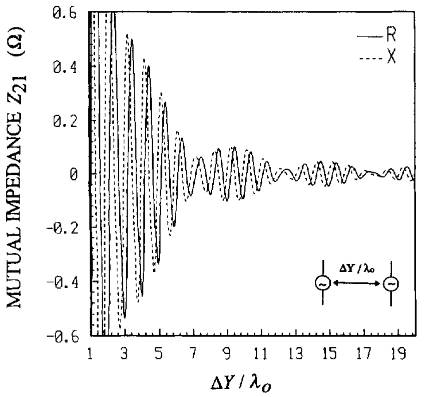

Fig. 6. H-plane mutual coupling ($Z_{21} = R + jX$) between dipoles versus normalized separation length, $\Delta Y/\lambda_o$, for $d = 0.15\ \lambda_o$, $\epsilon_r = 2.55$, $L = 0.295\ \lambda_o$.

V. Conclusion

This paper has presented a study of mutual coupling between two printed antennas. Emphasis was placed on the unexpected non-monotonic behavior of the mutual coupling magnitude observed primarily in the H-plane. This anomalous behavior underscores the point that the decay of mutual coupling with separation is a complicated function of the substrate parameters, and that a simple rule cannot be established in general. Fortunately, mutual coupling can be efficiently calculated using an asymptotic form of the Green's function. This work has implications for phased array design where it is useful to have some intuitive understanding of the behavior of mutual coupling as element separation and substrate parameters vary.

Finally, we would like to point out that the anomalies presented here were actually observed several years ago in numerically calculated coupling, but were mistakenly dismissed as inaccuracies caused by numerical integrations [12]. This work thus underscores the point that analytic results still have an important role to play in the understanding of electromagnetic phenomena.

Acknowledgment

The authors would like to acknowledge useful discussions with Professor P. Pathak and Dr. G. Somers of Ohio State University.

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Techniques for Evaluating the Uniform Current Vector Potential at the Isolated Singularity of the Cylindrical Wire Kernel

D. H. Werner, J. A. Huffman, and P. L. Werner

***Abstract*— The cylindrical wire kernel possesses a singularity which must be properly treated in order to evaluate the uniform current vector potential. Traditionally, the singular part of the kernel is extracted resulting in a slowly varying function which is convenient for numerical integration. This paper provides some new accurate and computationally efficient methods for evaluating the remaining singular integral. It is shown that this double integral may be converted to a single integral which no longer possesses a singular integrand and consequently may be efficiently evaluated numerically. This form of the integral is independent of the restrictions involving wire length and radius which are inherent in various approximations. Also presented is a highly convergent exact series representation of the integral which is valid except in the immediate vicinity of the singularity. Finally, a new approximation is derived which is found to be an improvement over the classical thin wire approximation. It is demonstrated that each of these methods provides extremely accurate as well as efficient results for a wide range of wire radii and field point locations.**

I. Introduction

The expression for the cylindrical wire kernel possesses a singularity which must be properly treated in order to evaluate the cylindrical antenna integral equation. Several approximations to the kernel in the electric field integral equation are discussed in [1]. More recently, an exact expansion for the cylindrical wire kernel has been found [2]. Common procedure is to extract the singular part of the kernel which results in a slowly varying function that is amenable to efficient numerical integration. Various techniques for

Manuscript received July 29, 1993; revised July 5, 1994. This work was supported by Cdr. Gus K. Lott, Chief Engineer, Code GX, Navy Security Group Command.

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IEEE Log Number 9406398.

evaluating the integral of the extracted singularity have been reported. Schelkunoff [3] transformed the singular part of the kernel into an elliptic integral of the first kind and showed that the asymptotic behavior in the immediate vicinity of the singularity is an integrable logarithmic term. Pearson [4] pointed out that by expanding the elliptic integral in a series form, the logarithmic term which gives rise to this asymptotic behavior is extracted. However, the resultant infinite series is complicated by the presence of additional integrals which must be evaluated numerically. Wilton and Butler [5] also avoided the singular nature of the kernel by directly replacing the singularity with the integrable asymptotic logarithm term and then extracting it from the kernel. The disadvantage of this approach is that the remaining integrand, although very smooth, cannot be evaluated at the singularity because the logarithm function as well as the elliptic integral are unbounded at this point. Also, evaluation at all other points requires double numerical integration. In another communication, Butler [6] presents a different form of the extracted integrable singularity and expands it in a highly convergent power series which is valid in the vicinity of the singularity for thin wires with piecewise constant current. For thin wires in which the radius is much less than the wire length and the wavelength, the reduced kernel approximation is widely used [7]. This form is independent of azimuthal variation and hence requires evaluation of only a single integral. When the current is uniform, the integrable singularity can be evaluated analytically.

This paper presents some computationally efficient and accurate alternatives for computing an integral of the extracted singularity associated with the vector potential of a uniform current cylindrical wire antenna. An intermediate approximation, which is valid for thin wires with piecewise constant current, is shown to be more accurate than the classical thin wire approximation while maintaining computational simplicity. For thicker wires, it is shown that the term containing the extracted singularity can be cast into a single integral form which is amenable to numerical integration. Also, a highly convergent exact series solution to the integral of the extracted singularity is presented which is valid for all cases except in the vicinity of the singularity. The results given here can be directly implemented in a moment method solution in which the unknown current is represented by pulses or by more complicated basis functions which contain a constant term, such as trigonometric or polynomial [8], [9].

II. THEORY

The vector potential associated with a cylindrical wire (segment) of length Δ and radius a centered about the z-axis is given by [7]

$$A_z = \frac{\mu}{4\pi}\int_{-\Delta/2}^{\Delta/2} I_z(z')K(z-z')dz' \tag{1}$$

where

$$K(z-z') = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{-j\beta R(z-z',\phi')}}{R(z-z',\phi')}d\phi' \tag{2}$$

represents the cylindrical wire kernel, in which

$$R(z-z',\phi') = \sqrt{(z-z')^2+\rho^2+a^2-2\rho a\cos\phi'}. \tag{3}$$

The integrand in (2) contains a singularity which may be extracted by expressing the cylindrical wire kernel in the form [3]

$$K(z-z') = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{d\phi'}{R(z-z',\phi')} + \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{-j\beta R(z-z',\phi')}-1}{R(z-z',\phi')}d\phi'. \tag{4}$$

For the situation in which the current is assumed to be uniformly distributed over the surface of the wire, i.e., $I_z(z') = I_0$, the expression for the vector potential (1) reduces to

$$A_z = \frac{\mu I_0}{4\pi}\left\{\frac{1}{2\pi}\int_{-\Delta/2}^{\Delta/2}\int_{-\pi}^{\pi}\frac{d\phi' dz'}{R(z-z',\phi')} + \frac{1}{2\pi}\int_{-\Delta/2}^{\Delta/2}\int_{-\pi}^{\pi}\frac{e^{-j\beta R(z-z',\phi')}-1}{R(z-z',\phi')}d\phi' dz'\right\}. \tag{5}$$

The second integral has a very slowly varying integrand and may be readily evaluated numerically. The first integral, however, possesses a singular integrand when $\rho = a$, $z = z'$ and $\phi' = 0$. For this reason, methods for evaluating integrals of the form

$$I(\rho, z, \Delta) = \frac{1}{2\pi}\int_{-\Delta/2}^{\Delta/2}\int_{-\pi}^{\pi}\frac{d\phi' dz'}{R(z-z',\phi')} \tag{6}$$

are of considerable interest from the computational point of view. This paper will introduce and compare several new techniques developed for evaluating integrals of the type given in (6).

If the change of variables $\xi = z - z'$ is applied to the integral (6), then it follows that

$$I(\rho, z, \Delta) = \frac{1}{\pi}\int_{\xi_1}^{\xi_2}\int_{0}^{\pi}\frac{d\phi' d\xi}{R(\xi,\phi')} \tag{7}$$

where $\xi_1 = z - \Delta/2, \xi_2 = z + \Delta/2$ and

$$R(\xi,\phi') = \sqrt{\xi^2+c^2} \tag{8}$$

$$c = \sqrt{\rho^2+a^2-2\rho a\cos\phi'}. \tag{9}$$

Interchanging the order of the integration in (7) and using the fact that

$$\int\frac{d\xi}{\sqrt{\xi^2+t^2}} = \ln(\xi+\sqrt{\xi^2+t^2}) \tag{10}$$

results in

$$I(\rho, z, \Delta) = \frac{1}{\pi}\int_0^{\pi}\ln\left[\frac{\xi_2+\sqrt{\xi_2^2+c^2}}{\xi_1+\sqrt{\xi_1^2+c^2}}\right]d\phi'. \tag{11}$$

Introducing the variable change, $v = (2/\pi)\phi' - 1$, (11) may be transformed into (12), shown at the bottom of the page. On the surface of the antenna where $\rho = a$, (7) reduces to

$$I(a, z, \Delta) = \frac{2}{\pi}\int_{\xi_1}^{\xi_2}\int_0^{\pi/2}\frac{d\theta d\xi}{R(\xi,\theta)} \tag{13}$$

where

$$R(\xi,\theta) = \sqrt{\xi^2+c^2} \tag{14}$$

$$c = 2a\sin\theta. \tag{15}$$

$$I(\rho, z, \Delta) = \frac{1}{2}\int_{-1}^{1}\ln\left[\frac{\xi_2+\sqrt{\xi_2^2+\rho^2+a^2-2\rho a\cos(v+1)\frac{\pi}{2}}}{\xi_1+\sqrt{\xi_1^2+\rho^2+a^2-2\rho a\cos(v+1)\frac{\pi}{2}}}\right]dv. \tag{12}$$

As before, by interchanging the order of integration in (13) and making use of (10), it can be shown that

$$I(a,z,\Delta) = \ln\left(\frac{\alpha_1}{\alpha_2}\right) + \frac{2}{\pi}\int_0^{\pi/2} \ln\left[1+\sqrt{1+\alpha_2^2\sin^2\theta}\right]d\theta - \frac{2}{\pi}\int_0^{\pi/2} \ln\left[1+\sqrt{1+\alpha_1^2\sin^2\theta}\right]d\theta, z > \Delta/2 \tag{16}$$

where $\alpha_1 = 2a/\xi_1$ and $\alpha_2 = 2a/\xi_2$. The restriction on the range of validity of (16) to positive values of z only, i.e., $z > \Delta/2$, can be easily removed to include negative values of z as well by making use of the symmetry relationship $I(a,z,\Delta) = I(a,|z|,\Delta)$, which is valid for $|z| > \Delta/2$. We next introduce another change of variables $v = (4/\pi)\theta - 1$ which may be used to transform (16) into

$$I(a,z,\Delta) = \ln\left(\frac{\alpha_1}{\alpha_2}\right) + \frac{1}{2}\int_{-1}^{1} \ln\left[\frac{1+\sqrt{1+\alpha_2^2\sin^2(v+1)\frac{\pi}{4}}}{1+\sqrt{1+\alpha_1^2\sin^2(v+1)\frac{\pi}{4}}}\right]dv, \quad z > \Delta/2. \tag{17}$$

A similar procedure can be followed for the special case where $z = 0$. The result for this case is found to be

$$I(a,0,\Delta) = 2\ln\left(\frac{2}{\alpha}\right) + \int_{-1}^{1} \ln\left[1+\sqrt{1+\alpha^2\sin^2(v+1)\frac{\pi}{4}}\right]dv \tag{18}$$

where $\alpha = 4a/\Delta$. Equations (12), (17), and (18) are in a convenient form for application of numerical integration techniques because their integrands are relatively smooth and do not contain singularities. The form of these integrals are particularly well-suited for numerical integration using a Gaussian quadrature technique [10], [11]. One significant advantage offered by (17) and (18), when compared with the conventional form of I [5], is that they do not require the evaluation of elliptic integrals.

A useful approximation to I may be obtained by applying the transformation $u = \sin\theta$ to the integrals contained in (16). This yields the expression

$$I(a,z,\Delta) = \ln\left(\frac{\alpha_1}{\alpha_2}\right) + \frac{2}{\pi}\int_0^1 \frac{\ln\left[1+\sqrt{1+\alpha_2^2u^2}\right]}{\sqrt{1-u^2}}du - \frac{2}{\pi}\int_0^1 \frac{\ln\left[1+\sqrt{1+\alpha_1^2u^2}\right]}{\sqrt{1-u^2}}du, z > \Delta/2. \tag{19}$$

Assuming that $(\alpha_1 u)^2 \ll 1$ and $(\alpha_2 u)^2 \ll 1$, we introduce the approximations

$$\sqrt{1+(\alpha_1u)^2} \approx 1 + \frac{1}{2}(\alpha_1u)^2 \tag{20}$$

$$\sqrt{1+(\alpha_2u)^2} \approx 1 + \frac{1}{2}(\alpha_2u)^2 \tag{21}$$

which may be used to reduce (19) to

$$I(a,z,\Delta) \approx \ln\left(\frac{\alpha_1}{\alpha_2}\right) + \frac{2}{\pi}\int_0^1 \frac{\ln\left[1+(\alpha_2/2)^2u^2\right]}{\sqrt{1-u^2}}du - \frac{2}{\pi}\int_0^1 \frac{\ln\left[1+(\alpha_1/2)^2u^2\right]}{\sqrt{1-u^2}}du, z > \Delta/2. \tag{22}$$

Use is now made of the fact that

$$\int_0^1 \frac{\ln[1+tu^2]}{\sqrt{1-u^2}}du = \pi\ln\left[\frac{1+\sqrt{1+t}}{2}\right], t \geq -1 \tag{23}$$

in order to arrive at the following simple result

$$I(a,z,\Delta) \approx \ln\left(\frac{\xi_1}{\xi_2}\right) + 2I_{\text{thin}}(a,z,\Delta), z > \Delta/2 \tag{24}$$

where I_{thin} is the well-known classical thin-wire approximation given by [12]

$$I_{\text{thin}}(a,z,\Delta) \approx \ln\left[\frac{\xi_2+\sqrt{\xi_2^2+a^2}}{\xi_1+\sqrt{\xi_1^2+a^2}}\right]. \tag{25}$$

Hence, (24) may be thought of as an extended thin-wire or intermediate-wire approximation. Similarly, it can be shown that when $z = 0$, the intermediate-wire approximation takes the form

$$I(a,0,\Delta) \approx 2\ln\left(\frac{a}{\Delta}\right) + 2I_{\text{thin}}(a,0,\Delta) \tag{26}$$

where

$$I_{\text{thin}}(a,0,\Delta) \approx 2\ln\left[\frac{\Delta+\sqrt{\Delta^2+(2a)^2}}{2a}\right] \tag{27}$$

is the corresponding thin-wire approximation.

A power series expansion of I was derived by Butler [6] for the $z = 0$ case, which converges provided $\Delta/a > 4$. This expansion is given by

$$I(a,0,\Delta) = 2\ln\left(\frac{\Delta}{a}\right) + 2\sum_{n=1}^{\infty}\binom{1/2}{n}(2n-1)\frac{(2n-1)!}{(n!)^2} \times \left[\frac{2a}{\Delta}\right]^{2n}, \Delta/a > 4 \tag{28}$$

A useful approximation may be obtained from (28) by retaining the logarithmic term and the first two terms of the series expansion. This leads to

$$I(a,0,\Delta) \approx 2\ln\left(\frac{\Delta}{a}\right) + 4\left[\frac{a}{\Delta}\right]^2 - 18\left[\frac{a}{\Delta}\right]^4. \tag{29}$$

An exact expression for the integral I may be found which is valid for $\rho \neq a$ or when $\rho = a$ and $|z| > \Delta/2$. The first step in the derivation of this exact solution is to make use of the fact that the singular part of the cylindrical wire kernel may be expressed in the form

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{d\phi'}{R(z-z',\phi')} = \frac{1}{\pi\sqrt{\rho a}}kF\left(\frac{\pi}{2},k\right), \quad 0 \leq k < 1 \tag{30}$$

where

$$F\left(\frac{\pi}{2},k\right) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} \tag{31}$$

is a complete elliptic integral of the first kind and

$$k = \frac{2\sqrt{\rho a}}{\sqrt{(z-z')^2+(\rho+a)^2}}. \tag{32}$$

A useful infinite series representation for $F\left(\frac{\pi}{2},k\right)$ is [13]

$$F\left(\frac{\pi}{2},k\right) = \frac{\pi}{2}\sum_{n=0}^{\infty}\binom{-1/2}{n}^2 k^{2n} \tag{33}$$

which converges, provided that $0 \leq k < 1$. Substituting (33) into (30), integrating term by term with respect to ξ and introducing the change of variables $x = \xi/(\rho + a)$ results in an exact series representation for I which is given by

$$I(\rho, z, \Delta) = \ln\left[\frac{x_2 + u_2}{x_1 + u_1}\right] + \sum_{n=1}^{\infty} \binom{-1/2}{n}^2 \left[\frac{2\sqrt{\rho a}}{\rho + a}\right]^{2n} \times F_n(\rho, z, \Delta), \rho \neq a \tag{34}$$

where

$$F_n(\rho, z, \Delta) = \int_{x_1}^{x_2} \frac{dx}{u^{2n+1}} \tag{35}$$

in which $u = \sqrt{1 + x^2}$, $x_1 = \xi_1/(\rho + a)$ and $x_2 = \xi_2/(\rho + a)$. A recurrence relation exists which provides a computationally efficient method for calculating the integrals F_n defined in (35). The form of this recurrence relation was found to be

$$F_n = \frac{1}{(2n-1)}\left[x_2(1/u_2)^{2n-1} - x_1(1/u_1)^{2n-1} + 2(n-1)F_{n-1}\right], \quad n \geq 2 \tag{36}$$

where

$$F_1 = \left[\frac{x_2}{u_2} - \frac{x_1}{u_1}\right] \tag{37}$$

in which $u_1 = \sqrt{1 + x_1^2}$ and $u_2 = \sqrt{1 + x_2^2}$. An approximation of I may be obtained by retaining the logarithmic term and the first two terms in the series expansion (34). The resulting expression is

$$I(\rho, z, \Delta) \approx \ln\left[\frac{x_2 + u_2}{x_1 + u_1}\right] + \frac{(\rho a)}{(\rho + a)^2}\left\{1 + \frac{3}{2}\frac{(\rho a)}{(\rho + a)^2}\right\}\left[\frac{x_2}{u_2} - \frac{x_1}{u_1}\right] + \frac{3}{4}\frac{(\rho a)^2}{(\rho + a)^4}\left[\frac{x_2}{u_2^3} - \frac{x_1}{u_1^3}\right]. \tag{38}$$

When $\rho = a$, (34) reduces to

$$I(a, z, \Delta) = \ln\left[\frac{x_2 + u_2}{x_1 + u_1}\right] + \sum_{n=1}^{\infty} \binom{-1/2}{n}^2 F_n(a, z, \Delta), |z| > \Delta/2 \tag{39}$$

where $x_1 = \xi_1/2a$ and $x_2 = \xi_2/2a$. The corresponding expression for (38) when $\rho = a$ is given by

$$I(a, z, \Delta) \approx \ln\left[\frac{x_2 + u_2}{x_1 + u_1}\right] + \frac{11}{32}\left[\frac{x_2}{u_2} - \frac{x_1}{u_1}\right] + \frac{3}{64}\left[\frac{x_2}{u_2^3} - \frac{x_1}{u_1^3}\right] \tag{40}$$

which is primarily useful when $\Delta/a \geq 4$ and $z\Delta \geq 1$.

III. Results

At the point $z = 0$, four methods were used to evaluate the uniform current vector potential integral of the isolated singularity associated with the cylindrical wire kernel. Equation (18) was computed using a three point Gaussian quadrature numerical integration technique, while (26), (27), and (29) give the intermediate approximation, the thin wire approximation and the three term approximation of the power series expansion derived by Butler [6], respectively. Plots of the relative percent error for the various methods versus the segment length-to-radius ratio, Δ/a, are shown in Fig. 1. As a basis of comparison, (18) was numerically integrated to a sufficiently high degree of accuracy. Clearly, the intermediate approximation and the three term Butler series have lower percent errors than the thin wire approximation across the entire range of Δ/a. This becomes significant as Δ/a approaches 4 (thicker wires) where the error associated with the thin wire approximation exceeds 1%. The three term Butler series proves to be extremely accurate and has the lowest error of the four methods for very thin wires ($\Delta/a \gtrsim 30$). However, the three point Gaussian quadrature and the intermediate approximation also give acceptable errors in this range. For thicker wires, the three point Gaussian quadrature is superior and, because no assumptions were made in modifying the integral to the form shown in (18), Δ/a can be extended below the ratio of 4 and still achieve very accurate results.

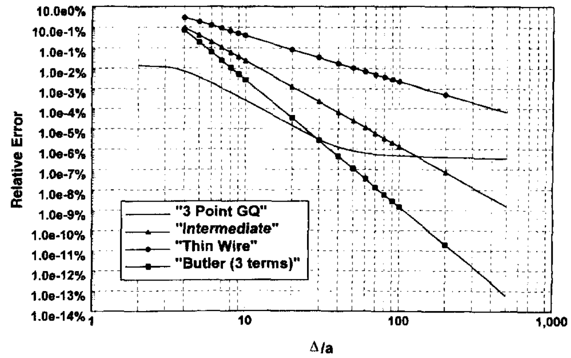

Fig. 1. Relative percent error versus Δ/a associated with various techniques for evaluating the integral (6) when $\rho = a$ and $z = 0$.

When z is not in the immediate vicinity of the singularity, the Butler series expansion is no longer valid, but can be replaced by the three term approximation of the exact series representation defined in (40). Also, a three point Gaussian quadrature numerical integration of (17) is valid as well as the intermediate and thin wire approximations of (24) and (25), respectively. Contour plots of the relative percent error as a function of Δ/a and z/Δ for the various methods of computing the integral are shown in Figs. 2–5. For all cases, the percent error decreases as z/Δ or Δ/a increases. The contour plot for the thin wire approximation (Fig. 2) depicts the highest errors across the entire range and exceeds 1% when z/Δ is less than 2 and Δ/a approaches 4. As shown in Fig. 3, the intermediate approximation is much more accurate and the error remains below 1% for most of the smaller Δ/a and z/Δ values. The contour plot of the three term approximation of the exact series representation is shown in Fig. 4. The percent error decreases rapidly as z/Δ or Δ/a increases and the error never exceeds 1% when $\Delta/a \geq 4$. Also, note that the error associated with the three term approximation decreases more rapidly for thinner wires and larger z/Δ than the errors associated with the other approximations. For the three point Gaussian quadrature, the range of Δ/a is extended to 2 in order to illustrate the validity of this method for thicker wires. Fig. 5 shows that very good accuracy is achieved for a wide range of z/Δ and Δ/a when using a simple numerical integration technique.

IV. Conclusion

Accurate evaluation of integrals in the form represented by (6) are important in the computation of cylindrical antenna integral equations. However, direct numerical integration of (6) is complicated by the singular nature of the integrand when $\rho = a$, $z = z'$ and $\phi' = 0$. This paper offers some new accurate as well as efficient methods for treating this fundamental integral. The intermediate approximation was shown to be superior to the classical thin wire formulation, achieving a high degree of accuracy while maintaining computational simplicity. This approximation was found to be primarily useful when $\Delta/a \geq 4$. For these reasons, it is recommended that the new intermediate approximation be used in place of the classical thin-wire approximation. The three term Butler series approximation

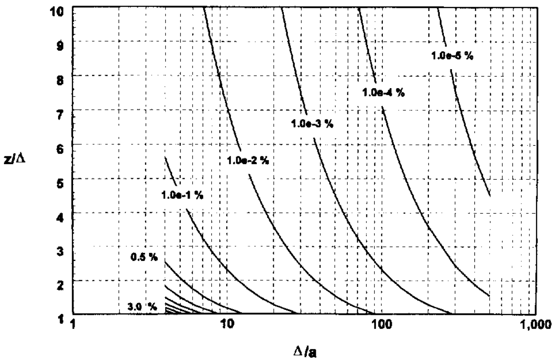

Fig. 2. Contour plot of the relative percent error associated with the thin-wire approximation (25).

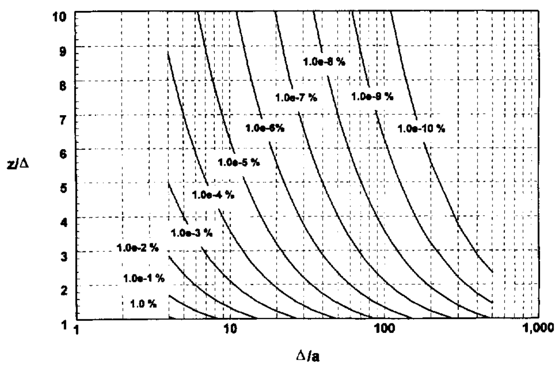

Fig. 3. Contour plot of the relative percent error associated with the intermediate approximation (24).

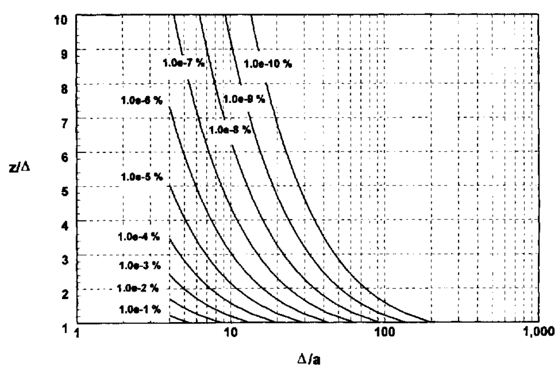

Fig. 4. Contour plot of the relative percent error associated with the three-term approximation (40).

also provides very accurate results for the special case when $z = 0$. A very useful form of the integral was presented for which the singularity in the integrand was removed by interchanging the order of integration and evaluating the inner integral. This approach resulted in a relatively smooth integrand which is well-suited for efficient numerical integration as well as avoids the need to evaluate elliptic integrals. Since no assumptions were made in transforming the integral, a significant advantage is that there are no restrictions placed on the ranges of Δ/a and z/Δ for which it is valid. A new exact

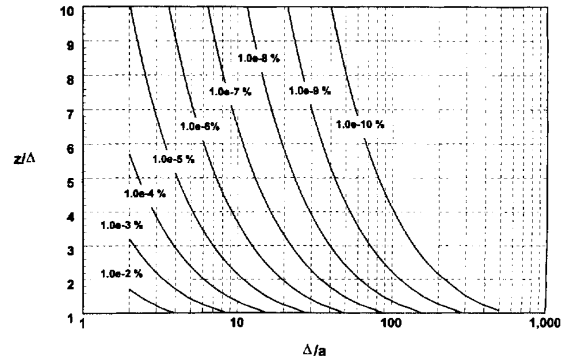

Fig. 5. Contour plot of the relative percent error associated with the three-point Gaussian quadrature approximation of (17).

series solution of the integral was presented which is valid provided $\rho \neq a$ or when $\rho = a$ and $|z| > \Delta/2$. The series converges rapidly and is computationally efficient due to a recurrence relation for computing the higher order terms. This series solution in conjunction with the Butler series solution, which is valid in the vicinity of the singularity, provides a complete exact solution when $\Delta/a > 4$ for integrals of the form given by (6). The three term approximation of the series solution offers another computationally efficient and accurate alternative to numerical integration when $|z| > \Delta/2$ and $\Delta/a \geq 4$.

Acknowledgment

The authors are grateful for the encouragement and support provided by James K. Breakall of the Pennsylvania State University and by Richard W. Adler of the Naval Postgraduate School.

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