1 Preliminaries from Real Differential Geometry

(There may be some sign differences between the notes and the lectures.)

1.1 Moving Frames in Euclidean Spaces

We equip $\mathbb{R}^N$ with the standard inner product $<,>$. By a *moving frame* in an open subset $U \subseteq \mathbb{R}^N$ we mean a choice of orthonormal bases $\{e_1(x), \cdots, e_N(x)\}$ for all $\mathcal{T}_x U$, $x \in U$. Taking exterior derivatives we obtain

$$dx = \sum_A \omega_A e_A, \quad de_A = \sum_B \omega_{BA} e_B \tag{1.1}$$

where ω_A's and ω_{AB}'s are 1-forms. Since ω_A and ω_{AB} depend on the point x and the choice of the moving frame $\{e_1, \cdots, e_N\}$, their natural domain of definition is the principal bundle $\mathcal{F}_g \to U$ of orthonormal frames on U. However, due to the functorial property of the exterior derivative $(f^\star(d\eta) = df^\star(\eta))$, the actual domain is immaterial for many calculations. One may use local parametrizations or sufficiently differentiable mappings in the actual computations and conclusions remain valid. The orthonormality condition implies $0 = d < e_A, e_B >=< de_A, e_B > + < e_A, de_B >$ and consequently

$$\omega_{AB} + \omega_{BA} = 0. \tag{1.2}$$

That is, the matrix valued 1-form $\omega = (\omega_{AB})$ takes values in the Lie algebra $\mathcal{SO}(N)$. From $ddx = 0$ and $dde_A = 0$ we obtain

$$d\omega_A + \sum_B \omega_{AB} \wedge \omega_B = 0; \quad d\omega_{AB} + \sum_C \omega_{AC} \wedge \omega_{CB} = 0. \tag{1.3}$$

These equations are often called the *structure equations* for Euclidean space or more precisely for the group of rigid motions of Euclidean space. The second set of equations is also known as the *structure equations* for the (proper) orthogonal group. These equations are a special case of Maurer-Cartan equations for (connected) Lie groups. Fixing an origin and an orthonormal frame at each point, the set of (positively oriented) frames on $\mathbb{R}^N$ can be identified with the group of (proper) Euclidean motions, and (1.3) becomes identical with what is generally called the Maurer-Cartan equations where we have represented the $(N+1) \times (N+1)$ matrix $U^{-1}dU$ in the form

$$\begin{pmatrix} \omega_{11} & \cdots & \omega_{1N} & \omega_1 \\ \vdots & \ddots & \vdots & \vdots \\ \omega_{N1} & \cdots & \omega_{NN} & \omega_N \\ 0 & \cdots & 0 & 0 \end{pmatrix}$$

The choice of (positively oriented) frame at each point of U means fixing a mapping of U into the group of (proper) Euclidean motions of $\mathbb{R}^N$ and the structure equations are just the pull-back of the matrix of left invariant 1-forms from the group to U. Much of basic differential geometry can be developed by exploiting these equations.

We introduce the fundamental concepts of Riemannian geometry by first looking at Euclidean space and its submanifolds, and determining which notions are dependent or independent of the embedding. This special case, besides being of intrinsic interest, will serve as a good example for the more abstract development of the general case. Let $M \subset U$ be a submanifold. It is no loss of generality and convenient to assume that a smooth function $F : U \to \mathbb{R}^{N-m}$ is given and the matrix of differentials DF has maximal rank everywhere so that $F^{-1}(c) = M_c$ is a submanifold of U. For convenience we often simply write M for the submanifolds M_c since the theory is applicable to all M_c's in a unified manner. To adapt the moving frame to this situation, that to the submanifolds M (i.e., all M_c's), we assume that x ranges over M and $e_1(x), \cdots, e_m(x)$ form an orthonormal basis for $\mathcal{T}_xM$. To simplify notation, we make the following convention on indices:

$$1 \le A, B, C, \cdots \le N, \quad 1 \le i, j, k, \cdots \le m, \quad m+1 \le a, b, p, q, \cdots \le N.$$

Since x ranges over M, $\omega_p = 0$, and hence $dx = \sum_i \omega_i e_i$. This simply expresses the fact that $\mathcal{T}_xM$ is spanned by $e_1, \cdots, e_m$. In a more cumbersome language this can be rephrased as follows: If $f : M \to U$ is a submanifold, then $f^\star(\omega_p)$ vanishes identically. By writing $\omega_p = 0$ we emphasize the point of view that M is regarded as the solutions to the Pfaffian system

$$\omega_{m+1} = 0, \quad \cdots, \quad \omega_N = 0. \tag{1.4}$$

By construction this Pfaffian system is integrable with submanifolds M_c.

Example 1.1 To better understand the meaning of this point of view, suppose we want to calculate the element of arc length (i.e., the Riemannian metric) ds^2 for the submanifolds M. From basic Calculus we know that the arc length ds^2 for a curve $\gamma : I \to \mathbb{R}^N$ is given by the inner product $< d\gamma, d\gamma >$. Let t be the parameter on the interval I. For γ to be a curve on a submanifold M is equivalent to $\gamma^\star(\omega_a) = 0$. Since $d\gamma = \sum_A \omega_A(\frac{\partial}{\partial t})$ the arc-length on a submanifold of the system (??) is given by

$$ds^2 \;=\; < d\gamma, d\gamma > \;=\; \sum_{i=1}^{m} \gamma^\star(\omega_i)^2.$$

Therefore we can simply say that the Riemannian metric is given by

$$ds^2 \;=\; \sum_{i=1}^{m} \omega_i^2. \tag{1.5}$$

This makes the calculation of the 1-forms ω_i and other quantities of geometric interest straightforward if the Riemannian metric of an abstract manifold is given and is in diagonal form relative to a given coordinate system. For example, for the upper half plane $\mathcal{H} = \{z = x + iy \mid y > 0\}$ the Poincaré metric is $ds^2 = \frac{dx^2+dy^2}{y^2}$ and we take $\omega_1 = \frac{dx}{y}$ and $\omega_2 = \frac{dy}{y}$ that will enable one to make many calculation by elementary algebra. Similarly the metric on the sphere of radius R in $\mathbb{R}^3$ is $ds^2 = R^2(d\varphi^2 + \sin^2\varphi d\theta^2)$ (in polar coordinates) and we take $\omega_1 = Rd\varphi$ and $\omega_2 = R\sin\varphi d\theta$. ■

The first set of structure equations becomes

$$d\omega_i + \sum_j \omega_{ij} \wedge \omega_j = 0, \quad \sum_i \omega_{pi} \wedge \omega_i = 0, \qquad \text{on } M. \tag{1.6}$$

It is convenient to decompose the matrix (ω_{AB}) in the form

$$\tilde{\omega} = \begin{pmatrix} (\omega_{ij}) & (\omega_{ip}) \\ (\omega_{pi}) & (\omega_{pq}) \end{pmatrix}.$$

The $m \times m$ matrix $\omega = (\omega_{ij})$ is called the *Levi-Civita connection* for the induced metric on $M \subset U$. Let us see how the connection ω transforms under a change of orthonormal frame. Let $A = (A_{ij})$ be an orthogonal matrix, and the frames $\{e_i\}$ and $\{f_i\}$ be related by the orthogonal transformation $e_j = \sum_i A_{ij} f_i$. Setting $f_p = e_p$ for $m + 1 \leq p \leq N$, and denoting the connection form relative to the f_A's by ω', we obtain after a simple calculation

$$\omega = A^{-1}\omega' A + A^{-1} dA \tag{1.7}$$

Notice that because of the additive factor $A^{-1}dA$, the connection ω is not a tensor but a collection of 1-forms transforming according to (1.7). As noted earlier, because of the dependence of ω on the choice of frame, its natural domain of definition is the principal bundle of orthonormal frames, however, we shall not dwell on this point. The matrix-valued function A effecting a change of frames is generally called a *gauge transformation*. Since the entries of $A^{-1}dA$ contain a basis for left invariant 1-forms on the special orthogonal group, for every point $p \in M$ there is a gauge transformation A defined in a neighborhood of p such that ω' vanishes at $p \in M$. In general, one cannot force ω' to vanish in a neighborhood of $p \in M$.

Before giving the formal definition(s) of curvature, let us give some general motivation for the approach we are taking. In analogy with the definition of the curvature of a curve in the plane, it is reasonable to try to define the curvature of a hypersurface in $\mathbb{R}^{m+1}$, or more generally of submanifolds of Euclidean spaces, by taking exterior derivatives of the normal vectors e_p. We shall show below that the exterior derivative de_p determines an $m \times m$ symmetric matrix $\mathsf{H}_p = (\mathsf{H}^p_{ij})$ for every direction e_p. The matrix H_p depends also on the choice of the frame $e_1, \cdots, e_m$ for the tangent spaces T_xM and therefore the individual components H^p_{ij} are not of geometric interest. However, the eigenvalues of H_p and their symmetric functions such as trace and determinant are independent of the choice of frames $e_1, \cdots, e_m$. Our first notions of curvature will be the trace and determinant of the matrices H_p. For the case of surfaces $M \subset \mathbb{R}^3$, Gauss made the fundamental observation (*Theorema Egregium*) that $\det(\mathsf{H}_3)$ (there is only one positively oriented normal direction e_3) curvature is computable directly in terms of the coefficients of the metric tensor ds^2 which is only the necessary data for calculating lengths of curves on the surface M. Gauss' theorem was taken up by Riemann who founded Riemannian geometry on the basis of the tensor ds^2 thus completely freeing the notion (or more precisely some notions) of curvature from the embedding. To achieve this fundamental point of view, we make use of the fact, which is far from obvious without hindsight, that the structure equations $d\omega_{AB}+\sum \omega_{AC}\wedge\omega_{CB} = 0$ express *flatness* (vanishing of curvature which will be elaborated on below) of Euclidean spaces, and the 2-forms $d\omega_{ij}+\sum_k \omega_{ik}\wedge\omega_{kj}$ (recall $1 \leq i, j \leq m$) which quantify the deviation of structure equations from being valid on M, contain much of the information about the curvature of the submanifold $M \subset \mathbb{R}^N$. The 2-form $d\omega_{ij} + \sum_k \omega_{ik} \wedge \omega_{kj}$ reduces to $d\omega_{12}$ for surfaces in $\mathbb{R}^3$ and it will be demonstrated shortly that

$$d\omega_{12} = -\det(\mathsf{H}_3)\omega_1 \wedge \omega_2. \tag{1.8}$$

The point is that once a Riemannian metric is specified, one can calculate the quantities ω_i and ω_{ij} although they depend on the choice of frames for the tangent spaces T_xM (see subsection on Levi-Civita Connection below). Therefore (1.8) contains Theorema Egregium. It should be pointed out that $\mathrm{Tr}(\mathsf{H}_3)$ is not computable from the data ds^2 alone, and it contains siginifcant geometric information which will be discussed in this chapter. In view of these facts, any quantity which is expressible in terms of ω_i's and ω_{ij}'s is called *intrinsic* to a Riemannian manifold M, and quantities which necessarily involve ω_p's or ω_{Ap}'s are called *extrinsic* in the sense that they depend on the embedding. Our immediate goal in this subsection is to make mathematics out of these remarks and specialize them to the case of surfaces in $\mathbb{R}^3$. Various notions of curvature, based on the above comments, will be introduced in the following subsections. We begin with the following algebraic lemma:

Lemma 1.1 (Cartan's Lemma) - *Let $v_1, \cdots, v_m$ be linearly independent vectors in a vector space V, and $w_1, \cdots, w_m$ be vectors such that*

$$v_1 \wedge w_1 + \cdots + v_m \wedge w_m = 0.$$

Then $w_j = \sum \mathsf{H}_{ij} v_i$ with $\mathsf{H}_{ij} = \mathsf{H}_{ji}$. The converse is also true.

Proof - Let $\{v_1, \cdots, v_m, \cdots, v_N\}$ be a basis for V, and set $w_j = \sum_i \mathsf{H}_{ij} v_i + \sum_p \mathsf{H}_{pj} v_p$. Then

$$\sum_{i=1}^{m} v_i \wedge w_i = \sum_{i,j=1}^{m} (\mathsf{H}_{ji} - \mathsf{H}_{ij}) v_i \wedge v_j + \sum_{i=1}^{m} \sum_{p=k+1}^{N} \mathsf{H}_{pi} v_i \wedge v_p.$$

Therefore $\mathsf{H}_{ij} = \mathsf{H}_{ji}$ and $\mathsf{H}_{pi} = \mathsf{H}_{ip}$. The converse statement is trivial. ∎

Applying Cartan's lemma to the second equation of (1.6), we can write

$$\omega_{ip} = \sum_j \mathsf{H}^p_{ij} \omega_j, \tag{1.9}$$

where (H^p_{ij}) is a symmetric matrix. The *Second Fundamental Form* of the submanifold M in the direction e_p is the quadratic differential given by

$$\mathsf{H}_p = \sum_{i,j} \mathsf{H}^p_{ij} \omega_i \omega_j \tag{1.10}$$

This means that the value of H_p on a tangent vector $\xi \in \mathcal{T}_x M$ is $\sum_{i,j} \mathsf{H}^p_{ij} \omega_i(\xi) \omega_j(\xi)$. The reason for regarding H_p as a quadratic differential (i.e., a section of the second symmetric power of $\mathcal{T}^* M$) is its transformation property which descibed below. (The *First Fundamental Form* is the metric ds^2.) Clearly H_p may also be regarded as the symmetric linear transformation, relative to the inner product induced from $\mathbb{R}^N$, of $\mathcal{T}_x M$ defined by the matrix (H^p_{ij}) with respect to the basis $\{e_1, \cdots, e_m\}$. Note that there is a second fundamental form for every normal direction to M.

Let us see how the second fundamental form transforms once we make a change of frames. First assume that $e_{m+1}, \cdots, e_N$ are kept fixed but $e_1, \cdots, e_m$ are subjected a transformation $A \in O(m)$. From the transformation property of the matrix (ω_{AB}) we obtain the transformation

$$\begin{pmatrix} \omega_{1p} \\ \vdots \\ \omega_{mp} \end{pmatrix} \longrightarrow A' \begin{pmatrix} \omega_{1p} \\ \vdots \\ \omega_{mp} \end{pmatrix}.$$

It follows that for fixed $e_{m+1}, \cdots, e_N$ the symmetric matrix $\mathsf{H}_p = (\mathsf{H}^p_{ij})$ transforms according

$$\mathsf{H}_p \longrightarrow A' \mathsf{H}_p A. \tag{1.11}$$

This transformation property justifies regarding the second fundamental form as a quadratic differential on M. Similarly, if we fix $e_1, \cdots, e_m$ and subject $e_{m+1}, \cdots, e_N$ to a transformation $A \in O(N-m)$, then the matrices H_p transform according as

$$\mathsf{H}_p \longrightarrow \sum_q A_{qp} \mathsf{H}_q. \tag{1.12}$$

While the matrix (H^p_{ij}) depends on the choice of the orthonormal basis for $\mathcal{T}_xM$, the symmetric functions of its characteristic values depend only on the direction e_p and not on the choice of basis for $\mathcal{T}_xM$. For example, the *mean curvature* in the direction e_p defined by $\mathsf{H}_p = \frac{1}{m}\text{trace}(\mathsf{H}^p_{ij}) = \sum_i \mathsf{H}^p_{ii}$ expresses a geometric property of the manifold $M \subset \mathbb{R}^N$ which we will discuss later especially in the codimension one case for surfaces. For a hypersurface $M \subset \mathbb{R}^{m+1}$, there is only one normal direction and we define the *Gauss-Kronecker curvature* at $x \in M$ as $K(x) = (-1)^{m+1}\det(\mathsf{H}_{ij})$ (in case $m = 2$ one simply refers to K as *curvature*). The eigenvalues of H are called the *principal curvatures* and are often denoted as $\kappa_1 = \frac{1}{R_1}, \cdots, \kappa_m = \frac{1}{R_m}$. If the eigenvalues of H are distinct, then (locally) we have m orthonormal vector fields on M diagonalizing the second fundamental form. The directions determined by these vector fields are called the *principal directions*, and an integral curve for such a vector field is called a *line of curvature*. Note that in the case of hypersurfaces the second fundamental form can also be written in the form

$$\mathsf{H} = - < dx, de_{m+1} > . \tag{1.13}$$

Example 1.2 Consider the sphere $S^n_r \subset \mathbb{R}^{n+1}$ of radius $r > 0$. Taking orthonormal frames as prescribed above, we obtain $x = re_{n+1}$, and consequently $\omega_{in+1} = \frac{1}{r}\omega_i$, $\mathsf{H}_{ij} = -\frac{\delta_{ij}}{r}\omega_i$, and $\Pi = -\frac{1}{r}\sum_i \omega_i^2$. Therefore the Gauss-Kronecker curvature of S^n_r is $K(x) = \frac{1}{r^n}$. ∎

Example 1.3 A simple case of a submanifold of codimension one is that of a surface $M \subset \mathbb{R}^3$. In this case the Levi-Civita connection is the matrix

$$\omega = \begin{pmatrix} 0 & \omega_{12} \\ -\omega_{12} & 0 \end{pmatrix}$$

The symmetric matrix (H_{ij}) in the definition of second fundamental form is defined by

$$\omega_{13} = \mathsf{H}_{11}\omega_1 + \mathsf{H}_{12}\omega_2, \quad \omega_{23} = \mathsf{H}_{12}\omega_1 + \mathsf{H}_{22}\omega_2.$$

Therefore

$$d\omega_{12} = -\omega_{13} \wedge \omega_{32} = (\mathsf{H}_{11}\mathsf{H}_{22} - \mathsf{H}^2_{12})\, \omega_1 \wedge \omega_2. \tag{1.14}$$

Therefore the measure of the deviation of the quantity $d\omega_{ij} + \sum \omega_{ik} \wedge \omega_{kj}$ from vanishing, which we had alluded to earlier, is the curvature K. It should be emphasized that the second fundamental form was obtained by restricting ω_p to M and therefore (1.14) is valid as an equation on M. Note that we have arrived at the curvature K of the surface via two different routes. The intrinsic approach where it is defined by $d\omega_{12} = K\omega_1 \wedge \omega_2$ (or the deviation of $d\omega_{12}$ from vanishing), and the extrinsic approach as the determinant of the matrix H of the second fundamental form. ∎

We have emphasized that the 1-form ω_{12} depends on the choice of the frame and therefore is naturally defined on the bundle of frames $\mathcal{P}M$. By fixing a frame (locally) we can express ω_{12} as a 1-form on M[1]. We can use this fact to advantage and deduce interesting geometric information as demonstrated in the following example:

Example 1.4 Consider a compact surface $M \subset \mathbb{R}^3$ without boundary and assume that ξ is nowhere vanishing vector field on M. From ξ we obtain a unit tangent vector field e_1 globally defined on S^2 and let e_2 be the unit tangent vector field to M such that e_1, e_2 is a positively oriented orthonormal frame. Let ω_{12} be the Levi-Civita connection expressed relative to the moving frame e_1, e_2 which is a 1-form on M. Since $\partial M = \emptyset$, Stokes' theorem implies

$$\int_M d\omega_{12} = 0.$$

On the other hand, $d\omega_{12} = K\omega_1 \wedge \omega_2$, and therefore

$$\int_M K\omega_1 \wedge \omega_2 = 0. \tag{1.15}$$

If we let $M = S^2$ be a sphere, then K is a positive constant and therefore (1.15) cannot hold. Therefore S^2 does not admit of a nowhere vanishing vector field ξ. On the other hand, it is easy to see that the torus T^2 admits of a nowhere vanishing vector field, and therefore no matter what embedding of T^2 in $\mathbb{R}^3$ we consider, still relation (1.15) remains valid. We shall return to this issue in the next chapter. ∎

The intrinsic description of the Gauss-Kronecker curvature K via the formula $d\omega_{12} = K\omega_1 \wedge \omega_2$ reduces the computation of K to straightforward algebra once the metric ds^2 is explicitly given. In fact, we have

Exercise 1.1 (a) - *Let $ds^2 = P^2(u,v)du^2 + Q^2(u,v)dv^2$. Show that the connection and curvature are given by*

$$\omega_{12} = \frac{1}{Q}\frac{\partial P}{\partial v}du - \frac{1}{P}\frac{\partial Q}{\partial u}dv, \quad K = -\frac{1}{PQ}\{\frac{\partial}{\partial v}(\frac{1}{Q}\frac{\partial P}{\partial v}) + \frac{\partial}{\partial u}(\frac{1}{P}\frac{\partial Q}{\partial u})\}.$$

(b) - *Let $M \subset \mathbb{R}^3$ be a surface, and L be a line of curvature on M. Show that the surface formed by the normals to M along L has zero curvature.*

[1] In more sophisticated language, the frame e_1, e_2 is a global section of the bundle of frames $\mathcal{P}M$ and ω_{12}, which is naturally defined on it, is pulled back to M by this section.

In view of the above considerations it is reasonable to define the *curvature matrix* $\Omega = (\Omega_{ij})$ *of a submanifold* $M \subset \mathbb{R}^N$ as

$$\Omega_{ij} = d\omega_{ij} + \sum_{k=1}^{m} \omega_{ik} \wedge \omega_{kj}.$$

For hypersurface $M \subset \mathbb{R}^{m+1}$, the curvature matrix Ω is then related to the second fundamental form by the important relation

$$\Omega_{ij} = -\omega_{im+1} \wedge \omega_{m+1j}, \tag{1.16}$$

for a $M \subset \mathbb{R}^3$. This formula follows immediately from the structure equations and the definition of Ω_{ij}. The definition of the curvature matrix will be extended and discussed in the following subsections.

Related to (1.16) is the concept of Gauss mapping which will be used exensively. Let $M \subset \mathbb{R}^{m+1}$ be a hypersurface and consider the mapping $\mathsf{G} : M \to S^m$ given by $\mathsf{G}(x) = e_{m+1}(x)$ called the *Gauss mapping*. Since $de_{m+1} = \sum \omega_{im+1} e_i$, we easily obtain

$$\mathsf{G}^*(dv_{S^m}) = \omega_{1m+1} \wedge \cdots \wedge \omega_{mm+1} = (-1)^m \det(\mathsf{H}) \omega_1 \wedge \cdots \wedge \omega_m. \tag{1.17}$$

1.2 Levi-Civita Connection and Curvature

The Levi-Civita connection (ω_{ij}) for a submanifold $M \subset \mathbb{R}^N$ is an anti-symmetric matrix with the property $d\omega_i + \sum \omega_{ij} \wedge \omega_j = 0$ (1.6). In general, for 1-forms $\theta_1, \cdots, \theta_m$ spanning the cotangent spaces to M, we can only assert the existence of a matrix of 1-forms (θ_{ij}) such that $d\theta_i + \sum \theta_{ij} \wedge \theta_j = 0$. A remarkable consequence of an inner product on $\mathbb{R}^N$ was that if we set $\theta_i = \omega_i$ then the matrix (θ_{ij}) can be replaced by the anti-symmetric matrix (ω_{ij}), i.e., a matrix of 1-forms taking values in the Lie algebra of $SO(m)$. The following proposition shows that the existence of a Riemanniann metric on M (and not an embedding) is all that is needed to ensure the existence and uniqueness of the matrix (ω_{ij}) with the required properties[2]:

Proposition 1.1 *Let* $\omega_1, \cdots, \omega_m$ *be a basis of one forms reducing the Riemannian metric to the identity matrix, i.e.,* $ds^2 = \sum_i \omega_i^2$. *Then there is a unique skew-symmetric matrix* $\omega = (\omega_{ij})$ (called the *Levi-Civita connection* for the given Riemannian metric) *such that*

$$d\omega_i + \sum_j \omega_{ij} \wedge \omega_j = 0.$$

[2]The remarkable property of the Levi-Civita connection becomes more evident when one studies geometric structures corresponding subgroups other than the orthogonal groups.

Proof - We have

$$d\omega_i = \sum_{j,k} a_{ijk}\omega_j \wedge \omega_k,$$

where the coefficients a_{ijk} satisfy the anti-symmetry condition

$$a_{ijk} + a_{ikj} = 0.$$

Since $a_{jki} + a_{kji}$ is symmetric in the indices (j, k), we have

$$\sum_{j,k} \omega_j \wedge (a_{jki} + a_{kji})\omega_k = 0,$$

and consequently

$$d\omega_i = \sum_{j,k} \omega_j \wedge (a_{ijk} + a_{jki} + a_{kji})\omega_k.$$

Now set

$$\omega_{ij} = \sum_k (a_{ijk} + a_{jki} + a_{kji})\omega_k$$

which satisfies the requirements of the proposition. To prove uniqueness, let (ω'_{ij}) be another such matrix, and set $\theta_{ij} = \omega_{ij} - \omega'_{ij}$. Applying Cartan's lemma to $\sum \omega_i \wedge \theta_{ij} = 0$, we obtain

$$\theta_{ij} = \sum_k b_{kij}\omega_k, \qquad b_{kij} = b_{ikj}.$$

On the other hand by anti-symmetry of θ_{ij}, we have $b_{kij} = -b_{kji}$. It follows easily that $b_{ijl} = 0$ thus completing the proof of the proposition. ∎

Exercise 1.2 *For the metric ds^2 in the diagonal form $ds^2 = \sum_i g_{ii}dx_i^2$, show that the Levi-Civita connection is given by*

$$\omega_{ij} = \frac{1}{\sqrt{g_{jj}}}\frac{\partial \log \sqrt{g_{ii}}}{\partial x_j}\omega_i - \frac{1}{\sqrt{g_{ii}}}\frac{\partial \log \sqrt{g_{jj}}}{\partial x_i}\omega_j,$$

where $\omega_i = \sqrt{g_{ii}}dx_i$.

The connection ω enables us to differentiate vector fields. More precisely, let $e_1, \cdots, e_m$ be an orthonormal frame on the Riemannian manifold M, and (ω_{ij}) be the Levi-Civita connection for the Riemannian metric g. Define

$$\nabla e_i = \sum_j \omega_{ji} e_j, \tag{1.18}$$

and we extend ∇ to a vector field $\xi = \sum_i b_i e_i$ by

$$\nabla \sum_i b_i e_i = \sum_{ij} \omega_{ji} b_i e_j + \sum_i db_i e_i. \tag{1.19}$$

The quantity $\nabla\xi$ is called the *covariant derivative* of the vector field ξ. For a vector field η

$$\nabla_\eta \xi = \sum_{ij} \omega_{ji}(\eta) b_i e_j + \sum_i db_i(\eta) e_i,$$

is the *covariant derivative of* ξ *in the direction* η. It is not difficult to verify that $\nabla_\eta\xi$ and $\nabla\xi$ are independent of the choice of orthonormal frame $e_1, \cdots, e_m$.

Another very useful operation on tensor is contraction. For every pair (i, j), with $1 \le i \le m$ and $1 \le j \le n$ the *contraction* operator

$$C_{ij} : \underbrace{V \otimes \cdots \otimes V}_{m \text{ times}} \otimes \underbrace{V^\star \otimes \cdots \otimes V^\star}_{n \text{ times}} \longrightarrow \underbrace{V \otimes \cdots \otimes V}_{m-1 \text{ times}} \otimes \underbrace{V^\star \otimes \cdots \otimes V^\star}_{n-1 \text{ times}}$$

is defined by

$$C_{ij}(v_1 \otimes \cdots \otimes v_m \otimes \xi_1 \otimes \cdots \otimes \xi_n) = \xi_j(v_i) v_1 \otimes \cdots \otimes \hat{v}_i \otimes \cdots \otimes v_m \otimes \xi_1 \otimes \cdots \otimes \hat{\xi}_j \otimes \cdots \otimes \xi_n,$$

where $\hat{v}_i$ means v_i is omitted.

We now can extend covariant differentiation to a derivation on the space of tensors by the requirements

1. $\nabla f = df$ for a smooth function f;

2. ∇ commutes with contractions.

An immediate consequence is

$$0 = dg(e_i, e_j) = \nabla(g)(e_i, e_j) + g(\nabla(e_i), e_j) + g(e_i, \nabla(e_j)) = \nabla(g)(e_i, e_j) + \omega_{ji} + \omega_{ij} = \nabla(g)(e_i, e_j).$$

Therefore

$$\nabla g = 0, \quad \text{or equivalently} \quad dg(\xi, \zeta)(\eta) = g(\nabla_\eta \xi, \zeta) + g(\xi, \nabla_\eta \zeta). \tag{1.20}$$

This equation expresses a fundamental property of the Levi-Civita connection.

Remark 1.1 We have followed the mathematical tradition of only considering Riemannian rather than *indefinite* metrics by which we mean the condition of positive definiteness of the symmetric matrix $g = (g_{ij})$ is replaced by that of nondegeneracy. We shall see in subsections on spaces of constant curvature and homogeneous spaces that indefinite metrics, besides being of intrinsic interest in physics, are useful in understanding the behavior of Riemannian metrics. For an indefinite metric ds^2 with r positive and $m - r$ negative eigenvalues we consider frames (also call them orthonormal) with the property

$$ds^2(e_i, e_j) = \pm\delta_{ij},$$

where $+$ or $-$ sign is chosen according as $i \leq r$ or $r + 1 \leq i \leq m$. The definition of Levi-Civita connection ω_{ij} is the same except that instead of skew symmetry we require $(\omega_{ij})J + J(\omega_{ij}) = 0$ where J is the diagonal matrix whose first r diagonal entries are 1, and the remaining diagonal entries are -1. In other words (ω_{ij}) takes values in the Lie algebra of the orthogonal group of J. The existence and uniqueness of the Levi-Civita connection is the same as in the Riemannian case. ∎

We stated earlier that the deviation of the quantity $d\omega_{ij} + \sum \omega_{ik} \wedge \omega_{kj}$ from vanishing reflects the curvature of the space, and can be calculated from the metric ds^2 alone. We set $\Omega_{ij} = d\omega_{ij} + \sum \omega_{ik} \wedge \omega_{kj}$, and call the matrix $\Omega = (\Omega_{ij})$ the *curvature form*. Ω is a skew symmetric matrix and depends on the choice of frame. Therefore it is defined on the bundle $\mathcal{F}_g$ of orthonormal frames and its individual entries are not of geometric interest. From (1.7) it follows easily that the dependence of Ω on the choice of frame is given by

$$\Omega = A^{-1}\Omega' A. \tag{1.21}$$

Since the entries of Ω are 2-forms, and 2-forms commute, we can manipulate the matrix Ω as if it were a matrix of scalars. Thus, for example, the various symmetric functions of the characteristic roots of Ω, which are polynomials in Ω_{ij}'s, are independent of the choice of the frame and are defined on the manifold M. This observation plays an important role in the differential geometry of Riemannian manifolds and understanding the connection between geometry and topology.

The identity $dd = 0$ implies certain relations among ω_i, ω_{ij} and Ω_{ij}. Indeed $dd\omega_i = 0$ implies the *first Bianchi* identity:

$$\sum_j \Omega_{ij} \wedge \omega_j = 0. \tag{1.22}$$

Similarly the relation $dd\omega_{ij} = 0$ implies the *second Bianchi* identity:

$$d\Omega_{ij} = \sum_k \omega_{ik} \wedge \Omega_{kj} - \sum_k \Omega_{ik} \wedge \omega_{kj}. \tag{1.23}$$

We set

$$2\Omega_{ij} = \sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l.$$

The scalar R_{ijij} is called the *sectional curvature* of the plane determined by the vectors e_i, e_j. R_{ijkl} is called the *curvature tensor*. The curvature tensor satisfies the relations

$$R_{ijkl} + R_{iklj} + R_{iljk} = 0, \quad R_{ijkl} = -R_{jikl} = R_{jilk}, \quad R_{ijkl} = R_{klij}. \tag{1.24}$$

The first identity is a consequence of (1.22), the second and third equations are trivial and the last equality follows from the preceding ones by simple manipulations.

The curvature of a Riemannian manifold may be interpreted as an endomorphism in more than one way. The endomorphism of $\mathcal{T}_xM$, for each $x \in M$, defined by $S_{ik}(e_l) = 2\sum_{j=1}^{m} \Omega_{ij}(e_k, e_l)e_j$ has trace R_{ik} (called *Ricci tensor*)

$$R_{ik} = \mathrm{Tr}(S_{ik}) = 2\sum_{j,l=1}^{m} < \Omega_{ij}(e_k, e_l)e_j, e_l >_x = \sum_{l=1}^{m} R_{ilkl}$$

where $<,>_x$ denotes the inner product (Riemannian metric) on $\mathcal{T}_xM$. Thus each component of the Ricci tensor at $x \in M$ is the trace of an endomorphism of $\mathcal{T}_xM$. Clearly the Ricci tensor is a symmetric matrix. It is a simple exercise to see that under a gauge transformation A, the Ricci tensor transforms according

$$(R_{ik}) = A^{-1}(R^{\star}_{ik})A, \tag{1.25}$$

where $(R^{\star}_{ik})$ denotes the Ricci tensor relative to the new orthonormal basis.

It is customary to define the curvature operator R as

$$R(e_i, e_j)e_k = 2\sum_{l=1}^{m} \Omega_{lk}(e_i, e_j)e_l.$$

It is useful to regard the curvature operator R and the curvature tensor R_{ijkl} as multilinear functions on $\mathcal{T}_xM$ or elements of the tensor algebra on $\mathcal{T}_xM$. For instance, if $v = \sum v_i e_i$ and $w = \sum w_i e_i$, then

$$R(v, w) = \sum_{i,j} v_i w_j R(e_i, e_j).$$

Similarly if $v' = \sum v'_i e_i$ and $w' = \sum w'_i e_i$, then

$$R(v, w, v', w') = \sum_{i,j,k,l} v_i w_j v'_k w'_l R_{ijkl}.$$

With this interpretation it is immediate that the sectional curvature can be regarded as the assignment of a number to each 2-plane in $\mathcal{T}_xM$. If $V \subset \mathcal{T}_xM$ is a 2-plane, and $e_1, \cdots, e_m$ is a basis for $\mathcal{T}_xM$ with e_1, e_2 spanning V, then

$$R_{1212} = \frac{R(v_1, v_2, v_1, v_2)}{g_x(v_1, v_1)g_x(v_2, v_2) - (g_x(v_1, v_2))^2}, \tag{1.26}$$

where v_1, v_2 is any basis for V and g_x denotes the inner product on $\mathcal{T}_xM$ (the Riemannian metric). The curvature tensor may be regarded as an element of $S^2(\bigwedge^2 W)$ where $W = \mathcal{T}_x^\star M$ (symmetric bilinear form on the second exterior power). Since a symmetric bilinear B form is uniquely determined by its values on the diagonal, i.e.,

$$2B(u, v) = B(u + v, u + v) - B(u, u) - B(v, v),$$

the curvature tensor is determined by the sectional curvatures.

A Riemannian manifold M is called *Einstein* if its Ricci tensor, when expressed relative to an orthonormal frame, is a multiple of the identity. This condition is equivalent to the requirement that relative to a coordinate system the Ricci tensor is multiple of the metric ds^2. In view of the transformation property (1.25), the Einstein property is independent of the choice of orthonormal frame. It expresses an intrinsic geometric property of the Riemannian manifold which is not as restrictive as being of constant sectional curvature.

Example 1.5 In this example we investigate the Einstein condition in the special case where $\dim M = 4$. We fix an orthonormal frame $\{e_1, \cdots, e_4\}$, and recall that $R_{ij} = \sum_k R_{ikjk}$. In particular, for an Einstein manifold we have

$$\sum_k R_{ikik} - \sum_k R_{jkjk} = 0,$$

for all i, j. This is a homogeneous system of three linear equations in six unknowns R_{ikik}. It is a simple matter to see that the solutions to this system are characterized by

$$R_{1212} = R_{3434}, \ R_{1313} = R_{2424}, \ R_{1414} = R_{2323}.$$

In other words, sectional curvatures of the planes determined by $\{e_1, e_2\}$ and $\{e_3, e_4\}$ are equal, etc. In view of the independence of the Einstein condition from the choice of frame and the transformation property (1.25), this conclusion can be restated as a four dimensional Riemannian manifold is Einstein if and only if its sectional curvatures are identical on orthogonal planes. ∎

Exercise 1.3 *By emulating the argument of example (1.5) show that for an Einstein manifold of dimension 3, sectional curvatures at a point $x \in M$ do not depend on the choice of the planes in T_xM, and $R_{1213} = 0$ etc. Thus $\Omega_{ij} = R(x)\omega_i \wedge \omega_j$ for some function $R : M \to \mathbb{R}$.*

The following example shows how part of exercise 1.3 generalizes to higher dimensions:

Example 1.6 Let M be a Riemannian manifold of dimension ≥ 3 and assume that the sectional curvatures at $x \in M$ do not depend on the choice of the plane (spanned by e_i, e_j). We show that the symmetries of the curvature tensor imply that M necessarily has constant curvature. Let $e_1, \cdots, e_m$ be a moving frame for M, and set

$$e_1^\theta = \cos\theta e_1 + \sin\theta e_3, \quad e_3^\theta = -\sin\theta e_1 + \cos\theta e_3.$$

Then $e_1^\theta, e_2, e_3^\theta, e_4, \cdots$ is a moving frame for M, and let $\omega_1^\theta, \omega_2, \omega_3^\theta, \omega_4, \cdots$ be the dual coframe. We denote the curvature form relative to this frame by (Ω_{ij}^θ). Let R_{1212}^θ denote the coefficient of $\omega_1^\theta \wedge \omega_2$ in Ω_{12}^θ. By 4-linearity of the curvature tensor

$$R_{1212}^\theta = \cos^2\theta R_{1212}^\circ + \sin^2\theta R_{3232}^\circ + \sin 2\theta R_{1232}^\circ.$$

The hypothesis implies that $R_{1212}^\theta = R_{3232}^\theta$ and is independent of θ, and consequently $R_{1232}^\circ = 0$. In other words, $R_{ijkl} = R_{ijkl}^\circ = 0$ if exactly three of the indices i, j, k, l are distinct. Similarly by looking at the coefficient of $\omega_1^\theta \wedge \omega_4$ in Ω_{12}^θ and using $R_{1214} = 0$ etc. we obtain $R_{1234}^\circ + R_{3214}^\circ = 0$, or

$$R_{ijkl}^\circ + R_{kjil}^\circ = 0 \tag{1.27}$$

This relation together with the first Bianchi identity $R_{4321}^\circ + R_{4213}^\circ + R_{4231}^\circ = 0$ imply

$$R_{4213}^\circ + 2R_{4132}^\circ = 0. \tag{1.28}$$

Equation (1.27) and skew symmetry of the curvature tensor in the last two indices imply

$$R_{4132} = -R_{3142} = -R_{4231} = R_{4213}$$

Substituting in (1.28) we obtain $R_{4213} = 0$. It follows that the curvature tensor form $\Omega_{ij} = \Omega_{ij}^\circ$ is of the form

$$\Omega_{ij} = R(x)\omega_i \wedge \omega_j. \tag{1.29}$$

Taking exterior derivative of Ω_{ij}, using the second Bianchi identity and substituting from (1.29) we obtain

$$\sum dR \wedge \omega_i \wedge \omega_j = 0,$$

which implies that $dR = 0$ and M has constant curvature. This example is due to *Schur*. ∎