1 Problem Set

1. **Exercise 1** - Let $\Lambda \subset \mathbb{R}^{2n} \simeq \mathbb{C}^n$ be a lattice.

   (a) Show that the standard Hermitian metric on $\mathbb{C}^n$ induces a Kähler metric on the n-dimensional complex manifold $T = \mathbb{C}^n/\Lambda$.

   (b) Explicity exhibit harmonic (p,q)-forms on T.

2. Let Γ be a discrete group acting holomorphically and properly discontinuously on the complex manifold M. Assume furthermore that the action is free so that the orbit space $\Gamma \setminus M$ is a complex manifold.

   (a) Consider the special case where $\Gamma \simeq \mathbb{Z}$ and $M = \mathbb{C}^2 \setminus \{0\}$ and $1 \in \mathbb{Z} = \Gamma$ acts as multiplication by 2 (and so -1 as multiplication by $\frac{1}{2}$). To what familiar 4-dimensional real manifold is the complex manifold $\Gamma \setminus M$ homeomorphic?

   (b) Does $\Gamma \setminus M$ admit of a Kähler metric?

   (c) Generalize to the case where $M = \mathbb{C}^n \setminus \{0\}$.

3. On $\mathbb{CP}(n)$ with homogeneous coordinates $[z_0, z_1, \ldots, z_n]$ consider the line bundle $\mathcal{L} = \mathbb{C}^{n+1} \to \mathbb{CP}(n)$ where the projection is the map

$$[\lambda z_0, \lambda z_1, \ldots, \lambda z_n) \longrightarrow [z_0, z_1, \ldots, z_n].$$

   (a) What are the transition functions $\{\rho_{\alpha\beta}\}$ for the bundle $\mathcal{L} \to \mathbb{CP}(n)$?

   (b) By working out the connecting homomorphism $H^1(\mathbb{CP}(n); \mathcal{O}^\star) \to H^2(\mathbb{CP}(n); \mathbb{Z})$, show that $\{\rho_{\alpha\beta}\}$, regarded as an element of $H^1(\mathbb{CP}(n); \mathcal{O}^\star)$, maps to a generator of $H^2(\mathbb{CP}(n); \mathbb{Z}) \simeq \mathbb{Z}$. (By general convention, this generator is taken to be -1 rather tha 1.)

   (c) Assume $n = 1$ and follow above convention. Let $\mathcal{T} \to \mathbb{CP}(1)$ be the holomorphic tangent bundle which is a holomorphic line bundle. Show that the line bundle $\mathcal{T} \to \mathbb{CP}(1)$ is holomorphically equivlent to the line bundle which is the tensor product of $\mathcal{L}$ with itself k times (k can be a negative integer which means the transitions functions of $\mathcal{L}$ should be replaced by their inverses) and determine k.

   (d) Let k be a negative integer. What is the space of global holomorphic sections of $\mathcal{L}^k \to \mathbb{CP}(1)$? (*For example, can it be identified naturally with the space of homogeneous polynomials in two of a certain degree?*) What is its dimension?

   (e) Can you generalize to $\mathbb{CP}(n)$?

4. Let ω_1 and ω_2 be two complex numbers linearly independent over $\mathbb{R}$, L the lattice generated by ω_1 and ω_2 and $T = \mathbb{C}/L$. Let $\mathcal{D}$ be the divisor on T consisting of a single point p and $\mathcal{L}_{\mathcal{D}}$ the corresponding line bundle.

   (a) What is the dimension of the space of sections of $\mathcal{L}_{\mathcal{D}} \to T$?

   (b) Consider the Weierstrass $\wp$ function as a doubly periodic function relative to L on $\mathbb{C}$. Can we regard $\wp$ as the meromorphic representation of a section of a line bundle? Which line bundle?

   (c) Is there a necessary relation between the locations of zeros and poles of a meromorphic function on T? Prove your answer.

5. (This problem requires looking for references and writing an essay that answers the questions raised below.) Let $\mathcal{G}(k, n)$ denote the Grassmann manifold of k-planes in $\mathbb{C}^{k+n}$, and $\mathcal{F}_n$ the flag manifold of sequences of linear subspaces

$$V_1 \subset V_2 \subset \ \ldots \ \subset V_{n-1} \subset \mathbb{C}^n,$$

   where $\dim V_j = j$.

   (a) Describe topological structure of $\mathcal{F}_n$ as a cell complex having only cells in even (real) dimensions and the set of cells is naturally parametrized by the symmetric group $\mathfrak{S}_n$ on n letters as explained in class.

   (b) What is the dimension of the cell corresponding to an element $\sigma \in \mathfrak{S}_n$?

   (c) What do the Hard Lefschetz Theorem and the Poincaré Dulaity tell us about the combinatorial structure of the symmetric group?

   (d) Show that $\mathcal{G}_{k,n}$ has natural cell decomposition with cells parametrized by the cosets $\mathfrak{S}_{k+n}/(\mathfrak{S}_k \times \mathfrak{S}_n)$.

   (e) What do the Hard Lefschetz Theorem and the Poincaré Dulaity tell us about the cosets representatives of $\mathfrak{S}_{k+n}/(\mathfrak{S}_k \times \mathfrak{S}_n)$ or certain card shuffles?