contributions

The Impedance Properties of Narrow Radiating Slots in the Broad Face of Rectangular Waveguide*

Part I—Theory

ARTHUR A. OLINER†

Summary—**Theoretical results, valid at and away from resonance, for the impedance properties of the rotated series slot, the displaced series slot, and the longitudinal shunt slot have been derived by the use of variational expressions coupled with certain stored power considerations. The additional influence of finite wall thickness, an appreciable factor, is taken into account by a microwave network treatment. The results for the zero-thickness resistive elements become identical with those of Stevenson when the slot length is made equal to a half wavelength.**

The theoretical derivations are presented in Part I. In Part II, comparison is made with experimental data both previously available and specially taken in this connection. The effect of wall thickness and the distinction between slots of rounded and rectangular ends are also considered. The agreement between theory and measurement is reasonably good.

A. Introduction

Although slots cut in the walls of rectangular waveguide are widely employed as radiators of microwave energy, relatively little theoretical material is available on their impedance properties. The well-known results of Stevenson[1] for the slot resistance or conductance apply only at resonance. The less well-known calculations of Pounder,[2] based on Stevenson's formulas, for the resonant length of a longitudinal shunt slot are admittedly extremely tedious and must therefore be considered impractical. Analytical expressions have been obtained by several authors, including Lewin[3] and the group at the Polytechnic Institute of Brooklyn,[4,5] for the impedance properties of *centered* slots of arbitrary aspect ratio located in various positions. Centered slots are not useful as radiators, however, since the conductance changes little with slot shape and therefore does not permit the control that is available with a shift or rotation of the slot. The additional influence of the slot wall thickness on its impedance properties is a significant factor and has been accounted for by the Polytechnic group[5] for certain centered slots. There remains the need, therefore, for a reasonably simple analytical procedure for determining the impedance properties of slots in actual use, including, of course, the effect of wall thickness.

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† Microwave Res. Inst., Polytechnic Inst. of Brooklyn, Bklyn., N. Y. This work was begun at the Hughes Aircraft Co. during a three-month leave of absence, and completed as part of consulting services since that time.

[1] A. F. Stevenson, "Theory of slots in rectangular wave-guides," *J. Appl. Phys.*, vol. 19, pp. 24–38; January, 1948.
S. Silver, "Microwave Antenna Theory and Design," vol. 12, M.I.T. Rad. Lab. Ser., McGraw-Hill Book Co., New York, N. Y., pp. 291–295; 1949.

[2] J. R. Pounder, "Theoretical Impedance of a Longitudinal Slot in the Broad Face of a Rectangular Wave Guide (Numerical)," Special Comm. on Appl. Math., Natl. Res. Council of Canada, Radio Rep., September, 1944. Quoted in W. H. Watson, "The Physical Principles of Wave Guide Transmission and Antenna Systems," Clarendon Press, Oxford, England, pp. 199–200; 1947.

[3] L. Lewin, "Advanced Theory of Waveguides," Iliffe and Sons, Ltd., London, England, pp. 88–97, 121–128; 1951.

[4] N. Marcuvitz, "The Representation, Measurement and Calculation of Equivalent Circuits for Waveguide Discontinuities with Application to Rectangular Slots," Polytechnic Inst. of Brooklyn; 1949. The report was a group project.

[5] A. A. Oliner, "Equivalent Circuits for Slots in Rectangular Waveguide," Polytechnic Inst. of Brooklyn; August, 1951. The report was a group project.

It is attempted here to furnish such an impedance description for radiating slots located in several positions in the broad face of rectangular waveguide. The three slot types considered are the rotated series slot, the displaced series slot, and the longitudinal shunt slot, illustrated in Fig. 1. Variational expressions are used to obtain the slot equivalent circuit parameters; the slot conductances are determined directly, while the susceptances are related by certain stored energy considerations to the corresponding *centered* slot result, the latter being obtained from the work of the Polytechnic group.[5] The effect of wall thickness is accounted for by microwave network considerations. It should be added that all theoretical results presented herein assume that the slot radiates into a half space (*i.e.*, bounded by an infinite baffle), and that the slot has rectangular ends. The effects of deviations in practice from these idealized conditions, particularly for rounded rather than rectangular slot ends, are discussed in Part II.

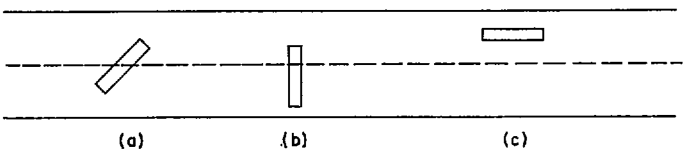

Fig. 1—Top view of slots located in the broad face of rectangular waveguide. (a) Rotated series slot, (b) displaced series slot, (c) longitudinal shunt slot.

The variational expressions are derived by following the standard procedure of first obtaining expressions for the magnetic field in the regions inside and outside of the waveguide, and then imposing the condition of continuity of the magnetic field in the slot aperture to obtain the appropriate integral equation. Restricted integral equations are then deduced, following a procedure due to Marcuvitz,[6] by applying antisymmetrical and symmetrical voltage (or electric field) excitation to series and shunt networks, respectively. These restricted integral equations are then cast into variational form in standard fashion. The resulting variational expressions are of the aperture type, thus requiring the insertion of a trial electric field in the slot aperture. The trial field chosen in all cases was cosinusoidal and is expected to be an excellent approximation. The stationary property of these expressions is not proved here since they are in standard form.

Since the conductance and resistance portions of the variational expressions are related only to the radiated power, and do not require the consideration of the rectangular guide Green's function, their determination is relatively simple. The results obtained are valid both at and away from resonance, and reduce identically to the results of Stevenson[1] when the slot length is made equal to a half wavelength.

[6] N. Marcuvitz, "Variational Calculations of Longitudinal Discontinuities," orally presented at URSI-IRE meeting, San Diego, April, 1950.

The direct evaluation of the variational expressions for the reactances and susceptances becomes a formidable task in view of the complexity of the waveguide dyadic Green's function. However, since the susceptance of a *centered* slot is already available,[5] an approach involving so-called "small" aperture or "stored power" considerations is used instead. These considerations relate the susceptance or reactance of the slots of Fig. 1 to the susceptance of the centered slot. The considerations are expected to be very good for rotated and displaced series slots, but are only approximately valid for longitudinal shunt slots.

Under the "small" aperture assumption, the numerators of the variational expressions for all three slots are identical and equal to that corresponding to the centered slot. The denominators, which are always positive definite, vary with the slot location. The implication is, therefore, that all three slots of Fig. 1 possess the same resonant length and the same Q value. A change in slot width will, of course, alter both the Q value and the resonant length. In the case of the rotated series slot, the validity of this conclusion is borne out by the fact that experimentally[7] one finds the resonant length independent of the angle of rotation. Experience with displaced transverse slots coupling identical waveguides indicates that the "small" aperture assumptions should also be very good for the displaced series slot. The dependence of the resonant length on displacement for the longitudinal shunt slot, however, implies that for this case one cannot neglect the influence of the closer side wall and that the "small" aperture results will be valid for small displacements only. Since the present theory does not account for this dependence, there remains the need for an improved theoretical expression for this case.[8] A more detailed evaluation of the usefulness and validity of these theoretical expressions is presented in Part II in connection with the comparisons between these expressions and the measured results.

B. Derivation of the Variational Expressions

Basic to the derivation of aperture type variational expressions are representations for the magnetic fields in the regions interior and exterior to the waveguide. Assuming the slot to radiate into a half space, the magnetic field in the exterior region is given by

$$\mathbf{H}_e(\mathbf{r}) = \iint_{\text{slot}} \mathbf{n} \times \mathbf{E} \cdot \mathcal{Y}_h dS \qquad (1)$$

where $\mathbf{E}$ is the electric field in the slot aperture, $\mathbf{n}$ is the normal pointing out of the guide into the half space, and

[7] L. A. Kurtz, "Design Applications of Series Slots," Hughes Aircraft Co. Tech. Memo. No. 273, Fig. 2 or p. 1; December, 1951. Also, see Sec. C, 3, of Part II of the present work.

[8] Since the completion of this work the author has been informed that Dr. W. K. Saunders of the Diamond Ordnance Fuze Labs. has obtained an expression for the longitudinal shunt slot which takes this dependence into account but contains slowly convergent infinite sums.

$\mathcal{Y}_h$ is the half-space dyadic Green's function,[9] or spatial admittance,

$$\mathcal{Y}_h = j\frac{\omega\epsilon}{2\pi}\left(\mathcal{E} + \frac{\nabla\nabla}{k^2}\right)\cdot\frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \qquad (2)$$

where $\mathcal{E}$ is the unit dyadic, and $k = 2\pi/\lambda$.

The representation chosen for the magnetic field in the interior region is[10]

$$\mathbf{H}_i(\mathbf{r}) = \tfrac{1}{2}(I_1 + I_2)\mathfrak{H}^{(1)}(\mathbf{r}) - j\tfrac{1}{2}Y_0(V_1 + V_2)\mathfrak{H}^{(2)}(\mathbf{r}) - j\iint_{\text{slot}} (\mathbf{n}\times\mathbf{E})\cdot\mathcal{B}\,dS \qquad (3)$$

where the composite (standing wave type) mode functions $\mathfrak{H}^{(1)}$ and $\mathfrak{H}^{(2)}$ are

$$\mathfrak{H}^{(1)}(\mathbf{r}) = \mathbf{h}(\boldsymbol{\rho})\cos\kappa z - j\mathbf{h}_z(\boldsymbol{\rho})\sin\kappa z$$
$$\mathfrak{H}^{(2)}(\mathbf{r}) = \mathbf{h}(\boldsymbol{\rho})\sin\kappa z + j\mathbf{h}_z(\boldsymbol{\rho})\cos\kappa z \qquad (4)$$

with $\boldsymbol{\rho} = (x, y)$ and $\kappa = 2\pi/\lambda_g$. The ordinary mode functions are

$$\mathbf{h}(\boldsymbol{\rho}) = \sqrt{\frac{2}{ab}}\cos\frac{\pi x}{a}\mathbf{x}_0,$$
$$\mathbf{h}_z(\boldsymbol{\rho}) = j\sqrt{\frac{2}{ab}}\frac{\pi}{a\kappa}\sin\frac{\pi x}{a}\mathbf{z}_0 \qquad (5)$$

where the geometry is indicated in Fig. 2.

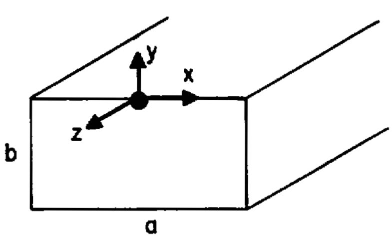

Fig. 2—Geometry of rectangular waveguide.

The voltages and currents V and I are discussed by Marcuvitz and Schwinger[10] and defined in their (3.1) and (3). Y_0 is the characteristic admittance of the dominant mode. The guide (internal) dyadic Green's function, or spatial susceptance, $\mathcal{B}$, is given in (3.41).[10]

Upon application of the condition of continuity of magnetic field in the slot aperture, the following integral equation for the electric field in the aperture is obtained.

$$\tfrac{1}{2}(I_1 + I_2)\mathfrak{H}^{(1)}(\mathbf{r}) - j\tfrac{1}{2}Y_0(V_1 + V_2)\mathfrak{H}^{(2)}(\mathbf{r}) - \iint_{\text{slot}} \mathbf{n}\times\mathbf{E}\cdot(\mathcal{Y}_h + j\mathcal{B})dS = 0. \qquad (6)$$

[9] H. Levine and J. Schwinger, "On the Theory of Electromagnetic Wave Diffraction by an Aperture in an Infinite Plane Conducting Screen," in "Theory of Electromagnetic Waves, A Symposium," Academic Press, New York, N. Y., pp. S31, S32; 1951.

[10] N. Marcuvitz and J. Schwinger, "On the representation of the electric and magnetic fields produced by currents and discontinuities in wave guides," *J. Appl. Phys.*, vol. 22, pp. 806–819; June, 1951. See (3.38b), (3.39), and (3.41).

The variational expressions are most conveniently deduced from a reduced form of the integral equation obtained by appropriate symmetric or antisymmetric voltage excitation. If the slot is representable by a purely shunt network as in Fig. 3(a), the application of symmetric voltage excitation produces an open circuit bisection of the network, with $I_1 = -I_2$, $V_1 = V_2 = V$. The integral equation then becomes

$$-jY_0V\mathfrak{H}^{(2)} = \iint_{\text{slot}} \mathbf{n}\times\mathbf{E}\cdot(\mathcal{Y}_h + j\mathcal{B})dS. \qquad (7)$$

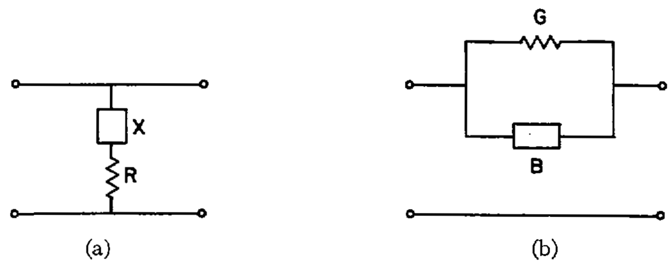

Fig. 3—Shunt and series networks. (a) Shunt network, (b) series network.

The variational form follows upon multiplication by $\mathbf{n}\times\mathbf{E}$ in dot product fashion and integration over the slot aperture, employing[11]

$$2I = jY_0\iint_{\text{slot}} \mathbf{n}\times\mathbf{E}\cdot\mathfrak{H}^{(2)}dS \qquad (8)$$

and dividing by $(2I)^2$

$$\frac{V}{2I} = R + jX = \frac{\iiiint_{\text{slot}} \mathbf{n}\times\mathbf{E}\cdot(\mathcal{Y}_h + j\mathcal{B})\cdot\mathbf{n}\times\mathbf{E}'dSdS'}{Y_0^2\left[\iint_{\text{slot}} \mathbf{n}\times\mathbf{E}\cdot\mathfrak{H}^{(2)}dS\right]^2} \qquad (9)$$

where all integrals are taken over the slot aperture. The normalized value is obtained by dividing both sides by the characteristic impedance $Z_0 (Z_0 = 1/Y_0)$.

If the slot is representable by a purely series network, as in Fig. 3(b), the application of antisymmetric voltage excitation, with $V_1 = -V_2$, $I_1 = I_2 = I$, yields a short circuit bisection of the network so that the integral equation reduces to

$$I\mathfrak{H}^{(1)} = \iint_{\text{slot}} \mathbf{n}\times\mathbf{E}\cdot(\mathcal{Y}_h + j\mathcal{B})dS. \qquad (10)$$

Again, the multiplication by $\mathbf{n}\times\mathbf{E}$ and integration over the slot aperture, together with the employment of[11]

$$2V = \iint_{\text{slot}} \mathbf{n}\times\mathbf{E}\cdot\mathfrak{H}^{(1)}dS \qquad (11)$$

[11] *Ibid.* See (3.42b).

and division by $(2V)^2$ casts the integral equation into variational form, *viz.*,

$$\frac{I}{2V} = G + jB$$

$$= \frac{\iiiint_{\text{slot}} \mathbf{n} \times \mathbf{E} \cdot (\mathcal{Y}_h + j\mathcal{B}) \cdot \mathbf{n} \times \mathbf{E}' dS dS'}{\left[\iint_{\text{slot}} \mathbf{n} \times \mathbf{E} \cdot \mathfrak{H}^{(1)} dS\right]^2}. \quad (12)$$

Again, the normalized value is obtained upon division by Y_0 on both sides.

C. The Resistance and Conductance Expressions

The resistance and conductance expressions, obtainable from the variational expressions (9) and (12), are particularly simple to evaluate as only the real part of the total dyadic Green's function $(\mathcal{Y}_h + j\mathcal{B})$ need be considered. Since all the slots treated here radiate into a half space, the numerators of the variational expressions for all of the slots will be identical, namely,

$$\iiiint_{\text{slot}} \mathbf{n} \times \mathbf{E}(\mathbf{r}) \cdot \mathcal{G}_h \cdot \mathbf{n} \times \mathbf{E}(\mathbf{r}') dS dS', \quad (13a)$$

where

$$\mathcal{G}_h(\mathbf{r}, \mathbf{r}') = \frac{\omega\epsilon}{2\pi}\left(\mathcal{E} + \frac{\nabla\nabla}{k^2}\right) \cdot \frac{\sin k|\mathbf{r} - \mathbf{r}'|}{|\mathbf{r} - \mathbf{r}'|} \quad (13b)$$

and is the real part of $\mathcal{Y}_h$. The integral (13a), which is related to the power radiated by the slot, has already been evaluated in previous work at the Polytechnic Institute of Brooklyn[12] in connection with a transverse slot radiating from the end of the guide into a half space. The result is

$$\text{"Power Radiated"} = \frac{16}{3\pi}\sqrt{\frac{\epsilon}{\mu}}\frac{1}{\lambda^2}(a'b')^2 \cdot \left[1 - 0.374\left(\frac{a'}{\lambda}\right)^2 + 0.130\left(\frac{a'}{\lambda}\right)^4\right] \quad (14)$$

neglecting terms of the order of $(b'/a')^2$, where a' and b' are the slot length and width, respectively. Eq. (14) is expected to be quite accurate for all narrow slots in general use.

The denominators of the various variational expressions, representing the square of the voltage or current, depending upon the slot in question, will vary with the slot location.

1) *Longitudinal Shunt Slot*

The equivalent network of Fig. 4(b) is valid at the terminal plane T. Appropriate variational expression for normalized resistance R/Z_0 follows from (9) as

$$\frac{R}{Z_0} = \frac{\text{"Power Radiated"}}{Y_0\left[\iint_{\text{slot}} \mathbf{n} \times \mathbf{E}(\mathbf{r}) \cdot \mathfrak{H}^{(2)}(\mathbf{r}) dS\right]^2} \quad (15)$$

where the numerator is given in (14). Using (4) and (5), $\mathfrak{H}^{(2)}$ is given by

$$\mathfrak{H}^{(2)}(x, z) = \sqrt{\frac{2}{ab}}\cos\frac{\pi x}{a}\sin\kappa z\,\mathbf{x}_0 - \sqrt{\frac{2}{ab}}\frac{\pi}{a\kappa}\sin\frac{\pi x}{a}\cos\kappa z\,\mathbf{z}_0. \quad (16)$$

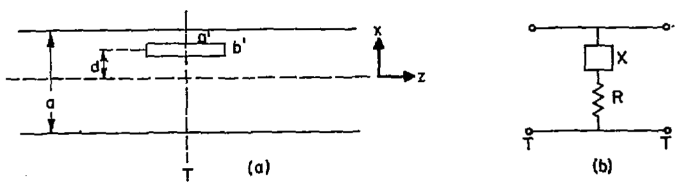

Fig. 4—Longitudinal shunt slot. (a) Physical structure, (b) equivalent network.

Since the dominant mode is an H mode, the characteristic admittance Y_0 is

$$Y_0 = \frac{\kappa}{\omega\mu} = \sqrt{\frac{\epsilon}{\mu}}\frac{\lambda}{\lambda_g}. \quad (17)$$

The trial aperture electric field $\mathbf{E}$ must be chosen as

$$\mathbf{n} \times \mathbf{E} = \mathbf{z}_0 \cos\frac{\pi z}{a'} \quad (18)$$

since that was the value chosen for the numerator in the result (14). For slots in general use this is expected to be an excellent approximation.

The integration over the slot aperture indicated in the denominator of (15) proceeds in straightforward fashion to yield

$$\left[\iint_{\text{slot}} \mathbf{n} \times \mathbf{E} \cdot \mathfrak{H}^{(2)} dS\right]^2 = \frac{2}{ab}\left[\frac{2\pi^2 b'}{aa'\kappa}\,\frac{\sin\frac{\pi d}{a}\cos\frac{\kappa a'}{2}}{\left(\frac{\pi}{a'}\right)^2 - \kappa^2}\right]^2. \quad (19)$$

When (19) is combined with (17) and (14), the result for the normalized resistance becomes

$$\frac{R}{Z_0} = \frac{8\pi a^3 b}{3\lambda^3\lambda_g} \frac{\left[1 - \left(\frac{2a'}{\lambda_g}\right)^2\right]^2\left[1 - 0.374\left(\frac{a'}{\lambda}\right)^2 + 0.130\left(\frac{a'}{\lambda}\right)^4\right]}{\sin^2\left(\frac{\pi d}{a}\right)\cos^2\left(\frac{\pi a'}{\lambda_g}\right)}. \quad (20)$$

[12] A. A. Oliner, *op. cit.* The particular integral in question was evaluated by H. Kurss.

Stevenson[1] presents a result for the normalized conductance of a longitudinal shunt slot at resonance, *i.e.*, when the network element is purely real, but with the slot length indicated as a half wavelength. If one takes the reciprocal of (20) after substituting $a'=\lambda/2$ and simplifying, one obtains

$$\frac{G}{Y_0} = 2.09\,\frac{a\lambda_g}{b\lambda}\cos^2\left(\frac{\pi\lambda}{2\lambda_g}\right)\sin^2\left(\frac{\pi d}{a}\right) \tag{21}$$

in exact agreement with Stevenson as quoted in (47) of Silver.[1] Even though the slot is generally not resonant at exactly a half wavelength, Stevenson's result is quite accurate because of the relative insensitivity of (20) with a'.

2) *Displaced Series Slot*

The equivalent network of Fig. 5(b) is valid at the terminal plane T. The appropriate variational expression for the normalized conductance G/Y_0 follows from (12) as

$$\frac{G}{Y_0} = \frac{\text{“Power Radiated”}}{Y_0\left[\iint_{\text{slot}} \mathbf{n}\times\mathbf{E}(\mathbf{r})\cdot\mathfrak{H}^{(1)}(\mathbf{r})dS\right]^2} \tag{22}$$

where the numerator is given in (14), and the mode function $\mathfrak{H}^{(1)}$ is obtained from (4) and (5) as

$$\mathfrak{H}^{(1)}(x,z) = \sqrt{\frac{2}{ab}}\left[\cos\frac{\pi x}{a}\cos\kappa z\,\mathbf{x}_0 + \frac{\pi}{a\kappa}\sin\frac{\pi x}{a}\sin\kappa z\,\mathbf{z}_0\right]. \tag{23}$$

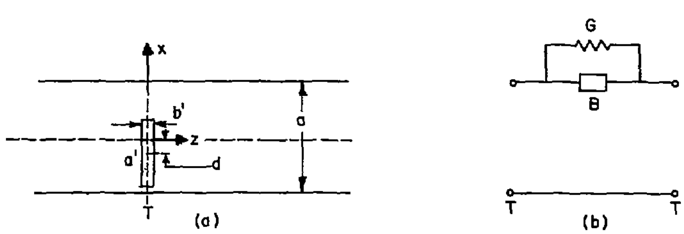

Fig. 5—Displaced series slot. (a) Physical structure, (b) equivalent network.

The trial aperture electric field has the same form as for the longitudinal shunt slot but is oriented at right angles so that

$$\mathbf{n}\times\mathbf{E} = \mathbf{x}_0\cos\frac{\pi x}{a'}\cdot \tag{24}$$

With this trial field, the denominator, which is essentially the square of a voltage, becomes

$$\left[\iint_{\text{slot}}\mathbf{n}\times\mathbf{E}\cdot\mathfrak{H}^{(1)}dS\right]^2$$

$$= \frac{8a^2}{\pi^2}\frac{(a'b')^2}{a^3b}\left[\frac{\cos(\pi a'/2a)}{1-(a'/a)^2}\right]^2\cos^2\frac{\pi d}{a} \tag{25}$$

and, when combined with (14), yields the following result for the normalized conductance

$$\frac{G}{Y_0} = \frac{\lambda_g}{\lambda^3}\frac{32ab}{3\pi}\left[\frac{\pi}{4}\frac{1-(a'/a)^2}{\cos(\pi a'/2a)}\right]^2 \cdot\left[1 - 0.374\left(\frac{a'}{\lambda}\right)^2 + 0.130\left(\frac{a'}{\lambda}\right)^4\right]\sec^2\frac{\pi d}{a}\cdot \tag{26}$$

As with the longitudinal shunt slot, agreement with Stevenson's resonant resistance for the displaced series slot is obtained by inverting (26) after a' is set equal to $\lambda/2$. The result is

$$\frac{R}{Z_0} = 0.522\,\frac{\lambda_g^3}{\lambda ab}\cos^2\frac{\pi\lambda}{4a}\cos^2\frac{\pi d}{a} \tag{27}$$

in agreement with (48) of Silver[1] (the factor there is given as 0.523).

3) *Rotated Series Slot*

The equivalent network of Fig. 6(b) is valid at the terminal plane T. The appropriate variational expression for the normalized conductance is the same as (22) for the displaced series slot, with the numerator again given by (14) and the mode function $\mathfrak{H}^{(1)}$ by (23). The trial aperture electric field has again a cosinusoidal form, but is oriented in the v direction [see Fig. 6(a)], so that

$$\mathbf{n}\times\mathbf{E} = \mathbf{u}_0\cos\frac{\pi u}{a'}\cdot \tag{28}$$

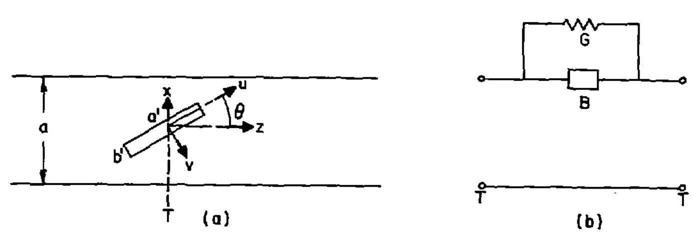

Fig. 6—Rotated series slot. (a) Physical structure, (b) equivalent network.

The angle of rotation of the slot is defined by the angle θ between the u and z axes, with

$$x = u\sin\theta,\qquad z = u\cos\theta$$

so that the voltage term in the denominator of the variational expression becomes

$$\iint_{\text{slot}}\mathbf{n}\times\mathbf{E}\cdot\mathfrak{H}^{(1)}dS$$

$$= \sqrt{\frac{2}{ab}}\,b'\int_{-a'/2}^{a'/2}\cos\frac{\pi u}{a'}\left[\sin\theta\cos\left(\frac{\pi u}{a}\sin\theta\right)\cos(\kappa u\cos\theta) + \frac{\pi}{a\kappa}\cos\theta\sin\left(\frac{\pi u}{a}\sin\theta\right)\sin(\kappa u\cos\theta)\right]du.$$

Upon performing the integrations and employing (14) the following result is obtained for the normalized conductance

$$\frac{G}{Y_0} = \frac{8\pi}{3} \frac{\lambda_g ab}{\lambda^3} \frac{\left[1 - 0.374\left(\frac{a'}{\lambda}\right)^2 + 0.130\left(\frac{a'}{\lambda}\right)^4\right]}{\left[A(\theta)\sin\theta + \frac{\lambda_g}{2a}B(\theta)\cos\theta\right]^2} \tag{29}$$

where

$$\left.\begin{matrix} A(\theta) \\ B(\theta) \end{matrix}\right\} = \frac{\cos\left(\frac{\pi}{2}\xi\right)}{1-\xi^2} \pm \frac{\cos\left(\frac{\pi}{2}\eta\right)}{1-\eta^2}$$

and

$$\left.\begin{matrix} \eta \\ \xi \end{matrix}\right\} = \left(\frac{a'}{a}\sin\theta \pm \frac{2a'}{\lambda_g}\cos\theta\right).$$

Again, the result of Stevenson for the normalized resistance at resonance may be checked by inversion of (29) after the substitution $a' = \lambda/2$ is made. The result

$$\frac{R}{Z_0} = 0.131 \frac{\lambda^3}{\lambda_g ab}\left[A(\theta)\sin\theta + \frac{\lambda_g}{2a}B(\theta)\cos\theta\right]^2 \tag{30}$$

wi h $A(\theta)$ and $B(\theta)$ defined as in (29), but with

$$\left.\begin{matrix} \eta \\ \xi \end{matrix}\right\} = \frac{\lambda}{2a}\sin\theta \pm \frac{\lambda}{\lambda_g}\cos\theta$$

is in exact agreement with (49) in Silver,[1] upon noting that A and B above are identical with I and J used :h re.

D. The Reactance and Susceptance Expressions

The evaluation of reactances and susceptances via the variational expressions is generally a rather difficult task. In particular, the difficulty is associated with the integrations in the numerator of the variational expressions, involving the guide (not half space) dyadic Green's function, $\mathfrak{G}$. However, with the recognition that the imaginary portion of the numerator is related to the stored power in the vicinity of the slot, it is sometimes possible to obtain the reactance or susceptance of a slot in a particular location from the corresponding already evaluated result for a slot in some other location. In particular, the available result for the susceptance of a *centered* transverse series slot, located in the broad face of rectangular waveguide, is related below to the corresponding quantity for a rotated series slot, a displaced series slot, and a longitudinal shunt slot.

1. *Rotated Series Slot*

The physical structure and the equivalent network for this slot are given in Fig. 6. The susceptance of this slot will be obtained from that of a centered transverse series slot; the two slots are sketched in Fig. 7. The variational expression for the normalized slot admittance for either slot is obtained from (12) as

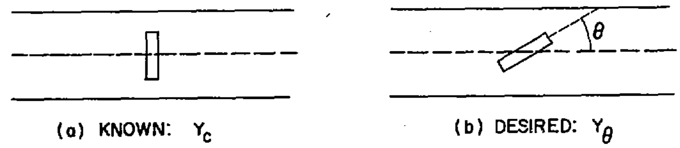

Fig. 7—The centered and rotated series slots.

$$\frac{G}{Y_0} + j\frac{B}{Y_0} = \frac{\text{"Power radiated"} + \text{"Power stored"}}{Y_0\left[\iint \mathbf{n}\times\mathbf{E}\cdot\mathfrak{H}^{(1)}dS\right]^2}. \tag{31}$$

With the highly reasonable assumption that the slot field is of the same form for either slot (a) or (b) of Fig. 7, the "power radiated" and the "power stored" in the external region are identical for both cases, since the slots are assumed to radiate into a half space. The denominators are quite different for the two cases as $\mathfrak{H}^{(1)}$ is a function of angle.

The assumption in the present treatment relates to the "power stored" in the interior, or guide, region. The "stored power," or, considered loosely, the distortion of the field lines in the neighborhood of the slot, will be most strongly affected by the guide walls closest by, which, for both of these slots, is the bottom wall directly opposite the slots. As the slot is rotated through angle θ, the influence of the bottom wall remains constant while that of the side walls changes somewhat. Since the side walls are relatively far away, we assume that the latter effect is negligible. The assumption that the interior "stored power" is independent of the angle of rotation of the slot therefore means that the variational expressions (31) for the two slots have identical numerators, and differ only by the denominators.

With this assumption, the susceptance for arbitrary θ is related to the known susceptance for the centered slot ($\theta = \pi/2$) by equating the numerators in the expressions (31) for the two slots and taking only the imaginary parts

$$\frac{B_\theta}{Y_0} = \frac{B_c}{Y_0}\frac{\left[\iint \mathbf{n}\times\mathbf{E}\cdot\mathfrak{H}_c^{(1)}dS\right]^2}{\left[\iint \mathbf{n}\times\mathbf{E}\cdot\mathfrak{H}_\theta^{(1)}dS\right]^2}. \tag{32}$$

The voltage terms multiplying B_c/Y_0 have already been evaluated in connection with the conductance expressions. Their ratio is denoted in Part II by V_θ^2 and presented as (16). The expression for B_c/Y_0 for a zero-thickness slot is[13]

$$\frac{B_c}{Y_0} = \frac{1}{2}\frac{B_t}{Y_0} + \frac{1}{n_j^2}\frac{B_{rj}}{Y_0} + \frac{2b}{\lambda g}\left[\ln 2 + \frac{\pi b'}{6b} + \frac{3}{2}\left(\frac{b}{\lambda_g}\right)^2\right] \tag{33}$$

[13] A. A. Oliner, *op. cit.* The result presented here for B_c/Y_0 is a modification of the original result evaluated by J. Blass.

where B_t/Y_0, B_{rj}/Y_0, and $1/n_j^2$ are presented in Part II as (8), (6), and (4), respectively. In contrast to the above, the symbol B_c/Y_0 in Part II refers to a centered series slot for which the slot thickness has been taken into account. An expression, analogous to (32), can be written for conductances, but this is unnecessary since the separate conductance expressions for zero-thickness slots are given in Section C.

The quantity B_t/Y_0 represents the normalized susceptance of a zero-thickness transverse slot coupling identical waveguides. It is seen in (8) of Part II that a closed form expression is presented for this quantity despite the fact that the waveguide dyadic Green's function, which occurs in the numerator of the variational expression, involves infinite sums. This result, derived previously,[5] was obtained by first expressing the dyadic Green's function in the form of a dyadic operator operating on a scalar function. This scalar function, an infinite double sum, separates naturally into a sum corresponding to what is obtained for a slot of full width (capacitive), and a sum over the remaining higher modes. The latter sum is then converted by means of a Poisson transformation into a series of Bessel functions of the second kind of imaginary argument; as a result, it converges rapidly enough to be approximated by a single, rather than a double, infinite sum. This transformed kernel, when employed in the variational integrals, will again contribute an infinite series whose sum, however, can be readily approximated. One thereby otains a closed form result which expresses the slot susceptance as that for the full width slot plus correction terms. Lewin's derivation[3] for the same slot also employs a Poisson transformation, but effects an inductive rather than a capacitive separation. The result (8) of Part II for B_t/Y_0 is therefore approximate but has yielded quite good agreement with measured data.

2. *Displaced Series Slot*

The physical structure and equivalent network are given in Fig. 5; one notes that the centered series slot is the centered position of the displaced series slot, *i.e.*, when $d=0$. The variational expression for the admittance of the displaced series slot is also given by (31), and the associated remarks given there apply here also. Since little of the stored field associated with the slot extends beyond the ends of the slot in its long dimension, the proximity of the side wall as the slot is displaced has relatively little influence on the numerator of the variational expression. The assumption made in subsection 1 concerning the constancy of the numerator as the slot is moved, in this case displaced, is therefore applicable.

An expression relating the susceptance of the displaced slot to that of the centered slot is obtained in a fashion similar to that for the rotated slot. The result, identical to (32) with rotated quantities replaced by displaced quantities, becomes, after the appropriate voltage terms are evaluated using (25),

$$\frac{B}{Y_0} = \frac{B_c}{Y_0} \sec^2 \frac{\pi d}{a}. \tag{34}$$

The susceptance B_c/Y_0 of the zero-thickness centered series slot is given in (33).

3. *Longitudinal Shunt Slot*

The physical structure and equivalent network are given in Fig. 4; the variational expression for the normalized slot impedance may be written from (9) as

$$\frac{R}{Z_0} + j\frac{X}{Z_0} = \frac{\text{“Power radiated”} + \text{“Power stored”}}{Y_0\left[\iint \mathbf{n} \times \mathbf{E} \cdot \mathfrak{H}^{(2)} dS\right]^2}. \tag{35}$$

Eq. (35) is to be compared to the corresponding relation (31) for the normalized admittance of the centered series slot in an attempt to relate the reactance of the former to the susceptance of the latter. Since the slot fields are assumed identical for the two cases, the "power stored" in the exterior region and the "power radiated" are identical for both slots since they both radiate into a half space. The assumption will be made now that the "power stored" in the interior region is also identical for both slots. This assumption is not as valid here as it was in the case of the rotated and the displaced series slots since here the side walls exert a nonnegligible influence on the stored power. Since the stored field extends out some distance away from the slot in its narrow dimension, the nearer side wall is expected to exert a noticeable effect here in contrast to the case of the displaced series slot. The effect of the side walls can be accounted for by considering appropriate image terms in the Green's function, but this has not been attempted. The influence of the side walls should manifest itself in a small shift in the resonant frequency of the slot as it is moved off-center; the constancy of the numerator in the case of the rotated and displaced series slots implies a constancy of resonant frequency with angle of rotation or displacement. These effects are borne out by the available experimental data.

With the assumption that the numerators of (31) and (35) are identical, one obtains, upon equating the numerators and taking the imaginary parts (the denominators are always purely real)

$$\frac{X}{Z_0} = \frac{B_c}{Y_0} \frac{\left[\iint \mathbf{n} \times \mathbf{E} \cdot \mathfrak{H}_c^{(1)} dS\right]^2}{\left[\iint \mathbf{n} \times \mathbf{E} \cdot \mathfrak{H}^{(2)} dS\right]^2}. \tag{36}$$

The voltage terms multiplying B_c/Y_0 have already been evaluated and are given in (19) and (25), the latter for the case of $d=0$. Their ratio is denoted in Part II by V_{sh}^2 and presented as (12). The susceptance B_c/Y_0 of the zero-thickness centered series slot is given in (33).

E. Effect of Wall Thickness

The variational expressions and the various slot parameter results obtained above are based on the assumption that the slot walls are of zero thickness. Since, in practice, the finite thickness of the walls exerts a noticeable effect on the slot admittance, this effect is theoretically taken into account here by microwave network considerations. The thick slot is viewed as a composite structure, consisting of a length of waveguide, equal to the slot wall thickness and of cross-sectional dimensions equal to those of the slot, connecting two junctions, one a radiating junction between the slot waveguide and a half space, and the other a Tee junction between the slot waveguide and the main guide. Since the slot wall thickness is usually comparable to the slot width, we can assume negligible higher mode interaction between the junctions and consider them as isolated.

For slots of the same cross-sectional dimensions and thickness, it is evident in this microwave network picture that the radiating junction and the connecting waveguide are identical for all three slot locations, since the slots radiate into a half space. Only the Tee junctions will be different for each.

The radiating junction, common to all three slot locations, is shown in Fig. 8. The Tee junction for the thick centered series slot is the centered E plane Tee pictured in Fig. 9 together with an equivalent network whose simple form is valid since the slot width is small compared to its length. When the component junctions are connected by a length of transmission line equal to the slot thickness, the composite representation of Fig. 10 is obtained for the complete thick centered series slot. In Fig. 10(a), the slot width b' and the thickness t are exaggerated for clarity. The subscripts j signify "junction."

It can be shown[5] that the parameter B_j/Y_0 of the Tee junction of Fig. 9 is given by variational expression (12) when $\mathcal{Y}_h$ is replaced by $j\mathcal{B}_s$, the dyadic Green's function for the stub guide of the Tee, which corresponds here to the slot guide. The term G/Y_0 then drops out, of course. Since the trial aperture electric field was chosen as a cosine, it is orthogonal to all the terms composing $\mathcal{B}_s$, and the stub guide contribution to B_j/Y_0 vanishes. The expression for parameter B_j/Y_0 then becomes identically equal to the interior contribution to B/Y_0 in the zero-thickness case. It is evident that the stored power considerations discussed above apply here in the same fashion, so that we may use the expressions derived in Section D to relate the admittance or impedance of the thick slot of interest to the admittance of the thick centered series slot.

The parameters of the composite representation of Fig. 10(b) are related to those of the over-all representation, the latter being a series network such as that in Fig. 3(b) with parameters G_c and B_c, the subscript signifying "centered," by comparing the two networks and using transmission line relations to reduce the former to the latter. By inspection, one finds

$$\frac{G_c}{Y_0} + j\frac{B_c}{Y_0} = j\frac{B_j}{Y_0} + \frac{1}{n_j^2}\frac{Y_0'}{Y_0} \cdot \frac{j + \frac{Y_0}{Y_0'}\left(\frac{G_{rj}}{Y_0} + j\frac{B_{rj}}{Y_0}\right)\cot \kappa' t}{\cot \kappa' t + j\left(\frac{G_{rj}}{Y_0} + j\frac{B_{rj}}{Y_0}\right)\frac{Y_0}{Y_0'}} \cdot \quad (37)$$

Upon rationalization, the separate expressions presented in the summary, Section B of Part II, for the conductance G_c/Y_0 and the susceptance B_c/Y_0 are obtained. Y_0' and κ' are seen to be the characteristic admittance and propagation wavenumber of the connecting slot waveguide. Theoretical expressions for the component parameters employed above are presented in (4) to (8) of Part II. The relations between G_c/Y_0 and B_c/Y_0 and the parameters of the other slots are also presented in Part II.

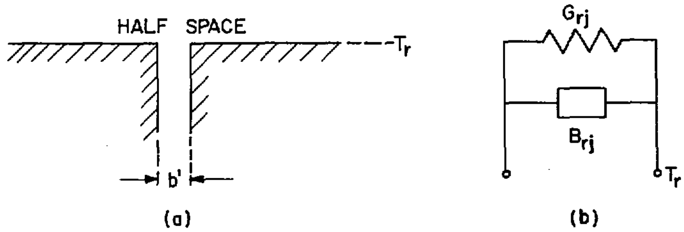

Fig. 8—Radiating junction. (a) Physical structure, (b) equivalent network.

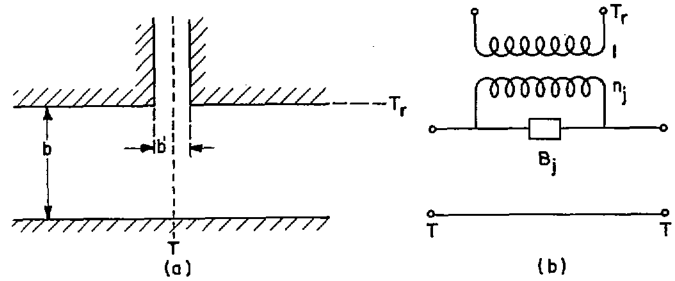

Fig. 9—E plane Tee junction. (a) Physical structure, (b) equivalent network.

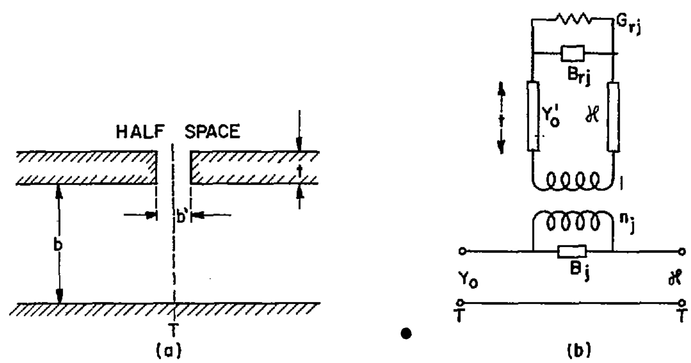

Fig. 10—The thick centered series slot. (a) Physical structure, (b) composite equivalent network.