The Admittance of Bare Circular Loop Antennas in a Dissipative Medium

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Summary—**The normalized input admittance of thin bare circular loop antennas has been evaluated from the theory of T. T. Wu. Computations have been made for loops in air and in an infinite homogeneous isotropic dissipative medium. A comparison is also made with Storer's theory of the loop. Numerical results are given in the form of graphs for several wire sizes and for loops up to two and one-half wavelengths in circumference. The properties of the medium are represented by the ratio α/β in the range from zero (perfect dielectric) to one (good conductor); α and β are the imaginary and real parts of the complex propagation constant $k=\beta-j\alpha=\omega\sqrt{\mu(\epsilon-j\sigma/\omega)}$ where μ is the permeability, ϵ the dielectric constant, and σ the conductivity of the medium.**

INTRODUCTION

THE FIRST general analysis of the circular loop as a transmitting antenna appears to be that of Hallen;[1] he used the method of expansion in Fourier series. However, owing to the occurrence of a singularity or a very large value when the number of terms in the summation is sufficiently great, Hallen concluded that the series was divergent. Storer[2] avoided the contribution from the large term by replacing the Fourier series by the corresponding integral and evaluating this in the sense of the Cauchy principal value. He provided extensive tables and graphs of the impedance, admittance, and distribution of current for loops up to a wavelength in circumference with a number of different wire sizes. Recently, Wu[3] re-examined the problem of evaluating the Fourier series. He considered Storer's expedient to be of doubtful validity and devised an alternative and improved method with approximations that are valid over larger ranges of the parameters.

Although formulated specifically for loops in air, the solutions of both Wu and Storer are readily applied to loops in an infinite homogeneous and isotropic medium by the introduction of the constitutive parameters of the medium at the appropriate points. It is the purpose of this paper to discuss the evaluation of the admittance of a loop antenna in an arbitrary dissipative medium from Wu's formula. A comparison with Storer's results is also provided.

Since loops up to a wavelength in circumference are considered, the present work is a significant extension of the earlier studies by Kraichman[4] and by Chen and King.[5] Kraichman's analysis is based on a postulated *uniform* distribution of current around the loop which is valid only for electrically extremely small loops if these are bare or covered with a very thin layer of dielectric. The work of Chen and King makes use of Storer's analysis but retains only the first two terms in the Fourier series. Although this is a considerably better approximation than that of Kraichman, it is also limited to electrically rather small loops. Indeed, even for loops with circumferences as small as 0.1λ or 0.3λ, a surprisingly large error in the normalized conductance is made when only the first two terms in the Fourier series are retained.

ANALYTICAL FORMULATION

The admittance of a circular loop of radius b when made of wire of radius a has been derived by Storer[2] and, in a somewhat more general form, by Wu[3] specifically for antennas in air. The generalized formula for the normalized admittance $Y/\Delta=G/\Delta+jB/\Delta$ of a loop in an infinite homogeneous and isotropic dissipative medium when driven by a delta-function generator is

$$\frac{Y}{\Delta}=\frac{-j(1-j\alpha/\beta)}{\pi\zeta_0}\left[\frac{1}{a_0}+2\sum_1^\infty\frac{1}{a_n}\right] \tag{1}$$

where $\zeta_0=(\mu_0/\epsilon_0)^{\frac{1}{2}}=120\pi$ ohms and

$$a_n=\frac{kb}{2}(K_{n+1}+K_{n-1})-\frac{n^2}{kb}K_n. \tag{2}$$

In this formula

$$K_0=\frac{1}{\pi}\ln\frac{8b}{a}-\frac{1}{2}\left[\int_0^{2kb}\Omega_0(x)dx+j\int_0^{2kb}J_0(x)dx\right] \tag{3}$$

$$K_n=K_{-n}=\frac{1}{\pi}\left[\mathcal{K}_0\left(\frac{na}{b}\right)\mathcal{I}_0\left(\frac{na}{b}\right)+C_n\right]$$

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[2] J. E. Storer, "Impedance of thin-wire loop antennas," *Trans. Am. Inst. Elec. Engrs.*, vol. 75, p. 606; 1956; also Cruft Laboratory, Harvard University, Cambridge, Mass., Tech. Rept. No. 212; May, 1955.

[3] T. T. Wu, "Theory of thin circular loop antenna," *J. Math. Phys.*, vol. 3, pp. 1301–1304; November–December, 1962.

[4] M. B. Kraichman, "Impedance of a circular loop in an infinite conducting medium," *J. Research Natl. Bur. Standards*, vol. 66d, pp. 499–503; July-August, 1962.

[5] C. L. Chen, and R. W. P. King, "The small bare loop antenna immersed in a dissipative medium," IEEE TRANS. ON ANTENNAS AND PROPAGATION, vol. AP-11, pp. 266–268; May, 1963.

$$-\frac{1}{2}\left[\int_0^{2kb}\Omega_{2n}(x)dx + j\int_0^{2kb}J_{2n}(x)dx\right] \quad (4)$$

$$C_n = \ln(4n) + 0.5772\cdots - 2\sum_{m=0}^{n-1}(2m+1)^{-1} \quad (5)$$

where $\mathcal{I}_0(z)$ and $\mathcal{K}_0(z)$ are the modified Bessel functions of the first and second kinds, and $\Omega(x)$ is the Lommel-Weber function defined by

$$\Omega_m(x) = \frac{1}{\pi}\int_0^{\pi}\sin(x\sin\theta - m\theta)d\theta. \quad (6)$$

In (2)

$$k = \beta - j\alpha = \omega\sqrt{\mu(\epsilon - j\sigma/\omega)} = \omega\sqrt{\mu\epsilon}[f(p) - jg(p)] \quad (7)$$

is the complex propagation constant. In (7)

$$p = \sigma/\omega\epsilon \quad (8)$$

is the loss tangent of the medium and the $f(p)$ and $g(p)$ functions are defined as follows:

$$f(p) \pm jg(p) = \sqrt{1 \pm jp}. \quad (9)$$

This is equivalent to

$$f(p) = \cosh(\tfrac{1}{2}\sinh^{-1}p), \quad g(p) = \sinh(\tfrac{1}{2}\sinh^{-1}p). \quad (10)$$

The functions $f(p)$ and $g(p)$ are extensively tabulated in the literature.[6,7]

In order to provide generally useful graphs and tables of the admittance of a loop antenna when immersed in a medium characterized by arbitrary values of σ and ϵ, it is convenient to introduce the normalizing factor

$$\Delta = \sqrt{\frac{\epsilon_r}{\mu_r}}f(p) \quad (11)$$

where ϵ_r and μ_r are the relative dielectric constant and permeability. This factor appears in (1); for antennas in air, it is equal to unity.

As with a dipole antenna when center driven by a delta-function generator, the admittance Y strictly does not exist, since its suceptance must become infinite owing to the knife-edge terminals with zero separation that characterize the delta-function generator. However, as shown for the dipole,[8,9] the representation of the current by continuous functions combined with the extreme localization of that part of the current that is associated with the knife edges of the generator effectively omits the latter for thin wires unless a very large number of terms in the Fourier series is taken. If the infinite sum in (1) is replaced by a sum over a finite number of terms, an approximate formula is obtained that is a good measure of the admittance of the antenna for use with a practical method of driving when combined with a suitable terminal-zone network.

Evaluation of the Tables

The formula (1) was evaluated on a high-speed computer using successively 8, 9, 10, 18, 19, and 20 terms in the series. The normalized conductance G/Δ and normalized susceptance B/Δ for $\alpha/\beta = 0$ are shown in Fig. 1 as a function of the number of terms in the series. Curves are shown for $\beta b = 0.5$ and 2.0 and for $\Omega = 2\ln(2\pi b/a) = 8$, 9, 10, 11, and 12. It is seen that the convergence is such that G/Δ does not change noticeably for both values of βb and for all values of Ω. On the other hand, B/Δ continues to increase with the number of terms. The rate of this increase is great when $\Omega < 10$ and when βb is large; it is quite small for $\Omega \geq 10$, especially when βb is small. In general, it may be concluded that when $\Omega \geq 10$, 20 terms in the Fourier series yield highly accurate values of G/Δ for $\beta b \leq 2.5$, quite good values of B/Δ for $\beta b < 1$, and fair values when $1 \leq \beta b \leq 2.5$.

The values of G/Δ and B/Δ evaluated from Storer's theory[2] are indicated on the right in Fig. 1. It is seen that they are in excellent agreement with Wu's results using 20 terms insofar as G/Δ is concerned, but that significant differences occur in B/Δ.

In Fig. 2 values of B/Δ are shown as functions of the number of terms in the Fourier series for $\beta b = 2.0$ with $\Omega = 8$, 12, and 20, when α/β is increased from zero to one. It is seen that except for $\alpha/\beta = 1$, the curves for the relatively thick-wire loop with $\Omega = 8$ or $2\pi b/a = 54.6$ increase significantly with the number of terms. On the other hand, the curves for the thinner loops with $\Omega = 12$ ($2\pi b/a = 403.4$) and $\Omega = 20$ ($2\pi b/a = 22{,}026$) are practically independent of the number of terms for all values of α/β.

These results indicate that a Fourier series solution in which 20 terms are retained is satisfactory for determining the admittance of thin-wire loops ($\Omega \geq 10$) that are not too large ($\beta b \leq 2.5$) when in air or an arbitrary dissipative medium. The approximation is excellent for the conductance, somewhat less accurate for the susceptance. Numerical values of the normalized admittance $Y/\Delta = G/\Delta + jB/\Delta$ for loops with $\Omega = 12$ are shown in Table I for $0 \leq \beta b \leq 1.5$ and for $0 \leq \alpha/\beta \leq 1$. More extensive tables that encompass the full range of Figs. 3–8 are available.[10]

[6] R. W. P. King, "Fundamental Electromagnetic Theory," Dover Publications Inc., New York, N. Y.; 1963. See Appendix II.

[7] D. W. Gooch, C. W. Harrison, Jr., R. W. P. King, and T. T. Wu, "Impedances and Admittances of the Long Antenna in Air and in Dissipative Media With Tables of the Functions $f(p) \pm ig(p) = \sqrt{1 \pm ip}$," Cruft Laboratory, Harvard University, Cambridge, Mass., Tech. Rept. No. 353; January, 1962.

[8] T. T. Wu and R. W. P. King, "Driving point and input admittance of linear antennas," *J. Appl. Phys.*, vol. 30, pp. 74–76; January, 1959.

[9] R. H. Duncan and F. A. Hinchey, "Cylindrical antenna theory," *J. Research Natl. Bur. Standards*, vol. 64D, pp. 569–584; September-October, 1960.

[10] R. W. P. King, C. W. Harrison, Jr., and D. G. Tingley, "The Admittance of Bare Circular Loop Antennas in a Dissipative Medium," Sandia Corp., Albuquerque, N. M., Monograph No. SCR-674, June, 1963; and Cruft Laboratory, Harvard University, Cambridge, Mass., Tech. Rept. No. 419, August, 1963.

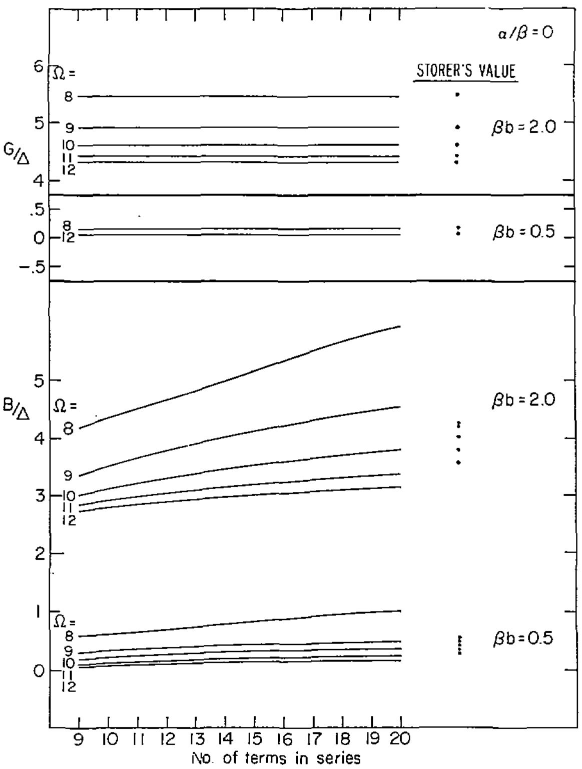

Fig. 1—Normalized conductance and susceptance of circular loop in air as a function of the number of terms: Wu's theory for $\beta b=0.5$ and 2.0, $\Omega=2\ \ln\ (2\pi b/a)=8$–12.

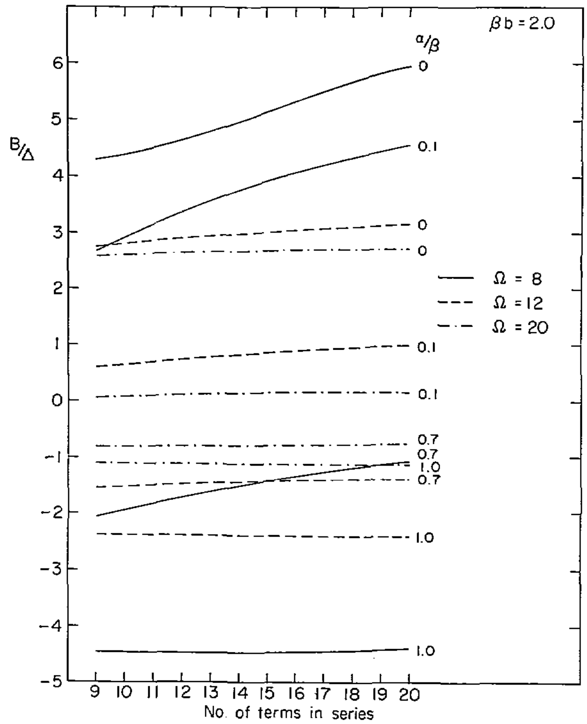

Fig. 2—Normalized susceptance of circular loop in a dissipative medium with propagation constant $k=\beta-j\alpha$ as a function of the number of terms: Wu's theory for $\beta b=2.0$, $\Omega=8, 12, 20$, $\alpha/\beta=0$–1.

TABLE I

Normalized Admittance Y/Δ in Millimhos of Loop Antennas in Dissipative Media

$\Omega=12$

βb	$\alpha/\beta=0.00$		$\alpha/\beta=0.01$		$\alpha/\beta=0.05$		$\alpha/\beta=0.10$		$\alpha/\beta=0.30$		$\alpha/\beta=1.00$	
	Y/Δ		Y/Δ		Y/Δ		Y/Δ		Y/Δ		Y/Δ	
0.05	0.0002	−12.3838	0.0026	−12.3839	0.0124	−12.3842	0.0246	−12.3851	0.0734	−12.3951	0.2433	−12.5069
0.10	0.0008	−6.0079	0.0057	−6.0079	0.0254	−6.0086	0.0500	−6.0107	0.1482	−6.0312	0.4857	−6.2596
0.15	0.0019	−3.7985	0.0094	−3.7986	0.0393	−3.7998	0.0767	−3.8031	0.2256	−3.8356	0.7265	−4.1891
0.20	0.0036	−2.6282	0.0137	−2.6284	0.0545	−2.6302	0.1053	−2.6349	0.3066	−2.6814	0.9644	−3.1724
0.25	0.0060	−1.8703	0.0191	−1.8706	0.0713	−1.8732	0.1364	−1.8799	0.3929	−1.9431	1.1973	−2.5867
0.30	0.0095	−1.3150	0.0258	−1.3154	0.0905	−1.3192	0.1710	−1.3283	0.4860	−1.4125	1.4220	−2.2256
0.35	0.0146	−0.8711	0.0343	−0.8717	0.1128	−0.8771	0.2103	−0.8895	0.5877	−1.0005	1.6350	−2.0003
0.40	0.0218	−0.4917	0.0455	−0.4928	0.1396	−0.5005	0.2561	−0.5175	0.7003	−0.6632	1.8323	−1.8652
0.45	0.0323	−0.1495	0.0605	−0.1511	0.1726	−0.1622	0.3105	−0.1858	0.8262	−0.3774	2.0105	−1.7931
0.50	0.0474	0.1742	0.0811	0.1716	0.2142	0.1554	0.3770	0.1224	0.9680	−0.1304	2.1670	−1.7652
0.55	0.0698	0.4934	0.1100	0.4893	0.2683	0.4652	0.4599	0.4184	1.1283	0.0835	2.3008	−1.7675
0.60	0.1033	0.8207	0.1516	0.8142	0.3407	0.7777	0.5660	0.7100	1.3092	0.2652	2.4126	−1.7890
0.65	0.1544	1.1692	0.2131	1.1587	0.4403	1.1020	0.7047	1.0025	1.5114	0.4119	2.5043	−1.8214
0.70	0.2347	1.5538	0.3068	1.5363	0.5814	1.4463	0.8897	1.2971	1.7328	0.5174	2.5789	−1.8585
0.75	0.3646	1.9931	0.4544	1.9631	0.7871	1.8161	1.1403	1.5888	1.9670	0.5739	2.6398	−1.8962
0.80	0.5833	2.5099	0.6961	2.4564	1.0952	2.2094	1.4814	1.8602	2.2021	0.5746	2.6903	−1.9322
0.85	0.9693	3.1257	1.1088	3.0265	1.5639	2.6016	1.9387	2.0703	2.4203	0.5172	2.7332	−1.9653
0.90	1.6821	3.8294	1.8378	3.6393	2.2692	2.9111	2.5192	2.1430	2.6016	0.4083	2.7709	−1.9953
0.95	3.0140	4.4372	3.1124	4.0862	3.2489	2.9404	3.1671	1.9724	2.7293	0.2644	2.8051	−2.0225
1.00	5.1747	4.1923	4.9896	3.7064	4.3203	2.3893	3.7210	1.4915	2.7960	0.1090	2.8369	−2.0471
1.05	6.9797	1.9691	6.4182	1.7662	4.9408	1.2056	3.9720	0.7912	2.8065	−0.0341	2.8671	−2.0699
1.10	6.3872	−0.7830	5.9627	−0.5402	4.7283	−0.0478	3.8479	0.1131	2.7747	−0.1480	2.8961	−2.0913
1.15	4.6500	−1.9270	4.5259	−1.6197	3.9992	−0.8072	3.4768	−0.3481	2.7184	−0.2245	2.9242	−2.1117
1.20	3.2716	−1.9702	3.2919	−1.7477	3.2274	−1.0562	3.0383	−0.5602	2.6537	−0.2641	2.9514	−2.1313
1.25	2.3889	−1.6551	2.4583	−1.5089	2.6099	−1.0065	2.6421	−0.5859	2.5928	−0.2721	2.9779	−2.1505
1.30	1.8314	−1.2655	1.9157	−1.1693	2.1616	−0.8194	2.3258	−0.4980	2.5435	−0.2559	3.0035	−2.1693
1.35	1.4688	−0.8811	1.5575	−0.8168	1.8473	−0.5770	2.0906	−0.3481	2.5095	−0.2232	3.0283	−2.1878
1.40	1.2260	−0.5180	1.3170	−0.4750	1.6321	−0.3156	1.9261	−0.1669	2.4921	−0.1809	3.0523	−2.2060
1.45	1.0616	−0.1741	1.1555	−0.1464	1.4909	−0.0495	1.8213	0.0286	2.4909	−0.1347	3.0755	−2.2241
1.50	0.9530	0.1578	1.0514	0.1730	1.4079	0.2164	1.7673	0.2287	2.5043	−0.0892	3.0979	−2.2419

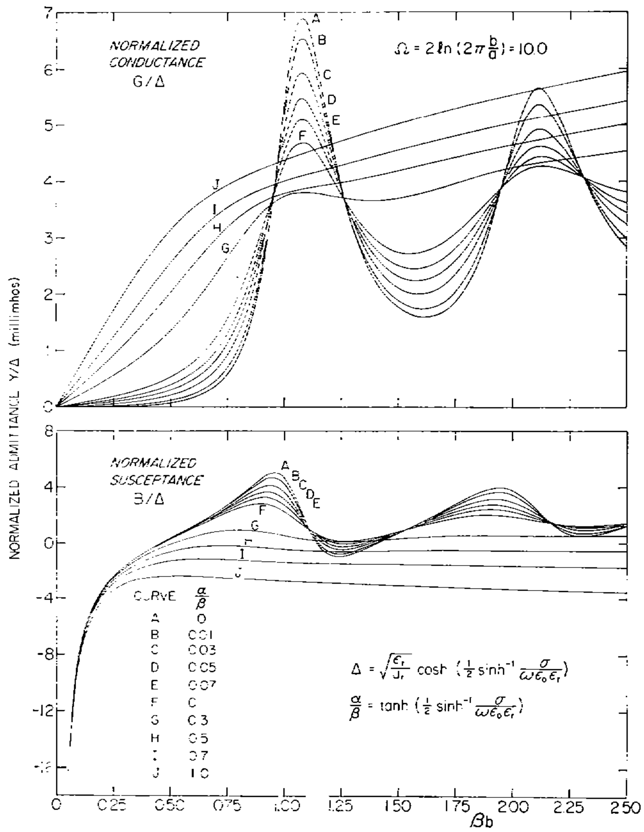

Fig. 3—Normalized admittance of circular loop antenna in a dissipative medium: Wu's theory, $\Omega = 10$.

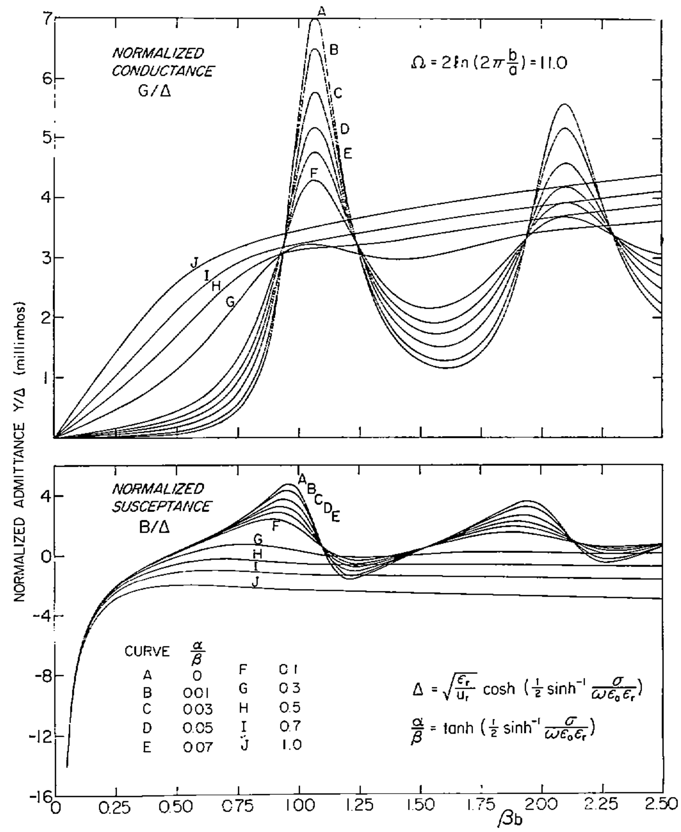

Fig. 4—Normalized admittance of circular loop antenna in a dissipative medium: Wu's theory, $\Omega = 11$.

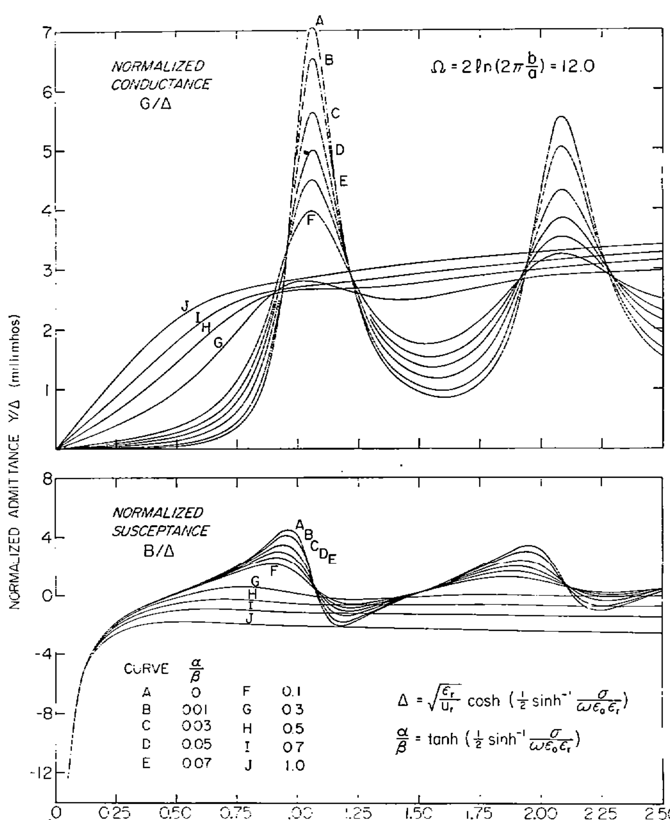

Fig. 5—Normalized admittance of circular loop antenna in a dissipative medium: Wu's theory, $\Omega = 12$.

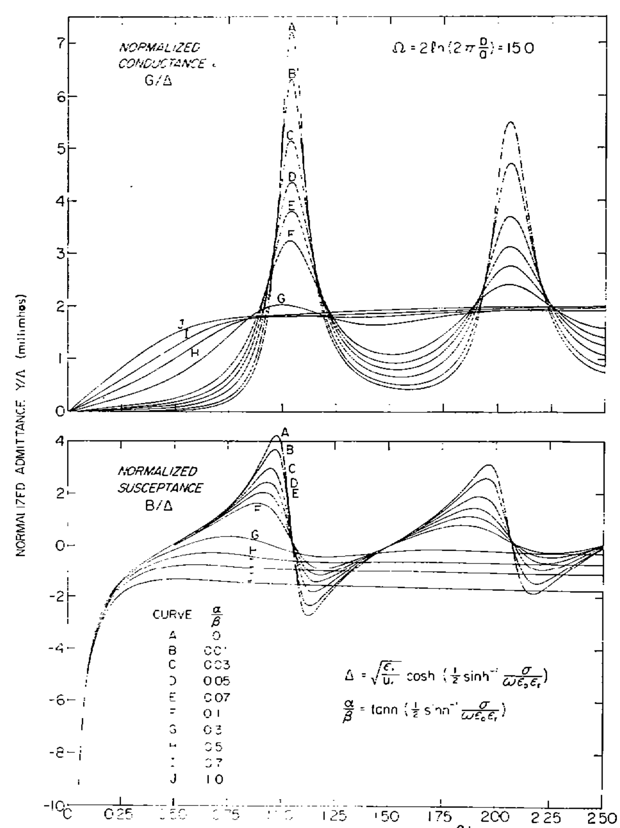

Fig. 6—Normalized admittance of circular loop antenna in a dissipative medium: Wu's theory, $\Omega = 15$.

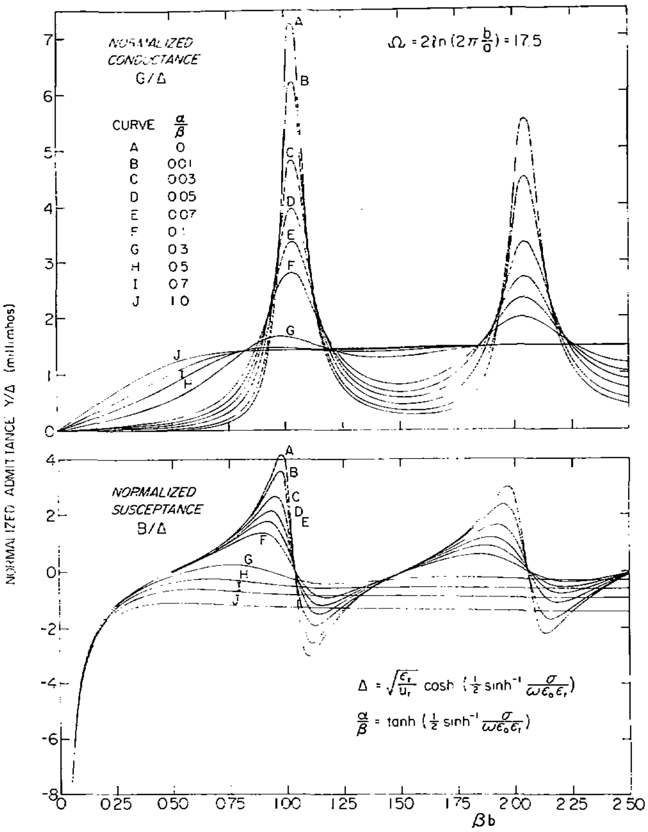

Fig. 7—Normalized admittance of circular loop antenna in a dissipative medium: Wu's theory, $\Omega=17.5$.

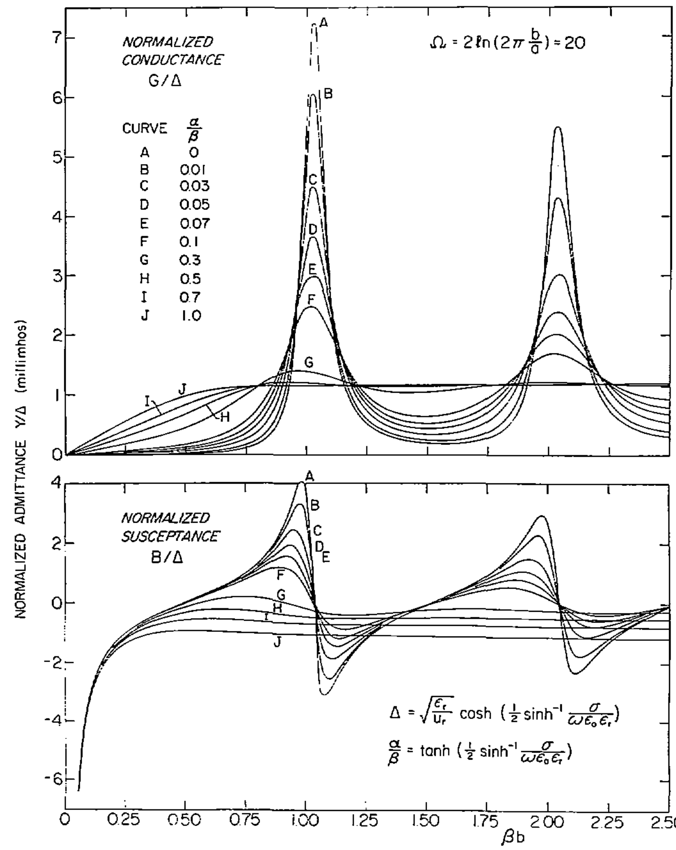

Fig. 8—Normalized admittance of circular loop antenna in a dissipative medium: Wu's theory, $\Omega=20$.

Graphical Representation

Graphs of the normalized conductance and susceptance of thin-wire loops as a function of the variable $\beta b=2\pi b/\lambda$ are shown in Figs. 3–8. The parameter α/β of the surrounding medium ranges from zero to one. Note how insensitive to the size of the loop the admittance becomes as α/β approaches unity. This is true particularly of loops near antiresonance that have a high driving-point impedance. Such an insensitivity merely means that most of the current has left the loop and has entered the surrounding dissipative medium.

Conclusion

The driving-point admittance of bare thin-wire loops up to two and one-half wavelengths in circumference when immersed in an arbitrary dissipative medium has been determined from Wu's theory using 20 terms in the Fourier series.