cylindrical Antennas," Aircraft Radiation Systems Lab., Stanford Res. Inst., Stanford, Calif., Tech. Rep. No. 12; 1950.

$$\gamma_{11}(\xi) = 2L2x + e^{2jx}(\ln 2 - \tfrac{1}{2}\Omega_\xi - L4x + 2L2x) + e^{-2jx}(-\ln 2 + \tfrac{1}{2}\Omega_\xi), \tag{12}$$

where

$$Lx = \int_0^x \frac{1 - e^{-ju}}{u}\, du = \overline{C}ix + Six$$

$$x = kl, \qquad \Omega_\xi = 2 \ln \frac{2l}{\xi} \cdot$$

Substituting (11) into (10) and performing the integration with respect to ξ, one notices that

$$\frac{2}{\pi}\int_0^{2a} \frac{d\xi}{\sqrt{(2a)^2 - \xi^2}} = 1$$

and

$$\frac{2}{\pi}\int_0^{2a} \frac{\Omega_\xi d\xi}{\sqrt{(2a)^2 - \xi^2}} = \Omega = 2 \ln \frac{2l}{a},$$

hence,

$$Z_i = \frac{\eta}{4\pi} \frac{[\gamma_{11} + 2\gamma_{12}A_t + \gamma_{22}A_t^2]}{[\sin x + A_t x \cos x]^2}, \tag{13}$$

where γ_{11}, γ_{12}, and γ_{22} are obtained by substituting Ω for Ω_ξ in $\gamma_{11}(\xi)$, $\gamma_{12}(\xi)$, and $\gamma_{22}(\xi)$. The interesting result of the present analysis is that (13) is the same as the expression previously derived using (4) as the basic equation. The fact that (13) can be derived using this alternative but more logical approach should remove the ambiguity that has often been attached to the approximation in deriving (4). As far as the final result for the input impedance is concerned the variational solution based upon (4) is therefore a good approximate solution of the exact integral equation defined by (1).

One further advantage of the present manipulation lies in the problem of thick antennas. If the condition specifying the characteristics of a thin antenna $l \gg a$ is not satisfied, (4) can not be used. On the other hand, (10) is still valid. The double integration with respect to z and z' can be expressed in terms of generalized sine and cosine integrals. The remaining integration with respect to ξ, however, can not be evaluated in a closed form although it seems feasible to do it numerically. Until some numerical values are compiled, we will not give any detailed discussion.