TABLE V
PHASE DELAY VERSUS DIRECTOR LENGTH

kh	Φ
4.28	2.16
4.30	2.18
4.32	2.20
4.34	2.23
4.36	2.26
4.38	2.30
4.40	2.36
4.42	2.44
4.44	2.60
4.46	2.94

Note: $a/h = 0.01$, $b/h = 0.5$, second passband.

TABLE VI
YAGI ARRAY DESIGN PARAMETERS

N	kh	D(db)	Band-width (%)	Array Size (λ)
6	4.34	5.8	2.7	2.07
8	4.35	7.4	3.2	2.77
10	4.35	8.9	3.2	3.46
12	4.35	10.1	3.2	4.15
14	4.35	10.1	3.2	4.85
16	4.35	12.0	3.2	5.55
18	4.35	12.8	3.2	6.23
20	4.35	13.6	3.2	6.91
24	4.34	14.8	3.0	8.30
28	4.34	15.8	3.0	9.65
32	4.34	16.6	3.0	11.05
36	4.34	17.3	2.5	12.40
40	4.33	17.9	2.3	13.80

Note: $b/h = 0.5$, second passband.

Unfortunately, the maximum directivity does not usually coincide with the central frequency of the passband. For example, for the Yagi array of $N = 6$, $a/h = 0.01$, and $b/h = 0.5$, the maximum directivity occurs at $kh = 1.46$ with $D = 9.3$ dB. The center of the passband, as seen from Table II, is at $kh = 1.35$ with $D = 7.6$ dB. Should the array be operated at the frequency associated with the maximum directivity, the frequency bandwidth would be only 4.1 percent since $kh = 1.46$ is very close to the cutoff frequency of $kh = 1.49$. Thus it should be noted that in Tables II–IV the kh values correspond to that of the central frequency, and the corresponding directivity is not necessarily equal to the maximum value. This is why for shorter arrays the theoretical directivity shown in Fig. 2 is lower than that of the measured value. For longer arrays the bandwidth is narrower; therefore, the discrepancy gradually disappears.

In Table V the kh versus Φ values are listed for the second passband. The data are used to calculate the directivity and bandwidth for Yagi arrays operated in this passband. The result is listed in Table VI. The information contained in Tables VI and II is useful in the design of a Yagi array to be operated at two frequency bands. An example is given in the next section to illustrate the design procedure.

III. DESIGN METHOD—EXAMPLES

Example 1

Design a Yagi array which is to be operated at 200 ± 10 MHz and is limited to 3 m in total length. Determine the array parameters which would give maximum directivity with minimum number of elements.

Solution: It is given that the array size is limited to 2λ, and the bandwidth is 10 percent. From Fig. 2, it is seen that the parameter $b/h = 1.0$ can be chosen. The corresponding directivity is roughly equal to 12 dB. From Table III, by interpolation, N is found to be equal to 9, $kh = 1.36$, $D = 11.9$ dB, and bandwidth = 10.3 percent. In term of physical lengths, $a = 0.0032$ m, $2h = 0.65$ m, $b = 0.324$ m, and array length = 2.92 m. The array will have one reflector, one feeder, and eight directors.

Example 2

Design a Yagi array subject to the same conditions as in the previous example, but it is to be operated also at another frequency about three times higher than the fundamental frequency.

Solution: Since the second passband occurs roughly at $kh = 4.49$ and kb must be less than 3.14, the separation of the element must be made smaller. Therefore, $b/h = 0.5$ is chosen. From Table II, $N = 19$, $kh = 1.28$, $D = 12.2$ dB, and bandwidth = 13.7 percent. From Table VI, for the same N, $kh = 4.35$, $D = 13.2$ dB, and bandwidth = 3.2 percent. In terms of physical lengths, $a = 0.0031$ m, $2h = 0.61$ m, $b = 0.153$ m, and array length = 2.90 m. The array will have one reflector, one feeder, and 18 directors. When operated at 200 MHz, the frequency bandwidth is ±14 MHz with $D = 12.2$ dB; when operated at 680 MHz, $D = 13.2$ dB and the bandwidth is equal to ±11 MHz.

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Formulation of Echelon Dipole Mutual Impedance for Computer

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***Abstract*—A concise formula suitable for computer calculations is given for mutual impedance of two dipoles with sinusoidal current in echelon. It is a rearrangement of that of King [2].**

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In the moment method for determining current distributions, piecewise sinusoidal segments are often used. For this situation the mutual impedance is needed between two dipoles, each with a sinusoidal current distribution with maximum in the center. This zeroth-order impedance is of course also directly useful in calculating mutual impedance between dipoles in an array provided they are thin and with length not near a multiple of a wavelength.

Carter [1] computed mutual impedance between half-wave dipoles in echelon, and King [2] extended this to two antennas of arbitrary and not necessarily equal lengths in 1957. King's result contains 24 pairs of different cosine and sine integrals (Ci and Si) and is tedious to implement on a computer. For the moment method and for array use, dipole lengths are usually equal, in which case King's formula reduces to 10 pairs of Ci/Si. This note provides a concise form which is easily programmed, for either a batch computer or for a remote terminal computer. It is somewhat easier to integrate from the beginning than to rearrange and simplify King's formula. Using the rigorous electric-field formulation of Schelkunoff and Friis [3], and the geometry of Fig. 1, the mutual impedance can be written as

$$Z = \frac{-j30}{\sin^2 kd} \int_0^d [\psi_1 - 2\cos kd\, \psi_2 + \psi_3 + \psi_4 - 2\cos kd\, \psi_s + \psi_6] \cdot \sin k(d-x)\, dx$$

where $k = 2\pi/\lambda$, $\psi = (\exp(-jkR)/R)$, and

$$R_1^2 = y_0^2 + (x_0 + d - x)^2$$
$$R_2^2 = y_0^2 + (x_0 - x)^2$$
$$R_3^2 = y_0^2 + (x_0 - d - x)^2$$
$$R_4^2 = y_0^2 + (x_0 - d + x)^2$$
$$R_5^2 = y_0^2 + (x_0 + x)^2$$
$$R_6^2 = y_0^2 + (x_0 + d + x)^2.$$

In condensed form this impedance becomes:

$$Z = \frac{-j30}{\sin^2 kd} \sum_{m=-1}^{1} \sum_{n=-1}^{1,2} C_m \int_0^d \psi_{mn} \sin k(d-x)\, dx$$

with

$$R_{mn}^2 = y_0^2 + (x_0 + md + nx)^2$$

and $C_{-1} = C_1 = 1$, $C_0 = -2\cos kd$, and the n sum indexes by 2. After substitutions and rearrangements which allow use of the Ci and Si, the result is:

$$Z = \frac{-15}{\sin^2 kd} \sum_{m=-1}^{1} \sum_{n=-1}^{1,2} \sum_{p=-1}^{1,2} \sum_{q=0}^{1} C_m D_q \exp(-jpku) E(k\alpha).$$

Here

$$u = n[x_0 + (m+n)d], \qquad D_0 = -1, \qquad D_1 = 1$$

$$\alpha = \{y_0^2 + [x_0 + (m+nq)d]^2\}^{1/2} - pn[x_0 + (m+nq)d]$$

and the exponential integral is $E(x) = \mathrm{Ci}(x) - j\,\mathrm{Si}(x)$. This form, however, has 24 Ci/Si terms of which many are redundant. When the coefficients of like Ci/Si are combined, the 4 sums have reduced to 2 sums, with 10 Ci/Si terms:

$$Z = \frac{15}{\sin^2 kd} \sum_{m=-2}^{2} \sum_{n=-1}^{1,2} A(m) \exp[-jkn(x_0 + md)] E(k\beta),$$

$$\beta = [y_0^2 + (x_0 + md)^2]^{1/2} - n(x_0 + md).$$

Here $A(1) = A(5) = 1$, $A(2) = A(4) = -4\cos kd$, $A(3) = 2(1 + 2\cos^2 kd)$ and the n sum steps by 2. The formula is quite easy to program and allows a short mutual impedance subroutine. When $y_0 = 0$, i.e., the dipoles are collinear, y_0 in the argument is replaced by radius a. For this case the $n = 1$ terms may be replaced by natural log approximations to the Ci. However, in most computers this is not necessary although there is a time saving as half of the Si terms are not calculated. When the dipoles are half-wave, the $m = 2$ and 4 terms disappear and the exponential simplifies.

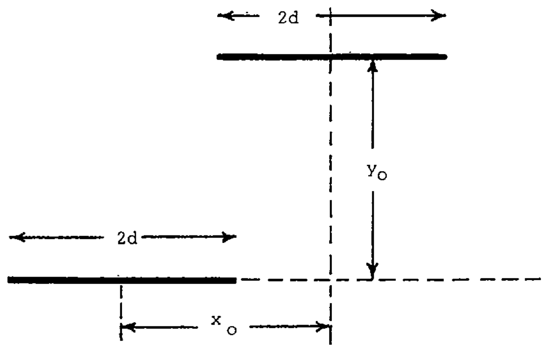

Fig. 1. Echelon dipoles.

This formulation is similar to that of Richmond [4] for two equal length thin dipoles with axes at an angle ψ. Both formulations compute quickly due to the use of economized series developed by Wimp and Luke [5]. A worst case Ci/Si typically requires 25 multiplications to yield 8 place accuracy, which makes the Ci/Si computation time comparable to that of sine/cosine. Thus the mutual impedance computation is quite fast and accurate.

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Trough Waveguide Dual-Frequency Antenna

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Abstract—**A new type of dual-frequency antenna using a hollow-fin trough waveguide is described. The hollow fin is used as a TE_{10} mode waveguide with sidewall radiating elements. These sidewall radiating elements constitute the high-frequency section of the antenna. The low-frequency section consists of a thick hollow-fin trough waveguide antenna. The attenuation due to radiation and waveguide wavelength of the trough waveguide section are analyzed. This analysis shows good agreement with experimental measurements. Existing data may be used as a guide in the design of the high-frequency section of this antenna, and the data presented may be used as a guide in the design of the low-frequency section.**

Introduction

Various forms of trough waveguide antennas have appeared in the literature over the past several years [1]–[5]. However, to the author's knowledge, the trough waveguide has not been previously used in the design of a dual-frequency antenna.

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