A Simple Solution to the Problem of the Cylindrical Antenna*

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Summary—It is recognized that the variational solution of the cylindrical antenna does not minimize $|Z_{in}-Z_0|$, as implied by Storer; but yields a minimax of the impedance function. Furthermore, the variational solution may be interpreted as simply placing two currents in parallel upon the antenna, following which, the solution for the magnitudes of the currents and for the driving point impedance may be found by elementary circuit analysis. Consequently, a first-order solution by the generalized circuit is found by postulating a linearly-attenuated traveling wave to be superimposed upon the sinusoidal standing wave. The results show excellent correlation with other theoretical and measured results for both the driving point impedance and the current distribution along the antenna.

Introduction

BROADLY SPEAKING, it may be stated that the calculus of variations in the theory of functions of real variables serves as a tool for discovering certain physical laws. The integrand of a definite integral is perturbed by adding to it a parameter times a function of the independent variable, subject to the condition that the function vanishes at both limits of the integration. The physical laws are obtained by requiring the first derivative of the perturbed integral with respect to the parameter to vanish. Perhaps the most useful laws are those for which the integral is made a minimum by the value of the parameter thus obtained.

Recently, the calculus of variations has been applied to many physical problems set up in terms of analytic functions of a complex variable. In particular, Storer[1] and Tai[2] have used it in obtaining a first-order solution to the symmetrically driven straight cylindrical antenna.

In agreement with Storer,[3] if

$$W = f(z) = U(x, y) + jV(x, y)$$

is analytic, and if W_0 is a constant, using the Cauchy-Riemann equations, it is not difficult to show that setting

$$\frac{\partial}{\partial x}\left| W - W_0 \right| = 0, \qquad \frac{\partial}{\partial y}\left| W - W_0 \right| = 0,$$

is equivalent to setting

$$\frac{dW}{dz} = 0,$$

or that the first term of a Taylor's expansion of the function vanishes.

However, by taking the second derivatives and using the Cauchy-Riemann equations in connection with the standard tests for maxima and minima of functions of two variables,[4] it can be shown that $|W-W_0|$ is neither a maximum nor a minimum when the first derivative with respect to z vanishes. That is, if Z_0 is the true input impedance of an antenna, and if $Z(\epsilon_r+j\epsilon_i)=Z(\epsilon)$ is an impedance obtained by using $I(\epsilon, x)=I(x)+\epsilon\eta(x)$ as an approximation to the true current distribution function $I(x)$, the requirement

$$\frac{dZ(\epsilon)}{d\epsilon} = 0$$

does not minimize $|Z-Z_0|$, as implied by Storer,[5] but merely yields a minimax[6] for the value $\epsilon=\epsilon_0$ found by solving the above equation.

Physical Significance of Variational Method

The variational method postulates the generalized Kirchoff's law,[7]

$$\frac{j30}{k}\int_{-l}^{l} I(x')G(x', x)dx' = Z_0 I(0)\delta(x) \tag{1}$$

in which

l = the half length of the antenna

a = the radius of the antenna

$$K = \frac{2\pi}{\lambda}$$

$$G(x', x) = \left(\frac{d^2}{dx^2} + k^2\right)\frac{e^{-jkr(x'-x)}}{r(x'-x)}$$

and $\delta(x)$ is the Dirac delta impulse function.

The driving point impedance is then chosen in the form

$$Z_0 = \frac{j30}{kI^2(0)}\int_{-l}^{l}\int_{-l}^{l} I(x)I(x')G(x', x)dx'dx. \tag{2}$$

It has been shown[8] that this form, using the approximate Green's function with $r=[(x-x')^2+a^2]^{1/2}$ yields the same result as that obtained by using the exact Green's function.

* Manuscript received by the PGAP, April 26, 1956. This paper is a revision of Tech. Rep. No. 15, U. S. Naval Postgraduate School, Monterey, Calif., January, 1956.

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[1] J. E. Storer, "Variational Solution to the Problem of the Symmetrical Antenna," Cruft Lab., Harvard Univ., Cambridge, Mass., Tech. Rept. No. 101; 1950.

[2] C. T. Tai, "A Variational Solution to the Problem of Cylindrical Antennas," Stanford Res. Inst., Stanford, Calif., Tech. Rep. No. 12; 1950.

[3] Storer, *op. cit.*, pp. 7–8.

[4] E. B. Wilson, "Advanced Calculus," Ginn and Co., New York, N. Y., pp. 114–115; 1912.

[5] Storer, *op. cit.*, p. 8.

[6] Wilson, *op. cit.*, p. 115.

[7] Tai, *op. cit.*, p. 4.

[8] C. T. Tai, "A new interpretation of the integral equation formulation of cylindrical antennas," IRE Trans., vol. AP-3; pp. 125–127, July, 1955.

The current function is perturbed by choosing

$$I(x, A) = I(x) + \epsilon\eta(x) = I_0[f(x) + A\eta(x)]. \quad (3)$$

If A_0 is a root of $dZ_0(A)/dA = 0$, the impedance is found to be[9]

$$Z_0(A_0) = 30 \frac{V_{11}V_{22} - V_{12}^2}{f^2(0)V_{22} - 2f(0)\eta(0)V_{12} + \eta^2(0)V_{11}} \quad (4)$$

with

$$V_{11} = \frac{j}{k}\int_{-l}^{l}\int_{-l}^{l} f(x)f(x')G(x', x)dx'dx$$

$$V_{12} = \frac{j}{k}\int_{-l}^{l}\int_{-l}^{l} f(x)\eta(x')G(x', x)dx'dx$$

$$V_{22} = \frac{j}{k}\int_{-l}^{l}\int_{-l}^{l} \eta(x)\eta(x')G(x', x)dx'dx. \quad (5)$$

By simple algebraic manipulation, (4) may be converted into

$$Z_{\text{in}} = \frac{Z_{11}Z_{22} - Z_{12}^2}{Z_{11} + Z_{22} - 2Z_{12}} \quad (6)$$

with

$$Z_{11} = \frac{j30}{kI^2(0)}\int_{-l}^{l}\int_{-l}^{l} I_1(x)I_1(x')G(x', x)dx'dx$$

$$Z_{12} = \frac{j30}{kI_1(0)I_2(0)}\int_{-l}^{l}\int_{-l}^{l} I_1(x)I_2(x')G(x', x)dx'dx$$

$$Z_{22} = \frac{j30}{kI_2^2(0)}\int_{-l}^{l}\int_{-l}^{l} I_2(x)I_2(x')G(x', x)dx'dx, \quad (7)$$

in which it is assumed that neither $I_1(0)$ nor $I_2(0)$ is zero.

Eq. (6) is recognized as the solution of the mesh equations,

$$\begin{aligned} Z_{11}I_1(0) + Z_{12}I_2(0) &= V_0 \\ Z_{21}I_1(0) + Z_{22}I_2(0) &= V_0. \end{aligned} \quad (8)$$

The current distribution becomes

$$I(x) = \frac{V_0}{I_1(0)}\frac{Z_{22} - Z_{12}}{Z_{11}Z_{22} - Z_{12}^2}I_1(x) + \frac{V_0}{I_2(0)}\frac{Z_{11} - Z_{12}}{Z_{11}Z_{22} - Z_{12}^2}I_2(x). \quad (9)$$

Suppose that $I_2(0) \neq 0$ with $I_1(0) = 0$. Let

$$\zeta_{11} = \lim_{x\to 0}\frac{I_1^2(x)Z_{11}}{I_{01}^2}, \qquad \zeta_{12} = \lim_{x\to 0}\frac{I_1(x)Z_{12}}{I_{01}}.$$

Then (6) and (9) become, respectively

$$Z_{\text{in}} = Z_{22} - \frac{\zeta_{12}^2}{\zeta_{11}}, \quad (10)$$

and

$$I(x) = \frac{V_0}{\zeta_{11}Z_{22} - \zeta_{12}^2}\left(\frac{\zeta_{11}}{I_2(0)}I_2(x) - \frac{\xi_{12}}{I_{01}}I_1(x)\right), \quad (11)$$

in which I_{01}^{-1} is a normalizing factor.

From (6), (8), and (9), it becomes evident that the vanishing of the first term in the Taylor's expansion of the perturbed current simply yields the physical information that two currents have been postulated to exist in parallel along the antenna. From (10) and (11), it becomes further evident that for singular cases one current becomes the *feed current* with the other parasitically excited.

This recognition greatly simplifies the variational theory of cylindrical antennas. In fact, the statement might be ventured that the variational method is not a distinct method *per se*. Any scheme for finding the self and mutual impedances of the two postulated current distributions may be used. Since $|Z_{\text{in}} - Z_0|$ is not minimized, there is no *a priori* reason for assuming that the driving point impedance computed by a method with a vanishing first term in its Taylor's expansion is any more accurate than that computed from the parallel circuit with the self and mutual impedances being computed by any recognized method.

First-Order Solution by the Generalized Circuit

For two straight currents, the mutual impedance by generalized circuit[10] scheme is given by

$$Z_{12} = \frac{j30}{k}\int_{-l}^{l}\int_{-l}^{l}\text{Re}\left[\frac{F_1(\gamma)f_2(x')^*}{f_1(0)f_2(0)^*}\right] \cdot \left[\frac{\partial^2}{\partial x^2} + k^2\right]\frac{e^{-jkr_{12}}}{r_{12}}dxdx', \quad (12)$$

in which the operator Re takes the real part of the product of one current by the complex conjugate of the other current. The current functions are normalized at the driving point. In case the current functions are different and at least one of them is not real, the reciprocity theorem requires the use of only the real part of the current product.

It is possible to show that the results obtained with the approximate Kernel are the same as those obtained when the exact Kernel is used in (12).

For a relatively low-loss transmission line, Tai[11] approximated the current distribution function by

$$I(x) = I_0[\sin\beta(l - x) - j\alpha(l - x)\cos\beta(l - x)] \quad (13)$$

from which he chose his pair of currents. The current on a lossy line also may be written

$$\begin{aligned} I(x) &= I_0[e^{\alpha x}\sin\beta(l - x) - je^{\alpha l}\sinh\alpha[(l - x)e^{j\beta(l-x)})] \\ &\approx I_0[\sin\beta(l - x) - j\alpha e^{\gamma l}(l - x)e^{-j\beta x}]. \end{aligned} \quad (14)$$

[9] Tai, Rep. No. 12, *op. cit.*, p. 11.

[10] J. G. Chaney, "A critical study of the circuit concept," *J. Appl. Phys.*, vol. 22, pp. 1429–1436; December, 1951.

[11] Tai, Rep. No. 12, *op. cit.*, p. 12.

To the writer, this arrangement intuitively seems preferable to that of (13). Hence, the current pair is chosen as

$$I_1(x) = I_{01} \sin k(l - |x|)$$
$$I_2(x) = I_{02}\alpha(l - |x|)e^{-jk|x|}. \quad (15)$$

The self impedance Z_{11} is then the well-known zero-order solution by the classical induced emf method.[12] For the others,

$$Z_{22} = \frac{j30}{k}\int_{-l}^{l}\int_{-l}^{l}\left(1 - \left|\frac{x}{l}\right|\right)\left(1 - \left|\frac{x'}{l}\right|\right) \cdot e^{jk(|x|-|x'|)}\left(\frac{\partial^2}{\partial x^2} + k^2\right)\frac{e^{-jkr_{12}}}{r_{12}}\,dx dx' \quad (16)$$

$$Z_{12} = \frac{j30}{k \sin k}\int_{-l}^{l}\int_{-l}^{l}\left(1 - \left|\frac{x}{l}\right|\right)\sin k(l - |x'|) \cdot \cos kx\left(\frac{\partial^2}{\partial x^2} + k^2\right)\frac{e^{-jkr_{12}}}{r_{12}}\,dx dx'. \quad (17)$$

It is not necessary to take the real part in (16) because it is a self impedance and the two current functions are identical. The resulting formulas are listed in the Appendix for $ka \ll 1$, $2a \ll l$.

The current distribution along the antenna then becomes

$$I(x) = \frac{V_0}{Z_{11}Z_{22} - Z_{12}^2}\left[\frac{Z_{22} - Z_{12}}{\sin kl}\sin k(l - |x|) + (Z_{11} - Z_{12})\left(1 - \left|\frac{x}{l}\right|\right)e^{-jk|x|}\right]. \quad (18)$$

Asymptotic Values

It is of interest to examine the asymptotic values of the current and of the impedance.

$$\text{For } \Omega = 2\ln\frac{2l}{a} \to \infty$$

$$Z_{11} = Z_{12} \to -j\,60\Omega \cot kl, \quad Z_{22} \to -j\frac{60}{kl}\Omega, \quad Z_{in} \to Z_{11},$$

and

$$I(x) \to \frac{V_0}{Z_{11}\sin kl}\sin k(l - |x|) = j\frac{V_0}{60\Omega}\frac{\sin k(l - |x|)}{\cos kl}.$$

For $kl \ll 1$,

$$Z_{11} = Z_{12} = Z_{22} \to 20(kl)^2 - j\frac{60}{kl}(\Omega - 2 - 2\ln 2)$$

and

$$I(x) \to \frac{V_0}{Z_{11}}\left(1 - \left|\frac{x}{l}\right|\right)$$

with $Z_{in} \approx Z_{11}$.

[12] S. A. Schelkunoff, "Electromagnetic Waves," D. Van Nostrand Co., New York, N. Y., p. 373; 1947.

For $kl \approx n\pi$, $n = 1, 2, 3, \cdots$,

$$Z_{in} = Z_{22} - \frac{\zeta_{12}^2}{\zeta_{11}}$$

and

$$I(x) = \frac{V_0}{\zeta_{11}Z_{22} - \zeta_{12}^2}\left[\zeta_{11}\left(1 - \left|\frac{x}{l}\right|\right)e^{-jk|x|} - \zeta_{12}\sin k(l - |x|)\right]. \quad (19)$$

From (19), it is seen that the attenuated traveling wave becomes the *feed current*, and that the standing wave becomes parasitically excited.

Thus asymptotically, this method is equivalent to the induced emf method with the removal of the singularities.

Results

Some values of the driving point impedance computed by this method (Tables I and II) were plotted on curves by Tai[13] (Figs. 1 and 2) for comparison with values computed by the King-Middleton, the variational, and the Schelkunoff methods. They show excellent correlation with the other methods, being very close to the results of Tai with perhaps about as many of the excursions being on the Schelkunoff side of the curves, as on the King-Middleton side of the curves.

For $\Omega = 10$ and $\Omega = 15$, the current distribution was computed for comparison with the sinusoidal current and with the current computed by King and Harrison[14]

TABLE I

kl	$\Omega=10$	$\Omega=15$
	Z_{in}	Z_{in}
$\pi/2$	$83.0+j41.8$	$77.5+j42.3$
2.2	$493 \; +304$	$360 \; +j654$
2.6	$883 \; +j94$	$1282 \; +j1101$
2.9	$633 \; -j492$	$2513 \; -j349$
π	$372 \; -j493$	$1345 \; -j1447$
$3\pi/2$	$78.3+j15.0$	$92 \; +j2$
5.1	$178 \; +j104$	$288 \; +j287$
5.6	$586 \; +j63$	$871 \; +j683$
6.0	$340 \; +j262$	$1699 \; -j130$
2π	$332 \; +j263$	$1094 \; -j904$

TABLE II

Ω	$kl=\pi/2$	$kl=\pi$
	Z_{in}	Z_{in}
10	$83.0+j41.8$	$372-j493$
12	$79.8+j42.0$	$683-837$
15	$77.5+j42.3$	$1345-1447$
22	$75.6+j42.3$	$3797-j3463$

[13] Tai, Rep. No. 12, *op. cit.*, Figs. 4–5.

[14] R. King and C. W. Harrison, Jr., "The distribution of current along a symmetrical center-driven antenna," Proc. IRE, vol. 31, pp. 548–656; October, 1943.

from a first-order solution of the Hallén integral equation (Figs. 3 and 4). In view of the known discrepancies existing in the King-Harrison method, good correlation is indicated.

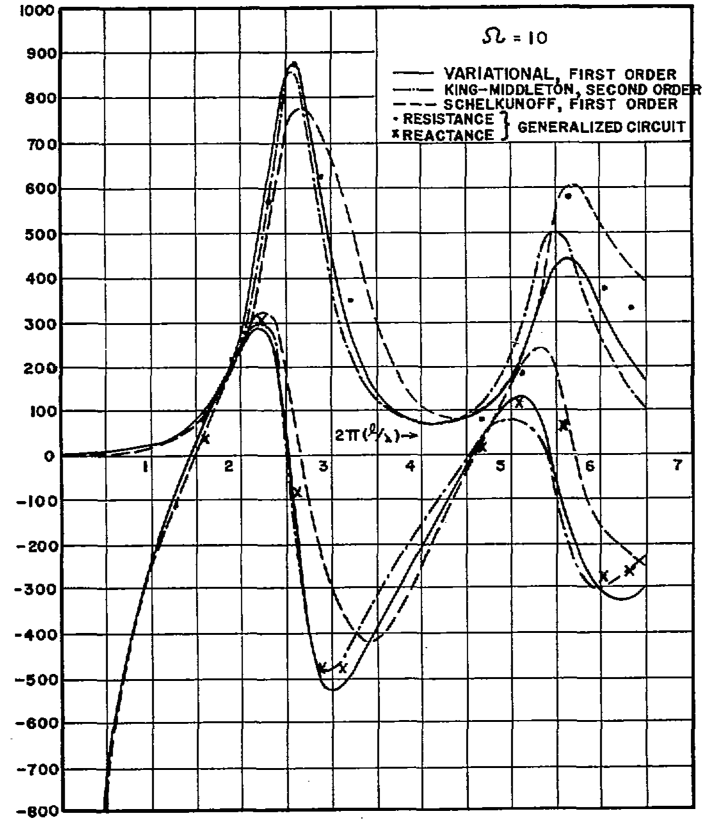

Fig. 1.

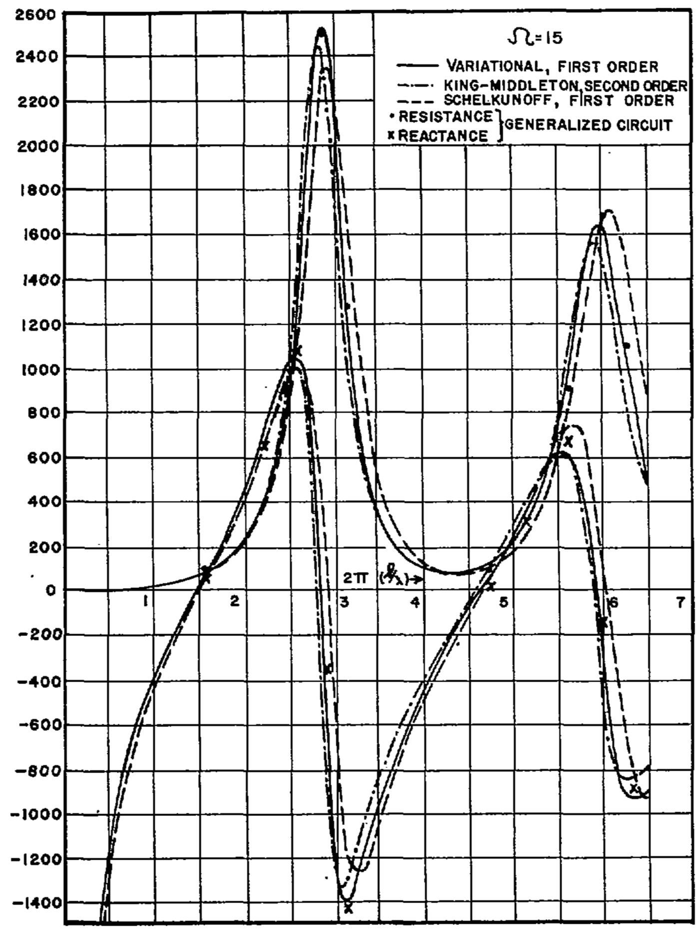

Fig. 2.

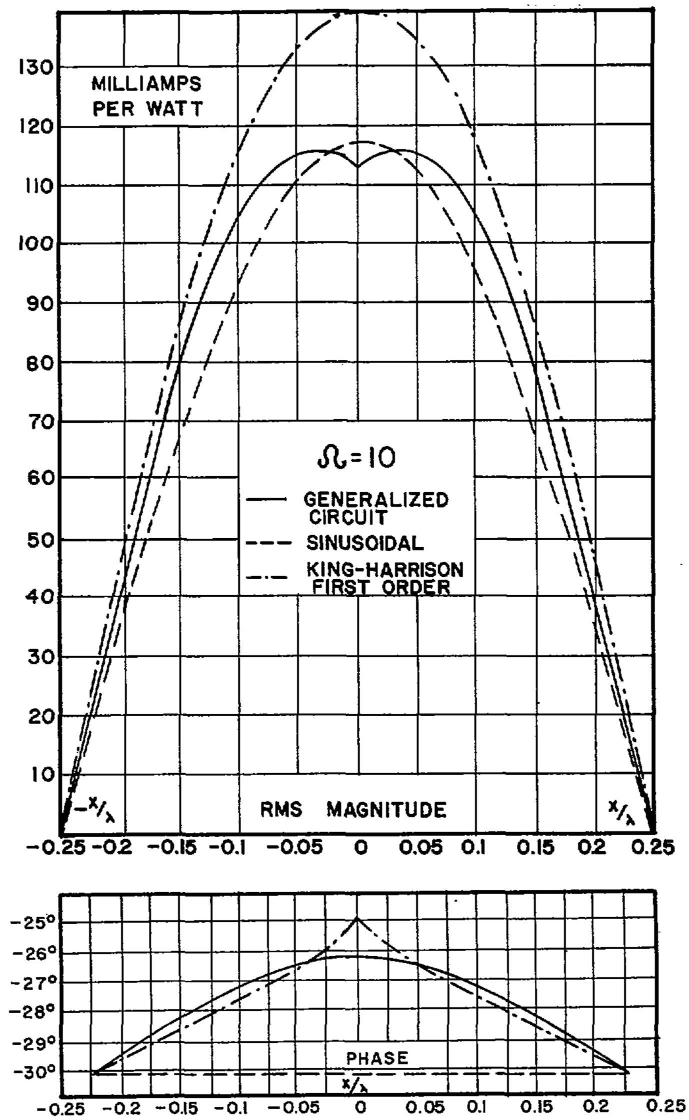

Fig. 3.

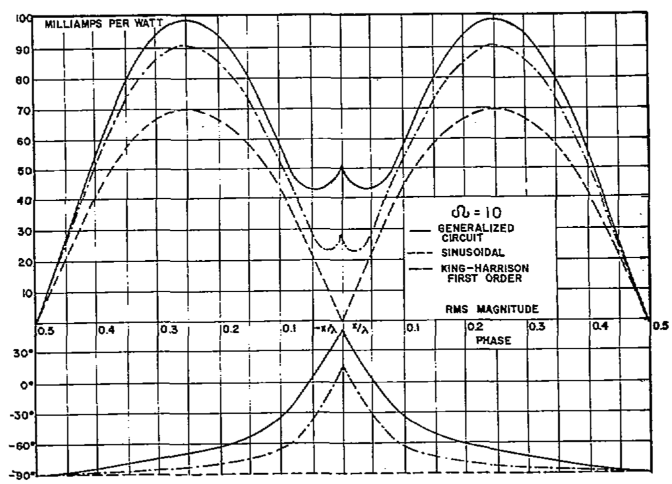

Fig. 4.

These computed values were also compared with the measured values of Barzilai[15] and of Morita.[16] Excellent correlation exists between the computed and measured results. The curves by Morita showed a comparison of the measured values with values computed by the King-Middleton first-order solution. The values by the generalized circuit first order show better correlation to the measured values than do those by the King-Middleton first-order method.

Conclusion

The comparative simplicity of the method of obtaining the first order generalized circuit solution, together with the excellent correlation with other solutions and with published measurements, should justify its introduction into the already crowded group of antenna theories.

It is believed that the simplicity of the method will greatly facilitate the teaching of the theory of the cylindrical antenna.

The method is being extended to the case of the receiving antenna.

It should be pointed out that this is not a variational method, but that the variational method as previously presented becomes a special case of this method because the current functions were real.

Appendix

$$Z_{11}\sin^2 kl = 60\,cin2kl \quad 30\sin 2kl(2si2kl - si4kl) + 30\cos 2kl(2cin2kl - cin4kl) + j\{60si2kl - 30\sin 2kl(\Omega - \ln 4 + cin4kl - 2cin2kl) + 30\cos 2kl(2si2kl - si4kl)\}$$

$$Z_{12}\sin kl = 60\cos kl(si4kl - 2si2kl) + 60\sin kl(cin4kl - cin2kl) + j\left\{60\cos kl(\ln 4 - \Omega + 1 - cin4kl + 2cin2kl) + 60\sin kl(si4kl - si2kl) + \frac{15}{kl}(3\sin kl + 2\sin 2kl - \sin 3kl)\right\}$$

$$Z_{22} = 60\,cin2kl - 30 + \frac{30}{kl}(\sin 2kl - 2si2kl) + \frac{15}{(kl)^2}[1 + 2cin2kl + \cos 2kl(2cin2kl - 2cin4kl - 1) + 2\sin 2kl(si4kl - si2kl)] + j\left\{60\,si2kl - \frac{30}{kl}(2\Omega - 2 - \cos 2kl - 2cin2kl) + \frac{15}{(kl)^2}[2si2kl + \sin 2kl(2\ln 4 + 1 + 2cin2kl - 2cin4kl) + 2\cos 2kl(si2kl - si4kl)]\right\}$$

$$six = \int_0^x \frac{\sin t}{t}\,dt, \qquad cix = \int_\infty^x \frac{\cos t}{t}\,dt,$$

$$cinx = \int_0^x \frac{1 - \cos t}{t}\,dt = \ln \gamma x - cix,\ \ln \gamma = 0.5772\cdots$$

[15] G. Barzilai, "Experimental determination of the distribution of current and charge along cylindrical antennas," Proc. IRE, vol. 37, pp. 825–829; July, 1949.

[16] T. Morita, "The Measurement of Current and Charge Distributions on Cylindrical Antennas," Cruft. Lab., Harvard Univ., Cambridge, Mass., Tech. Rep. No. 66, Figs. V-4, V-6; 1949.

Acknowledgment

The writer expresses his thanks to Capt. William J. Schreier, U.S.M.C., for lettering and inking the drawings.