$$-\frac{3\pi i}{2}\,\frac{J_1\left(k\sin\frac{\alpha}{2}\right)}{k^2\sin\frac{\alpha}{2}} + 3iM\,\frac{\sin\left(k\sin\frac{\alpha}{2}\right)}{k^2\sin\frac{\alpha}{2}}$$

$$+\frac{3\pi i}{4}\sin^3\frac{\alpha}{2}\,J_1\left(k\sin\frac{\alpha}{2}\right)$$

$$-\frac{3i}{2}M\sin^3\frac{\alpha}{2}\sin\left(k\sin\frac{\alpha}{2}\right).$$

Combine the first and fifth terms; their absolute value is approximately equal to

$$\frac{3}{4}\,\frac{\alpha}{k}\left[\cos\left(\frac{k}{2}\sin\alpha\right)-\cos\left(k\sin\frac{\alpha}{2}\right)\right] < 0.00007,$$

which we drop. The same can be done with terms six and eight, which are less than 0.00004. Dropping term nine introduces a maximum error of 0.0035 (in view of the maximum value of the Bessel function after its first minimum and in view of the maximum value of α), and term ten is less than 0.0002 so that it also can be dropped.

Terms two and three combine, with no error, to form

$$\frac{-3\cos^3\frac{\alpha}{2}}{k\sin\frac{\alpha}{2}}\,g^1\left(\frac{k}{2}\sin\alpha\right)$$

if we note that the first derivative of $\sin x/x$ is

$$g^1(x) = \frac{1}{x}\left(\cos x - \frac{\sin x}{x}\right).$$

Terms four and seven combine by the recurrence formula for the Bessel functions.

The maximum error is certainly not greater than the numerical sum of all the errors listed, viz., 0.004 ($\frac{1}{3}$ db at 20 db down).

A Note on Super-Gain Antenna Arrays*

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***Summary*—Numerical calculations have been made for linear broadside super-gain arrays. Using arrays having an over-all length of a quarter wavelength as an example, it is shown that as the required directive gain is increased, tremendous currents are required to produce only a small radiated field. For a 9-element array which produces a power gain of 8.5, the currents must be adjusted to their correct value to an accuracy of better than 1 part in 10^{11}. The efficiency is less than 10^{-14} per cent.**

THE THEORETICAL POSSIBILITY of obtaining arbitrarily high directivity from an array of given over-all length appears to have been pointed out first in 1943 by Schelkunoff[1] in connection with endfire arrays. In 1946, Bowkamp and de Bruijn,[2] discussing the problem of optimum current distribution on an antenna, concluded that there was no limit to the directivity obtainable from an antenna of given length (or broadside array). In 1947, Laemmel[3] gave source-distribution functions for small high-gain antennas. The extension of the results of Bowkamp and de Bruijn to a two-dimensional current distribution was considered by Riblet.[4] In a discussion of Dolph's[5] paper on an optimum current distribution for broadside arrays, Riblet[6] showed that if spacings less than one-half wavelength were considered, the Tchebyscheff distribution used by Dolph could be made to yield an array having as great a directivity as might be desired.

Arrays which are capable of producing arbitrarily sharp directive patterns for a given aperture or over-all length have become known as super-gain arrays. That such arrays might be possible in theory but quite impractical to build is almost to be expected. The papers mentioned above do not concern themselves with this aspect of the problem, but several other authors[7–9] have indicated some of the practical limitations. Wilmotte[8] pointed out the low radiation resistance and efficiency of such arrays, and later Chu[9] gave a completely general answer to the problem in terms of the maximum gain-Q ratio obtainable from a system of given size.

It is the purpose of this paper to carry through a typical super-gain array design in order to obtain numerical answers for some actual cases. These will serve to demonstrate the rapidity with which the design becomes impractical as the directivity is increased. Schelkunoff's method for the analysis of linear arrays is employed with the arrangement of nulls being made in accordance with the Tchebyscheff distribution as suggested by Riblet. The numerical results are rather sur-

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[1] S. A. Schelkunoff, "A mathematical theory of linear arrays," *Bell Sys. Tech. Jour.*, vol. 22 ,pp. 80–107; January, 1943.
[2] C. J. Bowkamp and N. G. de Bruijn, "The problem of optimum antenna current distribution," *Philips Res. Rep.*, vol. 1, pp. 135–158; January, 1946.
[3] A. Laemmel, "Source Distribution Functions for Small High-Gain Antennas," Microwave Research Institute, Polytechnic Institute of Brooklyn, Report R-137-47, PIB-88; April 4, 1947.
[4] H. J. Riblet, "Note on the maximum directivity of an antenna," Proc. I.R.E., vol. 36, pp. 620–624; May, 1948.
[5] C. L. Dolph "A current distribution for broadside arrays which optimizes the relationship between beam width and side-lobe level," Proc. I.R.E., vol. 34, pp. 335–348; June, 1946.
[6] H. J. Riblet, Discussion on "A current distribution for broadside arrays which optimizes the relationship between beam width and side-lobel level," Proc. I.R.E., vol. 35, pp. 489–492; May, 1947.
[7] C. E. Smith, "A critical study of two broadcast antennas," Proc. I.R.E., vol. 24, pp. 1329–1341; October, 1936.
[8] R. M. Wilmotte, "Note on practical limitations in the directivity of antennas," Proc. I.R.E., vol. 36, p. 878; July, 1948.
[9] L. J. Chu, "The physical limitations of directive radiating systems," *Jour. Appl. Phys.*, p. 1163; December, 1948.

prising and point up some interesting facts concerning such arrays.

It will be recalled that the basis of Schelkunoff's analysis is his fundamental theorem that every linear array with commensurable separation between its elements may be represented by a polynomial, and, conversely, every polynomial can be interpreted as a linear array. Furthermore, a recognition of the correspondence between the nulls of the array pattern and the roots of a complex polynomial on a unit circle in the complex plane leads to a combination analytical-graphical technique for determining antenna current amplitudes and phases and for obtaining a detailed plot of the directive field pattern.

In this analysis the relative amplitude of the electric field intensity for a linear array of n equispaced elements is represented by

$$E = \left| a_0 e^{j\alpha_0} + a_1 e^{j\psi + j\alpha_1} + a_2 e^{j2\psi + j\alpha_2} + \cdots + a_{n-2} e^{j(n-2)\psi + j\alpha_{n-2}} + e^{j(n-1)\psi} \right|, \quad (1)$$

where $\psi = \beta d \cos \phi + \alpha$ and $\beta = 2\pi/\lambda$.

In the above expression, d is the spacing between elements. The coefficients a_0, a_1, a_2, and so forth are proportional to the current amplitudes in the respective elements. The factor α is the progressive phase shift (lead) from left to right between successive elements; α_1, α_2, and so forth are the deviations from this progressive phase shift.

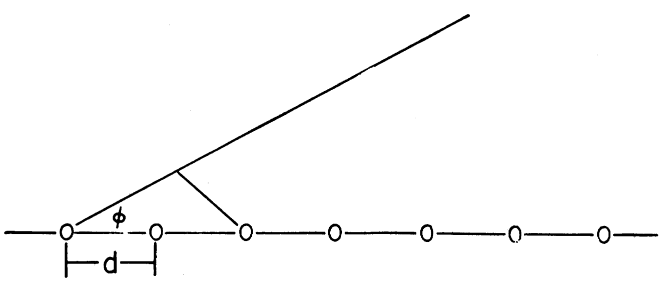

Fig. 1—A linear array.

By substituting $Z = e^{j\psi}$ and $A_m = a_m e^{j\alpha_m}$, (1) becomes

$$E = \left| A_0 + A_1 Z + A_2 Z^2 + \cdots + A_{n-2} Z^{n-2} + Z^{n-1} \right|. \quad (2)$$

The transform $Z = e^{j\psi}$ is a complex quantity whose modulus is one and whose argument ψ is real and a function of ϕ (the angle between the line of arrival of the signal and the line of the array). Hence, the plot of Z in the complex plane is always on the circumference of a unit circle. Each multiplication by $z = e^{j\psi}$ represents a displacement through an arc of ψ radians.

The polynomial (2) may be written also in the form of the product of $(n-1)$ binomials; thus,

$$E = \left| (Z - t_1)(Z - t_2) \cdots (Z - t_{n-1}) \right|, \quad (3)$$

where $t_1, t_2, \cdots, t_{n-1}$ are the zeros of the polynomial and correspond to the null points of the array. Accordingly, by (3), the relative amplitude of the radiated field intensity in any direction is given by the product of the distances from the null points on the unit circle to the point Z which corresponds to the chosen direction.

As a consequence of this correspondence between the nulls of an array and the roots of a complex polynomial, it is possible to obtain a relative field-strength plot and antenna-current ratios for a specified radiator spacing, and an arbitrary location of the nulls on the unit circle. The graphical-mathematical method involves the expansion of $(n-1)$ binomials.

In the expression $\psi = \beta d \cos \phi + \alpha$, ψ varies from $(\beta d + \alpha)$ to $(-\beta d + \alpha)$ as ϕ ranges from 0 to 180 degrees, and the range described by ψ is $(\beta d + \alpha) - (-\beta d + \alpha) = 2\beta d$. It is apparent that the current ratios and array pattern will vary with different location of the nulls in the range of ψ. For spacings less than a half wavelength, Schelkunoff has shown that an end-fire array with its null points equispaced in the range of ψ on the unit circle, has a narrower principle lobe and smaller secondary lobes than a uniform array (one whose null points are equispaced over the entire unit circle). If the current distribution of the array is always made to be that which corresponds to equispaced nulls in the range of ψ, it appears possible to increase indefinitely the directivity of an end-fire array of given length by increasing the number of elements and decreasing the spacing between them.

When this same technique of equispacing the nulls in the range of ψ is applied to a broadside array of given length, the pattern does not improve as the number of elements is increased, but, instead, deteriorates for spacings less than $\lambda/2$. However, if the nulls on the unit circle are spaced in the range of ψ according to a "Tchebyscheff distribution," a super-gain pattern results when a large number of elements at small spacings is used.

The properties of the Tchebyscheff polynomials[10] were used by Dolph[5] to obtain the optimum pattern of a broadside array for which the spacing between elements is equal to or greater than $\lambda/2$. The Tchebyscheff polynomials are defined by

$$T_n(Z) = \cos \left[n \cos^{-1} Z \right] \quad \text{for} \quad -1 < Z < +1$$

$$T_n(Z) = \cosh \left[n \cosh^{-1} Z \right] \quad \text{for} \quad |Z| > 1.$$

Graphically all the roots of $T_n(Z)$ occur between $Z = \pm 1$ and the maximum and minimum values of $T_n(Z)$ lying between $Z = \pm 1$ are alternately $T_n(Z) = \pm 1$ (see Fig. 2). For $|Z| > 1$, it can be shown that $|T_n(Z)|$ increases as $|Z|^n$. Derivations from the definition of the Tchebyscheff polynomials show that $T_0(x) = 1$; $T_1(x) = x$; $T_2(x) = 2x^2 - 1$, and so forth. Higher-order polynomials may be derived from the following equation:

$$T_{m+1}(x) = 2T_m(x)T_1(x) - T_{m-1}(x).$$

It is evident from an inspection of Fig. 2 that a pat-

[10] Courant and Hilbert, "Methoden der Mathe matischen Physik," Julius Springer, Berlin, Germany, vol. 1, p. 75; 1931.

tern may be obtained whose side lobes are all down an equal amount from the main lobe and that the ratio of the main lobe to the subordinate lobes may be determined by using the proper portion of the Tchebyscheff function.

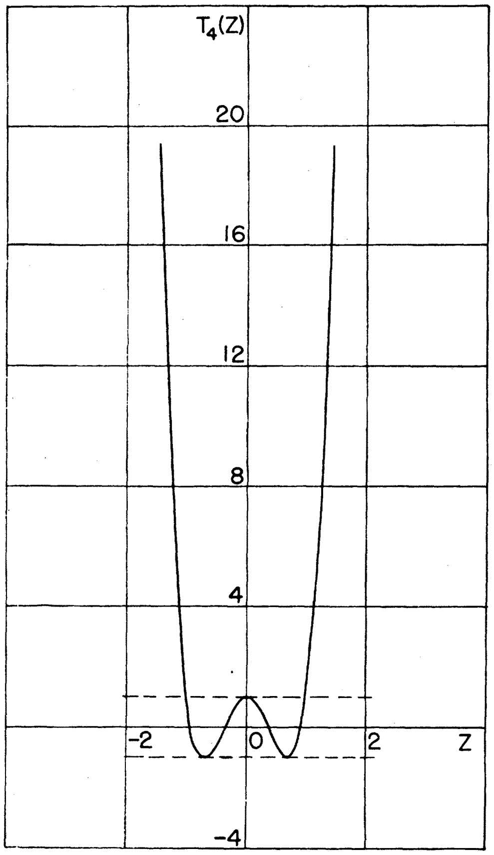

Fig. 2—Fourth-degree Tchebyscheff polynomial.

For $2n+1$ elements, symmetrical in spacing and current about the center element, the expression for the

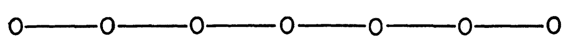

Fig. 3—Symmetrical linear array.

relative field strength can be found directly from (2). It is

$$|E| = A_0 + A_1 \cos\psi + A_2 \cos 2\psi + \cdots + A_n \cos n\psi, \quad (4)$$

with $\psi = \beta d \cos\phi$ since $\alpha = 0$ for broadside arrays. Substituting $x = \cos\psi$ in the above expression yields the polynomial

$$|E| = A_0 + A_1 x + A_2(2x^2 - 1) + \cdots + A_n T_n(x). \quad (5)$$

Expression (5) is a summation of polynomials, and it appears feasible, therefore, to design an array pattern exhibiting the useful properties of a Tchebyscheff plot. A linear shifting of the desired portion of the Tchebyscheff polynomial of correct degree into the range defined by $x = \cos\psi$ for the given array, and the equating of coefficients of (5) to those of the transformed Tchebyscheff function, permits the determination of the relative current distribution (A_1, A_2, A_3, A_n). Knowing the positions of the nulls, the Tchebyscheff pattern is obtained from (3) by the method previously indicated.

As an example, consider a nine-element broadside array whose total length is $\lambda/4$ (spacing between elements $d = \lambda/32$). The side-lobe level is to be 1/19.5 of the main lobe. Since there are nine elements, the array should have 8 nulls on the unit circle within the range of ψ, and 8 nulls in the range of the Tchebyscheff pattern which is used. It is possible to select the polynomial $T_4(x)$ (see Fig. 2) and use the whole range from $x = -1$ to x_0 (point at which $T_4(x) = 19.5$) and back to -1. With $d = \lambda . 32$, $\psi = \beta d \cos\phi = \pi/16 \cos\phi$; $x = \cos\psi = \cos(\pi/16 \cos\phi)$.

When

$$\phi = 0°, \quad x = \cos\pi/16$$
$$\phi = 90°, \quad x = 1$$
$$\phi = 180°, \quad x = \cos\pi/16.$$

Hence, the desired portion ($x = -1$ to x_0) of $T_4(x)$ must be shifted into the array range of $\cos\pi/16$ to 1. The point x_0 may be calculated from

$$T_4(x_0) = 8x_0^4 - 8x_0^2 + 1 = 19.5$$
$$x_0 \cong 1.449.$$

The shift of $T_4(x)$ is accomplished by the linear transform $x' = ax + b$.

Therefore, (5) for a nine-element array becomes

$$\begin{aligned} |E_R| &= A_0 + A_1 x + A_2(2x^2 - 1) + A_3(4x^3 - 3x) \\ &\quad + A_4(8x^4 - 8x^2 + 1) \\ &= 8(x')^4 - 8(x')^2 + 1 \\ &= 8(ax + b)^4 - 8(ax + b)^2 + 1. \end{aligned} \quad (6)$$

At this point the question arises as to what accuracy is required in the computations in order to achieve a reasonably accurate result. By using the general error formula,[11] it is shown below that the error in coefficients a and b must be less than 10^{-9} in order that current distribution A_1, A_2, A_3, A_4 be accurate to one decimal place.

In (6), $8(x')^4 = 8a^4x^4 + 32a^3x^3b + 48a^2x^2b^2 + 32axb^3 + 8b^4$ and $8(x')^2 = 8a^2x^2 + 16axb + 8b^2$. An error δa in a would have its greatest effect in the $8a^4x^4$ term ($a > 1$). Neglecting high-order infinitesimals

$$\delta N = \frac{\partial N}{\partial u_1}\delta u_1 + \frac{\partial N}{\partial u_2}\delta u_2 + \cdots + \frac{\partial N}{\partial u_n}\delta u_n,$$

where N denotes a function of several independent variables (u_1, u_2, u_n). Using the above expression, where $N = 8a^4$, for a variation in a, δN is $32a^3 \cdot \delta a$. For the example chosen, the approximate values of a and b determined from substitution in $x' = ax + b$ are 126.9 and 125.5, respectively. Hence, if the figures A_1, A_2, A_3, and

[11] J. B. Scarborough, "Numerical Mathematical Analysis," Johns Hopkins Press, Baltimore, Md., p. 7; 1946.

A_4 are to be accurate to 1 decimal place, $32a^3 \cdot \delta a < 0.1$ and

$$\delta a < \frac{0.1}{(32)(126.9)^3} \cong 10^{-9}.$$

Consequently, the first requirement is to determine a and b to this accuracy; $x = \cos \pi/16$ must therefore be found to this same degree of exactness. After carrying out the computation of (6) to this degree of accuracy, the current distribution (A_1, A_2, A_3, and A_4) may be considered to be accurate to one decimal place. Any numerical calculations not maintaining this degree of accuracy are worthless, as may be seen from the following calculations for the example. For this particular array of nine elements, the current ratios are as follows:

$$A_0 = 8{,}893{,}659{,}368.7,$$
$$A_1 = -14{,}253{,}059{,}703.2,$$
$$A_2 = 7{,}161{,}483{,}126.6,$$
$$A_3 = -2{,}062{,}922{,}999.4,$$
$$A_4 = 260{,}840{,}226.8,$$

and (4) becomes

$$\begin{aligned} E_R = {} & 8{,}893{,}659{,}368.7 \\ & - 14{,}253{,}059{,}703.2 \cos\left(\frac{\pi}{16}\cos\phi\right) \\ & + 7{,}161{,}483{,}126.6 \cos\left(\frac{\pi}{8}\cos\phi\right) \\ & - 2{,}062{,}922{,}999.4 \cos\left(\frac{3\pi}{16}\cos\phi\right) \\ & + 260{,}840{,}226.8 \cos\left(\frac{\pi}{4}\cos\phi\right). \end{aligned} \tag{7}$$

The antenna pattern (portion of $T_4(x)$) is available from direct substitution of ϕ in the above expression. The accuracy requirements in the computations can now be demonstrated. Broadside to the array (direction of maximum radiation) where $\phi = 90$ degrees, (7) is evaluated as

$$\begin{aligned} E_R = {} & 8{,}893{,}659{,}368.7 - 14{,}253{,}059{,}703.2 \\ & + 7{,}161{,}483{,}126.6 - 2{,}062{,}922{,}999.4 \\ & + 260{,}840{,}226.8. \end{aligned}$$

The sum of the positive terms is 16,315,982,722.1 and the sum of the negative terms is $-16{,}315{,}982{,}702.6$. The resultant $|E_R|$ is 19.5, the value specified for the ratio of the major lobe to the side lobes. End-fire to the array $\phi = 0$ or 180 degrees, and (7) takes the form

$$\begin{aligned} |E_R| = {} & 8{,}893{,}659{,}368.7 - 14{,}253{,}059{,}703.2 \cos\frac{\pi}{16} \\ & + 7{,}161{,}483{,}126.6 \cos\frac{\pi}{8} \\ & - 2{,}062{,}922{,}999.4 \cos\frac{3\pi}{16} \\ & + 260{,}840{,}226.8 \cos\frac{\pi}{4} \cdot \end{aligned}$$

The numerical values for the trigonometric functions must be accurate to the same number of significant figures as are the coefficients. The resultant $|E_R|$ for the end-fire direction is 1.0. Calculations having an accuracy that would normally be considered adequate—four or five significant figures—cannot be used to obtain the correct current distribution and array pattern from (7). The resultant $|E_R|$ in (7) is the difference between large numbers, nearly equal, such that significant figures (to the left) are lost. In this example, 12-figure accuracy was necessary at the start in order to end up with only 2 or 3 figures.

It is interesting to note that the pattern obtained by the graphical method from the location of the nulls on the unit circle does not require this high degree of accuracy. Solution of (6) maintaining accuracy to 4 significant figures, and the determination of the nulls from the roots of (6) affords sufficient information to plot a fairly accurate pattern. This is because this method involves the product of the lengths from the $(n-1)$ null points on the unit circle to the point Z corresponding to a chosen direction. The final result is as exact in significant figures as are contained in the least accurate factor. In Fig. 4 there is shown the pattern for the nine-element array of this example as calculated (graphically) from (3), using 4 figures, or alternatively from (7) using 12 figures.

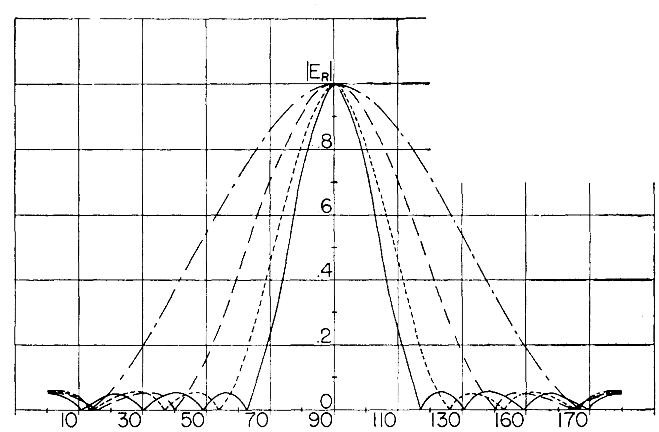

Fig. 4—Broadside "super-gain" patterns for three-, five-, seven-, and nine-element arrays with an over-all array length of quarter wavelength.

— - —	Three elements, $d = \lambda/8$
—— ——	Five elements, $d = \lambda/16$
- - - - - - -	Seven elements, $d = \lambda/24$
————	Nine elements, $d = \lambda/32$

Examination of the numerical results for the 9-element example indicates some of the practical shortcomings of super-gain arrays. In this example, currents of the order of 14 million amperes are required in the individual elements in order to produce in the direction of maximum radiation a field intensity equivalent to that which would be produced by 19.5 milliamperes flowing in a single element. Moreover, it would be necessary to maintain these currents, as well as the spacing between antenna elements, to an accuracy of about 1 part in 10^{11} if the super-gain pattern is to be obtained.

In order to illustrate the rapidity with which the design becomes impractical, the curves of Figures 4, 5, 6, and 7 have been drawn. Fig. 4 shows the directivities obtainable from an array that has an over-all length of one-quarter wavelength when the number of elements used is 3, 5, 7, and 9, respectively. In Fig. 5 these directivities are expressed in terms of directive gain over a single element. As the number of elements is increased

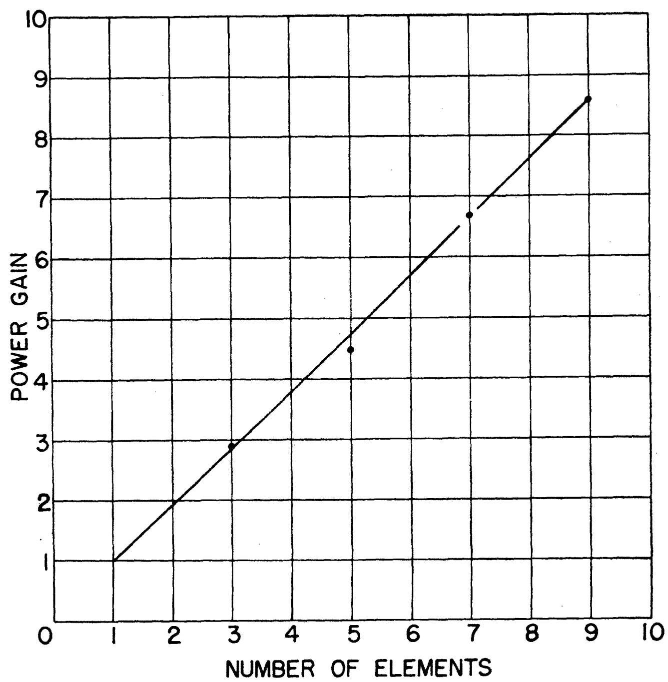

Fig. 5—Power gain of Tchebyscheff arrays over a single element. The array length is λ/4 in each case.

and the spacing between elements is correspondingly decreased, the required currents become very large for any appreciable radiation. The accuracy with which these currents must be adjusted in order to obtain the calculated super-gain patterns within 0.5 per cent is shown in Fig. 6.

The exceedingly large currents required cause large ohmic losses with resultant low efficiencies. In general, most of these losses will occur in the coupling and matching networks (and in the ground system if monopole antennas erected on a finitely conducting earth are being considered). However, for the efficiency calculations, the results of which are shown in Fig. 7, only the ohmic losses in the antenna elements themselves have been considered. For the purpose of illustration, the antennas have been assumed to be half-wave dipoles at 10 mc, constructed of copper with a diameter of 1 cm. The resultant efficiencies under the assumptions are shown in Fig. 7. When matching network losses are considered, the actual efficiencies would be much lower.

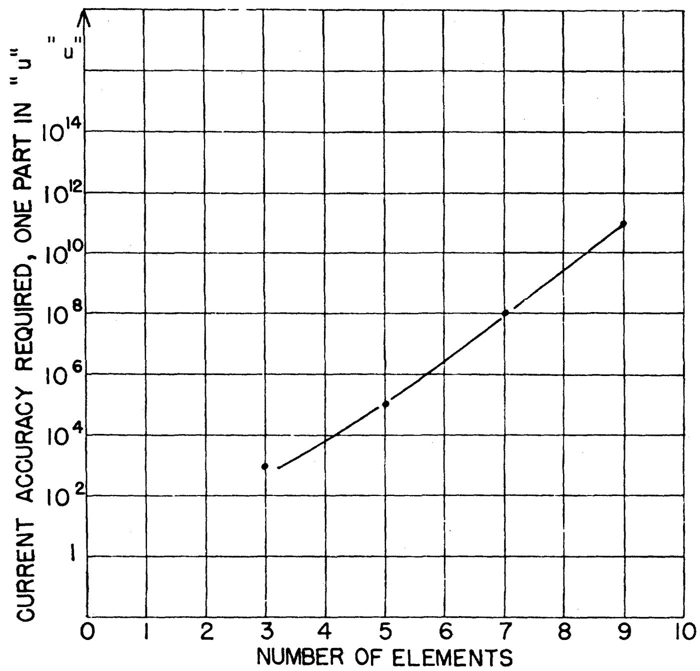

Fig. 6—Current accuracy required to obtain Tchebyscheff super-gain patterns correct to 0.5 per cent.

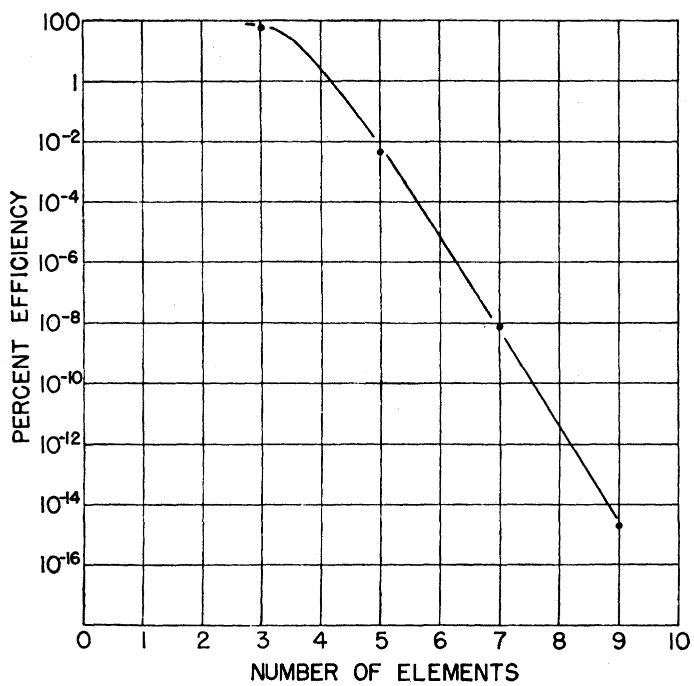

Fig. 7—Per cent efficiency of Tchebyscheff arrays versus number of elements in a quarter-wavelength array.

Acknowledgments

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