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delta-function generator, as mentioned in the introduction. This procedure, in particular, takes account properly of the effects of the size of the coaxial line. In order to use this procedure, the approximate theory used to evaluate the difference $Y_a - Y_{a\infty}$ must be valid for the infinitely long dipole. In other words, a theory of the long dipole antenna is needed. Furthermore, for comparison with experimental results, this theory of the long dipole antenna must also be reasonably accurate even when the antenna is not excessively long. A theory with precisely this purpose in mind has been given previously.[3]

[3] T. T. Wu, J. Math. Phys. 2, 550 (1961).

This point of view is also useful in some cases more complicated than that of the dipole antenna. For example, it is applicable to the case of the thin circular loop antenna provided that the radius of the loop is large compared with b. If (11) is replaced by $|kb| \ll 1$, this entire calculation is also valid for dissipative media, where k is complex.

Procedures very similar to the present one can easily be found for various other problems, such as a discontinuity in radius.

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Theory of the Thin Circular Loop Antenna

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The current distribution on a thin circular loop transmitting antenna driven by a delta-function generator is determined approximately by Fourier series expansion. A difficulty encountered in previous analysis is shown to be due to an inadequate approximation.

I. INTRODUCTION

As early as 1897, Pocklington[1] studied the excitation of a thin loop antenna by a plane wave. Using methods very similar to that of Pocklington, Hallén[2] and later Storer[3] considered the case of the driven antenna. All these authors used Fourier series expansion, as appropriate for the geometry under consideration, and the latter authors found a difficulty in this approach. Their difficulty takes the form of the appearance of either a singularity or a very large term in the Fourier series expansion when the index n is close to a certain large number determined by the geometry. Hallén then concluded that the series is divergent and attributed this divergence to the approximation of a "one-dimensional" equation, whereas Storer avoided the contribution from this large term by first replacing the Fourier series by the corresponding integral and then evaluating the integral in the sense of the Cauchy principle value. This procedure of Storer seems at best to be of doubtful validity. More recently, a similar difficulty has been found to appear also in the case of dipole antennas.[4,5] In this case, however, it is clear from the derivation that the trouble has nothing to do with the originally posed problem, but instead, is a consequence of the approximations used. It is the purpose of this paper to point out that this is also the case with the thin loop antenna, and a procedure is proposed that avoids this difficulty by invoking approximations that are valid over larger ranges of the parameters. It may be added that the difficulty under consideration does not have anything to do with the so-called "gap problem."

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[1] H. C. Pocklington, Proc. Cambridge Phil. Soc. 9, 324 (1897).
[2] E. Hallén, Nova Acta Regiae Soc. Sci. Upsaliensis 2, No. 4 (1938).
[3] J. E. Storer, Trans. A. I. E. E. 75, Part I, 606 (1956).
[4] T. T. Wu, J. Math. Phys. 2, 550 (1961).
[5] T. T. Wu, J. Research Natl. Bur. Standards (to be published).

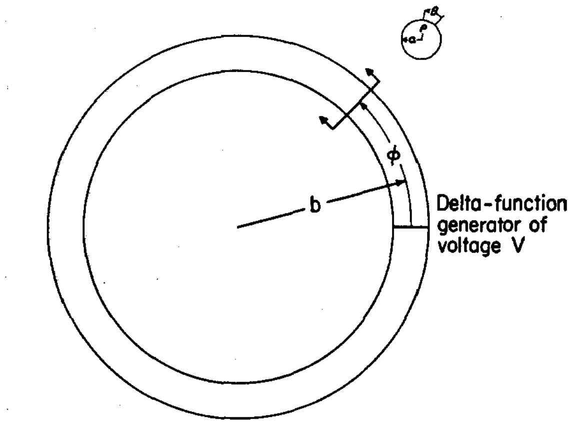

FIG. 1. Geometry of the problem.

2. FORMULATION OF THE PROBLEM

In Fig. 1 is shown the geometry and the coordinate system for a loop antenna driven by a delta-function generator. If the loop is assumed to be perfectly conducting, then there are two components for the surface current density, namely, $J_\phi(\phi, \theta)$ and $J_\theta(\phi, \theta)$. By setting the tangential components of the electric field to zero in the surface of the loop, the following integral equations are obtained for J_ϕ and J_θ:

$$\begin{aligned}(\partial/\partial\phi) \int d\phi'\, d\theta' (4\pi R)^{-1} e^{ikR} \{a(\partial/\partial\phi') J_\phi(\phi', \theta') \\ + (\partial/\partial\theta')[(b + a\cos\theta') J_\theta(\phi', \theta')]\} \\ + k^2 a (b + a\cos\theta) \\ \times \int d\phi'\, d\theta' (4\pi R)^{-1} e^{ikR} (b + a\cos\theta') \\ \times [J_\phi(\phi', \theta') \cos(\phi - \phi') \\ - J_\theta(\phi', \theta') \sin(\phi - \phi') \sin\theta'] \\ = ik\zeta_0^{-1} V\, \delta(\phi) \qquad \text{(1a)}\end{aligned}$$

and

$$\begin{aligned}(\partial/\partial\theta) \int d\phi'\, d\theta' (4\pi R)^{-1} e^{ikR} \{a(\partial/\partial\phi') J_\phi(\phi', \theta') \\ + (\partial/\partial\theta')[(b + a\cos\theta') J_\theta(\phi', \theta')]\} \\ + k^2 a^2 \int d\phi'\, d\theta' (4\pi R)^{-1} e^{ikR} (b + a\cos\theta') \\ \times \{J_\phi(\phi', \theta') \sin(\phi - \phi') \sin\theta + J_\theta(\phi', \theta') \\ \times [\cos(\phi - \phi') \sin\theta \sin\theta' + \cos\theta\cos\theta']\} \\ = 0. \qquad \text{(1b)}\end{aligned}$$

Here, all integrals are from $-\pi$ to π, R is the Euclidean distance between the points (ϕ, a, θ) and (ϕ', a, θ'), ζ_0 is the characteristic impedance of free space, and V is the voltage supplied by the delta-function generator.

As a first approximation to these rather complicated integral equations, the so-called one-dimensional equation is obtained under the following circumstances

$$a \ll b, \quad \text{and} \quad ka \ll 1 \qquad \text{(2a)}$$

so that

$$J_\phi(\phi, \theta) \sim (2\pi a)^{-1} I(\phi), \qquad \text{(2b)}$$

and

$$J_\theta(\phi, \theta) \sim 0. \qquad \text{(2c)}$$

Equation (1b) is omitted altogether and (1a) is approximated by

$$\begin{aligned}(\partial/\partial\phi) \int d\phi' K(\phi - \phi')(\partial/\partial\phi') I(\phi') \\ + k^2 b^2 \int d\phi' K(\phi - \phi') \cos(\phi - \phi') I(\phi') \\ = 4\pi ik\zeta_0^{-1} V\, \delta(\phi), \qquad \text{(3)}\end{aligned}$$

where

$$\begin{aligned}K(\phi) = (2\pi)^{-1} \int_{-\pi}^{\pi} d\theta [(2b \sin\tfrac{1}{2}\phi)^2 + (2a \sin\tfrac{1}{2}\theta)^2]^{-1/2} \\ \times \exp\{ik[(2b\sin\tfrac{1}{2}\phi)^2 + (2a\sin\tfrac{1}{2}\theta)^2]^{1/2}\}. \qquad \text{(4)}\end{aligned}$$

Equation (3) is to be studied by Fourier series expansion.

The integral equation used previously differs from the present one in that $K(\phi)$ is replaced by

$$\begin{aligned}\bar{K}(\phi) = [(2b\sin\tfrac{1}{2}\phi)^2 + a^2]^{-1/2} \\ \times \exp\{ik[(2b\sin\tfrac{1}{2}\phi)^2 + a^2]^{1/2}\}. \qquad \text{(4')}\end{aligned}$$

This difference has the following consequence. When Fourier series expansion is used, as given by (5) and (6) below, it is easily verified that the series for $I(\phi)$ converges for $\phi \neq 0$. If $\bar{K}$ had been used, the corresponding Fourier coefficients κ_n decreases exponentially for large n, and consequently the series for $I(\phi)$ diverges everywhere. In the work of Hallén and Storer, this violent divergence is avoided by an approximation on the coefficients κ_n which does not decrease exponentially for large n. Equation (4) is quite similar to the expression of the kernel in the case of the dipole antenna.[6]

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[6] T. T. Wu and R. W. P. King, J. Appl. Phys. 30, 76 (1959).

3. FOURIER SERIES EXPANSION

Let

$$I(\phi) = \sum_{n=-\infty}^{\infty} I_n e^{in\phi}, \qquad (5)$$

and

$$K(\phi) = \sum_{n=-\infty}^{\infty} \kappa_n e^{in\phi}, \qquad (6)$$

then

$$I_n = ikV\pi^{-1}\zeta_0^{-1}[\tfrac{1}{2}k^2b^2(\kappa_{n+1} + \kappa_{n-1}) - n^2\kappa_n]^{-1}. \qquad (7)$$

Thus, the major task here is to evaluate the coefficients κ_n.

If

$$M_n(A) = \int_{-\pi}^{\pi} d\phi e^{-in\phi}[(2b\sin\tfrac{1}{2}\phi)^2 + A^2]^{-1/2} \times \exp\{ik[(2b\sin\tfrac{1}{2}\phi)^2 + A^2]^{1/2}\} \qquad (8)$$

for $n \geq 0$, then

$$\kappa_n = \pi^{-2}\int_0^{2a} dA(4a^2 - A^2)^{-1/2}M_{|n|}(A). \qquad (9)$$

Thus, it is sufficient to consider the case $0 < A \ll b$. If

$$A \ll b/n, \qquad (10)$$

then the analysis of Oseen[7] and Storer[3] applies with the result

$$bM_n(A) = 2\ln(8b/A) - 4\sum_{m=0}^{n-1}(2m+1)^{-1} - \pi\int_0^{2kb} dx[\Omega_{2n}(x) - iJ_{2n}(x)] \qquad (11)$$

approximately, where Ω is the Lommel-Weber function

$$\Omega_m(x) = \pi^{-1}\int_0^{\pi}\sin(x\sin\theta - m\theta)\,d\theta. \qquad (12)$$

In particular, (11) implies that

$$bM_0(A) = 2\ln(8b/A) - \pi\int_0^{2kb} dx[\Omega_0(x) - iJ_0(x)]. \qquad (13)$$

On the other hand, the integral

$$N_n(A) = \int_{-\infty}^{\infty} d\phi e^{-in\phi}[b^2\phi^2 + A^2]^{-1/2} \times \exp\{ik[b^2\phi^2 + A^2]^{-1/2}\} \qquad (14)$$

may be readily computed to be given by

$$bN_n(A) = \pi i H_0^{(1)}[(k^2 - n^2/b^2)^{1/2}A]. \qquad (15)$$

When (10) holds, (15) is approximated by

$$bN_n(A) = -2\gamma - 2\ln[(-k^2 + n^2/b^2)^{1/2}A/2], \qquad (16)$$

where γ is Euler's constant, numerically about 0.57722. At least for n not too large, the difference $M_n(A)$ and $N_n(A)$ is approximately independent of A so that it may be computed for small values of A. Thus it follows from (15) and a comparison of (11) with (16) that

$$bM_n(A) = 2K_0[(-k^2 + n^2/b^2)^{1/2}A] + \ln(1 - k^2b^2/n^2) + 2C_n - \pi\int_0^{2kb} dx[\Omega_{2n}(x) - iJ_{2n}(x)], \qquad (17)$$

where

$$C_n = \ln(4n) + \gamma - 2\sum_{m=0}^{n-1}(2m+1)^{-1}. \qquad (18)$$

Since A is small, (17) may be slightly simplified to

$$bM_n(A) = 2K_0(nA/b) + 2C_n - \pi\int_0^{2kb} dx[\Omega_{2n}(x) - iJ_{2n}(x)]. \qquad (19)$$

It is worth noting that both C_n and the integral on the right-hand side of (19) are of the order of magnitude of n^{-2} as $n \to \infty$. Thus (19) does not hold if $n \gg b/A$. Fortunately, this does not cause any trouble. Finally the substitution of (13) and (19) into (9) gives

$$\kappa_0 = (\pi b)^{-1}\left\{\ln(8b/a) - \tfrac{1}{2}\pi\int_0^{2kb} dx[\Omega_0(x) - iJ_0(x)]\right\} \qquad (20a)$$

and

$$\kappa_n = \kappa_{-n} = (\pi b)^{-1}\left\{K_0(na/b)I_0(na/b) + C_n - \tfrac{1}{2}\pi\int_0^{2kb} dx[\Omega_{2n}(x) - iJ_{2n}(x)]\right\} \qquad (20b)$$

for $n \geq 1$. This is the desired answer. The current distribution may be found from (5), (7), and (20).

The difficulty of Hallén and Storer is simply not encountered, since for large n,

$$I_n^{-1} = -(ikV)^{-1}\pi\zeta_0 n^2\kappa_n \qquad (21)$$

approximately, and this is never very small.

4. THE INPUT ADMITTANCE

Similar to the known case of the dipole antenna,[6] $\lim_{\phi\to 0} I(\phi)$ does not exist. Let Y_∞ be the admittance

[7] C. W. Oseen, Arkiv Mat. Astron. Fys. 9, No. 12 (1913).

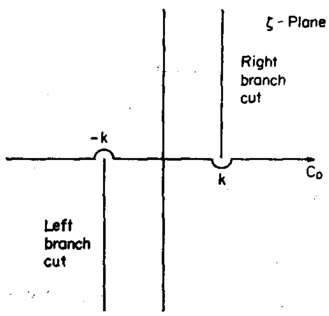

FIG. 2. The ζ plane and the contour C_0.

of an infinite dipole antenna of radius a, then the difference between $Y = I(0)/V$ and Y_∞ is finite. More specifically

$$Y - Y_\infty = \lim_{\phi\to 0}\Big(V^{-1}\sum_{n=-\infty}^{\infty} I_n e^{in\phi} + (2/\pi)k\zeta_0^{-1} \times \int_{C_0} d\zeta e^{i\zeta\phi b}(\zeta^2 - k^2)^{-1} \times \{J_0[a(k^2-\zeta^2)^{1/2}]H_0^{(1)}[a(k^2-\zeta^2)^{1/2}]\}^{-1}\Big), \quad (22)$$

where the contour of integration C_0 is shown in Fig. 2. Equation (22) may be simplified to

$$Y - Y_\infty = \lim_{N\to\infty} ik\pi^{-1}\zeta_0^{-1} \times \Big(\sum_{n=-N}^{N}[\tfrac{1}{2}k^2b^2(\kappa_{n+1}+\kappa_{n-1}) - n^2\kappa_n]^{-1} - 2i\int_{C_0(N)} d\zeta(\zeta^2-k^2)^{-1} \times \{J_0[a(k^2-\zeta^2)^{1/2}]H_0^{(1)}[a(k^2-\zeta^2)^{1/2}]\}^{-1}\Big), \quad (23)$$

where $C_0(N)$ is the part of C_0 with $|\zeta| < N + \frac{1}{2}$.

5. DISCUSSIONS

Within the framework of one-dimensional approximation, the current distribution on a thin circular loop transmitting antenna driven by a delta-function generator has been determined completely. Within the same approximation, the receiving antenna can be treated in a very similar manner without any new difficulty.

One-dimensional approximations, however, are never entirely satisfactory. Although difficult, it is highly desirable to study the complete Eqs. (1a) and (1b) directly, expanding the various quantities in the small parameters a/b and ka. This remains to be done.

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