Lecture 1

1. Remarks on Basic Concepts

In this lecture we discuss several topics related to the material covered so far.

1.1. **Area Formula.** Let $f : U \to V$ be a C^1 orientation preserving homeomorphism of domains in $\mathbb{C}$. Then he area of the domain V can be computed from the formula

$$A(V) \;=\; \frac{i}{2}\int_U df \wedge d\overline{f}. \tag{1}$$

This is just the change of variable formula in dimension 2 since $(i/2)df \wedge d\overline{f} = Jdx \wedge dy$ and J is the Jacobian of change of variable by f. Let us see some other applications of this simple observation. Assuming that the domain U is bounded and has C^1 boundary C and F extends to a neighborhood of U, we apply Stokes theorem to the right hand side to obtain

$$A(V) \;=\; \frac{i}{2}\int_C f\; d\overline{f}.$$

The right hand side is now a line integral. Writing f as $f = u + iv$ and denoting denoting the image of the boundary curve C under f by Γ, we obtain

$$\frac{i}{2}\int_C f\; d\overline{f} \;=\; \int_\Gamma udv. \tag{2}$$

(In deriving this formula note that $\int_\Gamma (udu + vdv) = 0$ since the integrand is the exact differential $(1/2)(d(u^2) + d(v^2))$.) Note that this equation has no reference to the domain bounded by C or Γ but only to the boundary curves. The boundary curve (e.g. a Jordan curve) decomposes the extended into two regions. So which area does this integral represent? In Stokes theorem the region is required to be bounded so that integrals exist and furthermore correct orientation for the bounding curve is essential for otherwise one obtains negative of the area. If the region is unbounded (i.e., contains the point at ∞) then the integral $\int udv$ represents the area of the bounded region by extending $u + iv$ to a C^1 homeomorphism onto the bounded region and the bounding curve has counter-clockwise orientation.

Now let U and V be the simply connected regions in the extended complex plane, one bounded and the other unbounded (containing ∞)

defined as

$$U: \ ||z|| < 1; \ V: \ ||z|| > 1 \text{ and point at } \infty$$

Let f_U and f_V be an analytic function defined on U or V normalized so that they have expansions

$$f_U(z) = z + a_2z^2 + a_3z^3 + \dots, \ f_V(z) = z + \frac{b_1}{z} + \frac{b_2}{z^2} + \frac{b_3}{z^3} + \dots$$

Assuming f is univalent (i.e., 1-1) what can we say about the coefficients a_n and b_n. It is known that that $|a_n| \leq n$ (Bieberbach conjecture proven by de Brange) but the proof is very difficult and relies on techniques very different from the ideas in this course. However, the case for b_n can be easily settled with the aid of the area formula. In fact, we will show

$$\sum_{n=1}^{\infty} n|b_n|^2 \ \leq \ 1 \tag{3}$$

Let C_r denote the circle of radius $r > 1$ and Γ_r denote its image under the mapping. Set $f_V = u + iv$ as before. In view of (2) and the remark following it, the integral

$$I \ = \ \int_{\Gamma_r} udv$$

is just the area of the bounded region enclosed by Γ_r and therefore $I > 0$. On the other hand direct substitution from $f_V(z) = z + \frac{b_1}{z} + \dots$ in $I = (i/2)\int_{C_r} fd\overline{F}$ yields after a simple calculation

$$I` = \ \pi[r^2 - \sum_{n=1}^{\infty} n|b_n|^2 r^{-2n}]. \tag{4}$$

Therefore for all $r > 1$ we have $\sum n|b_n|^2 r^{-2n} < r^2$ which proves (3).

This calculation can be used to show that $|a_2| \leq 2$ which is what Bieberbach had been able to prove. In fact, let $F(z) = \sqrt{f(z^2)}$ then F is well defined and is univalent on the disc $|z| < 1$. For if $F(z_1) = F(z_2)$ then univalence of f implies $z_1 = \pm z_2$. It is a simple calculation that the power series expansion of F is of the form

$$F(z) \ = \ z + \frac{1}{2}a_2z^3 + \dots,$$

and has only odd powers of z. Therefore F is odd function and $F(z_1) = -F(z_2)$ proving univalence of F.

Now set $g(z) = \frac{1}{F(z^{-1})}$. Then

$$g(z) \ = \ z - \frac{a_2}{2}\frac{1}{z} + \dots \text{ (higher negative powers of } z).$$

Then (3) implies $|\frac{1}{2}a_2|^2 \leq 1$ or $|a_2| \leq 2$ as expected.

Another consequence of the area calculation above is

Theorem 1. (Koebe $\frac{1}{4}$ Theorem) *Let $f(z) = z + a_2z^2 + \dots$ be defined and univalent in the unit disc. Then image of f contains the open ball of radius $\frac{1}{4}$.*

Proof - Let w be a value not assumed by f, then

$$\frac{wf(z)}{w - f(z)} \;=\; z + (a_2 + \frac{1}{w})z^2 + \dots,$$

and it is immediate that f is univalent in the unit disc. Therefore $|a_2 + \frac{1}{w}| \leq 2$ which implies $\frac{1}{|w|} \leq |a_2| + 2 \leq 4$. The required result follows. □

1.2. **Schwarz-Pick Lemma.** The standard version of the Schwarz (or Schwarz-Pick) Lemma states that a holomorphic map $f : \mathbb{D} \to \mathbb{D}$ with $f(0) = 0$ satisfies

(1) $|f(z)| \leq |z|$.
(2) $|f'(0)| \leq 1$.
(3) If $|f(z)| = |z|$ for a single point $z \neq 0$ then f is a rotation $f(z) = e^{i\theta}z$.
(4) $|f'(0)| = 1$, then $f(z) = e^{i\theta}z$.

The Schwarz Lemma can be restated in the language of differential geometry and the reformulation has had far-reaching consequences. The unit disc $\mathbb{D}$ carries a metric of constant negative curvature -1 known as the *Poincaré metric* given by

$$ds^2 \;=\; \frac{4dzd\overline{z}}{(1 - |z|^2)^2}.$$

Given a map $f : \mathbb{D} \to \mathbb{D}$ then $f^\star(ds^2)$ is also a Riemannian metric on $\mathbb{D}$ except at the points where df fails to be an isomorphism. In particular it makes sense to compare distances relative to the metrics ds^2 and $f^\star(ds^2)$. First let us look at some examples. If $f(z) = \frac{az+b}{\overline{b}z+\overline{a}}$ where $|a|^2 - |b|^2 = 1$. (Fractional linear transformation under the group $SU(1,1)$.) Then it is a routine calculation that $f^\star(ds^2) = ds^2$ so that $SU(1,1)$ acts on $\mathbb{D}$ as a group of isometries relative to the Poincaré metric. Using invariance of the metric under $SU(1,1)$ it is a straightforward calculation to give formulae for non-Euclidean distances of points in the disc. In fact the distance between $z, w \in \mathbb{D}$ is given by

$$d_{\mathbb{D}}(z, w) \;=\; \log\left(\frac{1 + |\frac{|z-w|}{|1-\overline{z}w|}|}{1 - |\frac{|z-w|}{|1-\overline{z}w|}|}\right). \tag{5}$$

The tricks in proving this formula are as follows:

(1) First one shows that Euclidean straights line through the origin are geodesics in $\mathbb{D}$ and therefore realize distances between points.
(2) By an element of $SU(1,1)$ we can move z to 0 and w to a point on the imaginary axis.
(3) By straightforward integration one obtains the required formula in this case.
(4) Verify that formula (5) is invariant under the action of $SU(1,1)$ to complete the proof.
(5) Now one can also in addition easily verify that the geodesic (up to parametrization) connecting z to w (and necessarily realizing the distance between them is given by the equation

$$t \longrightarrow \frac{z + t\frac{w-z}{1-\bar{z}w}}{1 + t\bar{z}\frac{w-z}{1-\bar{z}w}}. \tag{6}$$

The metric of the disc $\mathbb{D}$ is of the form $ds^2 = \rho_{\mathbb{D}}^2(dx^2 + dy^2)$ (in real coordinates (x, y)). Coordinates where the metric takes this form are called *isothermal* coordinates in differential geometry in 2D. It is classic fact that in dimension 2 every Riemannian metric admits of isothermal coordinates (locally). Notice that in isothermal coordinates measure of angles are the same as those in Euclidean coordinates (x, y) because the metric is a multiple of the Euclidean one $dx^2 + dy^2$. With this in mind, one expects that a consequence of isothermal coordinates on an orientable surface is that the Riemannian metric (put in isothermal coordinates) allows one to define complex coordinates on it by setting $z = x + iy$. (The expression $\rho_{\mathbb{D}}^2(dx^2 + dy^2)$ is symmetric in x and y and so one could have defined the complex coordinates as $y + ix$ instead of $x + iy$. The problem is that we want the orientation to be the standard counterclockwise orientation of the (x, y)-plane and $y + ix$ gives the opposite orienttion.) This is in fact the case. We will not rigorously prove this but use this idea freely.

To apply these notions to the Schwarz Lemma, let $f : \mathbb{D} \to \mathbb{D}$ be an analytic map and $z \in \mathbb{D}$. Define automorphisms g and h of $\mathbb{D}$ by

$$g(\zeta) = \frac{\zeta + z}{\bar{z}\zeta + 1}, \; h(\zeta) = \frac{\zeta - f(z)}{-\overline{f(z)}\zeta + 1}.$$

Then the composition $F = h \circ f \circ g$ satisfies the hypothesis of the Schwarz Lemma ($F(0) = 0$). Therefore $|F'(0)| \leq 1$ and since $F'(0) =$

$\frac{1-|z|^2}{1-|f(z)|^2}f'(z)$ we obtain

$$\frac{|f'(z)|}{1-|f(z)|^2} \;\leq\; \frac{1}{1-|z|^2}. \tag{7}$$

This equation simply means

$$f^\star(ds^2_{\mathbb{D}}) \leq ds^2_{\mathbb{D}}, \tag{8}$$

that is, analytic maps of the disc to itself are distance decreasing relative to the Poincaré metric. This is the first geometric reformulation of the Schwarz Lemma.

To put this geometric form into a more useful one we need the notion of curvature from differential geometry of surfaces. The (Gaussian) curvature of a metric $ds^2 = \rho^2(dx^2 + dy^2)$ is given by

$$\kappa \;=\; -\frac{1}{\rho^2}\frac{\partial^2 \log\rho}{\partial z \partial \overline{z}} \;==\; -\frac{1}{4\rho^2}\left(\frac{\partial^2 \log\rho}{\partial x^2} + \frac{\partial^2 \log\rho}{\partial y^2}\right), \tag{9}$$

where $\frac{\partial}{\partial z} = \frac{1}{2}\big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\big)$ and $\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\big)$.

Now we can state a more general and useful version of the Schwarz Lemma. On the disc $\mathbb{D}_r = \{z \in \mathbb{C} \mid |z| < r\}$ consider the Riemannian metric

$$ds^2_r \;=\; \frac{4}{A(r^2 - |z|^2)^2}\, dz d\overline{z},$$

where $A > 0$. The curvature of this metric is the constant $-A$.

Theorem 2. *Let M be a Riemann surface with a Riemannian metric ds^2_M whose curvature is bounded above by a negative constant $-B$. Then a holomorphic map $f : \mathbb{D}_r \to M$ is distance decreasing and more precisely*

$$f^\star(ds^2_M) \;\leq\; \frac{A}{B} ds^2_r.$$

The proof is not difficult but before proving it we show some of its consequences.

Corollary 1. (Liouville) *A bounded entire function is constant.*

Proof - We can assume the values of a bounded entire function are in the unit disc. In Theorem 2 let $M = \mathbb{D}$ with the standard Poincaré metric so that $B = 1$ and set $A = 1$. Then

$$f^\star(ds^2_{\mathbb{D}} \;\leq \frac{4}{(r^2 - |z|^2)^2} dz d\overline{z}.$$

Now let $r \to \infty$ to get $f^\star(ds^2_{\mathbb{D}} = 0$ or f is a constant. □

We can prove more, namely Picard's (little) theorem that an entire function missing two values is necessarily a constant. We csn assume the missing values are 0 and 1 and and let us assume that $\mathbb{C} \setminus \{0, 1\}$ admits of a metric of constant curvature -1 (to be proven below). Then the proof of Corollary 1 works with $M = \mathbb{C} \setminus \{0, 1\}$ to give

Corollary 2. (Picard) *An entire function missing two values is necessarily a constant.*

It remains to show that $\mathbb{C} \setminus \{0, 1\}$ admits of a metric of constant negative curvature. We give a more topological and algebraic proof of this that will be more in the spirit of later material. In fact we exhibit a discrete group of isometries and conformal transformations of the upper half plane $\mathbb{H}$ such that its orbit space is $\mathbb{C} \setminus \{0, 1\}$ and the map $\mathbb{H} \to \Gamma(2) \setminus \mathbb{H}$ is the universal cover projection. Let $\Gamma'(N)$ (*principal congruence subgroup of level* N) be

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a \equiv d \equiv 1 \bmod N, \text{ and } b \equiv c \equiv 0 \bmod N \right\}$$

Let $\Gamma(2) = \Gamma'(2)/(\pm I)$ and $\Gamma(N) = \Gamma'(N)$ for $N \geq 3$. $\Gamma(2)$ is generated by the matrices $\begin{pmatrix} 1 & \pm 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ \pm 2 & 1 \end{pmatrix}$. (This is no longer true for general N.) A fundamental domain for $\Gamma(2)$ in $\mathbb{H}$ is shown in Figure 1.2.1 where the vertical lines $x = \pm 1$ are mapped to each other by $\begin{pmatrix} 1 & \mp 2 \\ 0 & 1 \end{pmatrix}$. So the quotient space is homeomorphic to the sphere with three points removed (note the point $i\infty$). One verifies easily that the action of $\Gamma(2)$ on $\mathbb{H}$ has no fixed points and consequently $\mathbb{H} \to \Gamma(2) \setminus \mathbb{H}$ is a covering projection. By stereographic projection we can identify it with the $\mathbb{C} \setminus \{\text{two points}\}$. We can assume these two points are 0 and 1 by a fractional linear transformation. Since the action of $SL(2; \mathbb{R})$ by fractional linear transformations are conformal isometries the metric will descend to one on $\mathbb{C} \setminus \{0, 1\}$. (This can be done explicitly by using the modular function but it takes too far away from our subject.) Since the metric is induced from $ds^2 = \frac{dz d\bar{z}}{y^2}$ which has constant negative curvature -1, the same is true for the metric on $\mathbb{C} \setminus \{0, 1\}$.

The metric of constant negative curvature -1 on $\mathbb{C} \setminus \{0, 1, \infty\}$ has representation $ds^2 = \rho(z)^2 dz d\bar{z}$ where ρ has integral representation

$$\rho(z)^{-1} = \frac{|z(z-1)|}{4\pi} \int_{\mathbb{C}} \frac{1}{|\zeta(\zeta - 1)(\zeta - z)|} d\xi d\eta \tag{10}$$

where $\zeta = \xi + i\eta$. The proof of this formula requires some tools from quasiconformal maps that we have not yet introduced. It is mentioned here because the rational fraction

$$\frac{z(z-1)}{\zeta(\zeta-1)(\zeta-z)}$$

occurs prominently in the analytic theory of quasiconformal maps.

It remains to prove Theorem 2.

Proof of Theorem 2 - We have $f^\star(ds_M^2) = g(z)ds_r^2$ and the assertion is equivalent to $g(z) \leq \frac{A}{B}$. Assume first that the maximum of $g(z)$ occurs in the interior of $\mathbb{D}_a$, say at z_0. We can assume $g(z_0) > 0$ and so the differential *df* is an isomorphism in a neighborhood of z_0 and f is an analytic diffeomorphism in a neighborhood U of z_0. Set $f^\star(ds_M^2) = \eta^2 dz d\overline{z}$ and $ds_r^2 = \rho^2 dz d\overline{z}$. Then $g = \eta^2/\rho^2$. The curvature of the metrics ds_M^2 and ds_r^2 are

$$k_M = -\frac{1}{\eta^2}\frac{\partial^2 \log \eta}{\partial z \partial \overline{z}},\ k_{\mathbb{D}_r} = -\frac{1}{\rho^2}\frac{\partial^2 \log \rho}{\partial z \partial \overline{z}}.$$

By assumption $k_M \leq -B$ and therefore

$$\begin{array}{rcl} \frac{\partial^2 \log g}{\partial z \partial \overline{z}} & = & \frac{\partial^2 \log \eta}{\partial z \partial \overline{z}} - \frac{\partial^2 \log \rho}{\partial z \partial \overline{z}} \\ & = & -k_M \eta^2 - A\rho^2 \\ & \geq & B\eta^2 - A\rho^2. \end{array}$$

Since $\log g$ assumes its maximum at z_0 the second derivative $\frac{\partial^2 \log g}{\partial z \partial \overline{z}}$ is non-positive at this point and the required result follows. Therefore it remains to show that the maximum of g occurs in the interior. Looking at the expression for the metric $ds^2_{\mathbb{D}_r}$ we see that ρ^2 tends to ∞ at the boundary and therefore $g(z)$ tends to zero. Thus the maximum occurs in the interior. ☐

1.3. **Analytic definition of quasi-conformality.** The geometric definition of a quasiconformal map is not adequate for some important analytical developments that center around solving the Beltrami equation

$$\frac{\partial u}{\partial \overline{z}} \ = \ \mu(z)\frac{\partial u}{\partial z}, \tag{11}$$

that will be discussed later in the course. Here μ is an L^∞-function of norm < 1. It is natural that a more elaborate analytical definition is suitable for the investigation of this equation. First we recall some notions from real analysis.

A function f on an interval $[a, b]$ is *absolutely continuous* (or simply *AC*) if given $\epsilon > 0$ there is $\delta > 0$ such that for any finite sequence

$a \le a_1 < b_1 < a_2 < b_2 < \ldots < a_n < b_n \le b$ with the property

$$\sum_{j=1}^{n}(b_j - a_j) \; < \; \delta,$$

we have

$$\sum_{j=1}^{n}|f(b_j) - f(a_j)|) \; < \; \epsilon.$$

Important properties of AC function are summarized as follows:

(1) Absolute continuity is preserved by arithmetical operations on functions with the obvious restriction of non-vanishing denominators.
(2) An absolutely continuous function is almost everywhere differentiable.
(3) If the derivative of an AC function f is zero almost everywhere then f is a constant.
(4) The indefinite integral of an L^1-function g on $[a, b]$

$$f(x) \; = \; C + \int_a^x g(y)dy$$

is absolutely continuous, and the derivative f' of f is almost everywhere equal to g.
(5) Every absolutely continuous function f is an indefinite integral of its derivative f' that exists almost everywhere (Item 2 above).
(6) An absolutely continuous function maps sets of Lebesgue measure 0 to sets of Lebesgue measure 0.

(For s detailed discussion of absolute continuity see, for example, I. P. Natanson - *Theory of Functions of a Real Variable*, Volume 1, Chapter 9.)

An important feature of the geometric definition of quasiconformality was that a 1-quasiconformal map is conformal. Whatever the analytic definition maybe, it should preserve this form of relationship with conformality. With this in mind, the relevance of the concept of absolute continuity to quasiconformal maps is demonstrated by the following example and the definition given below:

Example 1. We construct an example of a function F defined on an open subset of $\mathbb{C}$ such that $\frac{\partial f}{\partial \bar{z}}$ exists and vanishes almost everywhere but f is not quasi-conformal. It will be clear that absolutely continuity is violated in this example. First we construct a sequence of nondecreasing functions $\{f_n\}$ on the interval $[0, 1]$ converging monotomically to a function f. The sequence is best demonstrated in Figure 1.3.1

where f_1, f_2, f_3 are shown and the general construction is clear from the picture. f has also an analytical description. Let $\mathfrak{C} \subset [0,1]$ denote the Cantor set. A point $x \in \mathfrak{C}$ is characterized by having a tertiary expansion of the form

$$x = \sum_i 2.3^{-n_i},$$

where $n_1 < n_2 < \ldots$. On $\mathfrak{C}$, the function f is given by $f(x) = \sum_i 2^{-n_i}$. On the complement of $\mathfrak{C}$, f is uniquely determined by the requirement to be nondecreasing on $[0,1]$. Consequently on each connected component of the complement of $\mathfrak{C}$ it will be a constant. Now extend f to a function on $(0,1) \times \mathbb{R}$ as

$$f(x+iy) = f(x) + (x+iy).$$

It is immediate that f is a homeomorphism of $(0,1) \times \mathbb{R} \subset (C)$ onto $(0,2) \times \mathbb{R}$. On each connected component of the complement of $\mathfrak{C} \times \mathbb{R} \subset (0,1) \times \mathbb{R}$ (discard the endpoints 0, 1 of the interval $[0,1]$), f ian analytic function of $z = x + iy$. It follows that $\frac{\partial f}{\partial \bar{z}} = 0$ almost everywhere in $(0,1) \times \mathbb{R}$.

Clearly f is not analytic since $\frac{\partial f}{\partial \bar{z}} = 0$ only almost everywhere and the derivative does not even exist on $\mathfrak{C} \times \mathbb{R}$. Whatever definition we want to adopt for quasiconformality should have the the property that if a homeomorphism f is K-quasiconformal and $\frac{\partial f}{\partial \bar{z}} = 0$ almost everywhere than f is conformal. Obviously this is not satisfied for the function f constructed above. □

Let $U \subset \mathbb{C}$ and $R \subset U$ be a rectangle with sides parallel to the axes. A homeomorphism f of U onto its image $f(U) \subset \mathbb{C}$ is *absolutely continuous on lines* (ACL) if it is absolutely continuous on almost every horizontal and vertical line segment in R. If this property holds for every rectangle then we say f is ACL. With this in mind, we give the following analytic definition of quasiconformality: An orientation preserving homeomorphism f of an open subset $U \subset \mathbb{C}$ onto its image $f(U) \subset \mathbb{C}$ is *quasiconformal* if

(1) f is ACL.
(2) $|\frac{\partial f}{\partial \bar{z}}| \leq k |\frac{\partial f}{\partial z}|$ almost everywhere in in U where $k < 1$.

It is a theorem that this definition is equivalent to the geometric definition of K-quasiconformality where $K = \frac{1+k}{1-k}$.

First we explore what properties we like establish for quasi-conformality on the basis of its analytic definition and postpone the the proofs after we have the direction. In doing analysis, one generally needs a function space together with an operator (or operators) acting on it. What is the appropriate function space here? Condition (1) in the analytic definition of quasi-conformality is problematic in the sense that it does not specify a "good" function space as stated. A more suitable definition requires the notion of distributional derivative that I assume is familiar. The distributional derivative of a function is in general a distribution. Recall also that a function f defined on an open subset $U \subset \mathbb{R}^n$ is *locally in* $L^p(U)$ (or $f \in L^p_{\text{loc}}(U)$) if $\phi f \in L^p(U)$ for every compactly supported (in U) smooth function ϕ. This is equivalent to the requirement that every point has a compact neighborhood $V \subset U$ such that the restriction of f to V is in $L^p(V)$. With this in mind we can restate conditions (1) and (2) in the analytic definition as follows:

(1) The distributional derivatives of f are in $L^1_{\text{loc}}(U)$.
(2) $|\frac{\partial f}{\partial \bar{z}}| \leq k|\frac{\partial f}{\partial z}|$ almost everywhere in in U where $k < 1$.

Therefore it is natural to introduce the function spaces $W^{1,p}(U)$ (resp. $W^{1,p}_{\text{loc}}(U)$) as the space of complex valued functions f whose distributional gradient ∇f is in $L^p(U)$ (resp. in $L^p_{\text{loc}}(U)0$.

The analogue of 1-quasiconformal map being conformal in the analytic setting is

Proposition 1. *If* $\frac{\partial f}{\partial \bar{z}} = 0$ *almost everywhere for a quasi-conformal map, then* f *is consformal.*

Another consequence of the analytic definition is

Proposition 2. (Removable Singularities*) Let* $f : U \to \mathbb{C}$ *be a homeomorphism onto its image and* $\gamma : S^1 \to f(U)$ *be a* C^1 *Jordan curve. If* f *is* K*-quasiconformal on* $U \setminus \text{Image}(\gamma)$*, then* f *is* K*-quasiconformal on* U*.*

There are subtle properties of homeomorphisms $f : U \to U'$ lying in $W^{1,p}_{\text{loc}}(U)$ that become useful in the analysis of quasi-conformal maps. A key technical result (not easy to prove) is the Gehring-Lehto theorem that assets that a continuous open mapping of $U \to \mathbb{C}$ is differentiable almost everywhere if and only if it has finite first partial derivatives almost everywhere. This looks like a technical result and to see why it is interesting we note some consequences of it whose relevance is more understandable.

(1) A homeomorphism $f \in W^{1,1}_{\text{loc}}(U)$ is differentiable almost everywhere.
(2) Let $f \in W^{1,1}_{\text{loc}}(U)$ be a homeomorphism $f : U \to U'$. Then the Jacobian determinant $J(z, f)$ does not change sign. (We are looking at $U,\ U'$ as subsets of $\mathbb{R}^2$.)
(3) Let $f \in W^{1,1}_{\text{loc}}(U)$ be a homeomorphism $f : U \to U'$. Then $J(z, f)$ is locally integrable and for every Borel set $E \subset U$ we

have

(12) $$\int_E J(z,f)dxdy \;\le\; \text{meas.}(f(E))$$

(Notice that we are only stating an inequality instead of equality that was in an integral formula that we had previously used.)

(4) With the hypothesis of the previous item assume in addition that $f \in W^{1,2}_{\text{loc}}(U)$, then the area formula

(13) $$\int_E J(z,f)dxdy \;=\; \text{meas.}(f(E))$$

is valid and in particular, sets of measure 0 are mapped to sets of measure 0.

Remark 1. While the geometric definition of quasi-conformality is equivalent to the analytic definition based on $W^{1,1}_{\text{loc}}$, for analytical problems it is perhaps more convenient to require the stronger condition $f \in W^{1,2}_{\text{loc}}$. This is demonstrated in the analogue of Area Formula below. Note also that $W^{1,2}(U)$ has the structure of a Hilbert space and thus it is a more convenient space than $W^{1,1}(U)$ which is a only Banach space. □

The main tool in establishing the existence of limits of sequences in function spaces in elementary analysis is the Arzela-Ascoli theorem that roughly speaking it states that boundedness plus equicontinuity imply relative compactness in the appropriate function space. The condition of equicontinuity is fulfilled if one can establish uniform boundedness of derivatives or the weaker statement of uniformly Hölder will work as well. For quasi-conformal maps there are various ways of stating the Hölder continuity condition.

Theorem 3. *Let $f : U \to U'$ be a K-quasi-conformal homeomorphism.*

(1) *If $U = U' = \mathbb{D}$ and f is surjective, then*

$$|f(z_1) - f(z_2)| \;<\; 16\;|z_1 - z_2|^{1/K}.$$

(2) *More generally, if a disc B and $2B$ (disc with same center and twice radius) lie in U, then*

$$|f(z_1) - f(z_2)| \;\le\; C(K)\frac{\text{diam}(f(B)}{(\text{diam}B)^{1/k}}|z_1 - z_2|^{1/K},$$

where the constant $C(K)$ depends only on K.

A couple of consequences of this theorem are

Corollary 3. *Let $\mathcal{F}$ be the family of K-quasi-conformal homeomorphisms of $\mathbb{D}$ fixing points 0. Then $\mathcal{F}$ is compact in the topology of uniform convergence.*

Corollary 4. *Let $\mathcal{F}$ be the family of K-quasi-conformal homeomorphisms of $\mathbb{CP}(1) \simeq S^2$ fixing three points (say $0, 1$ and ∞). Then $\mathcal{F}$ is compact in the topology of uniform convergence.*

2. Exercises

- **Exercise 1** - Let $h(z) = z + a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots$ be an analytic function which is conformal in the exterior of unit disc. Let $w \in \mathbb{C}$ and assume $h(z) \neq w$ for any z in the exterior of the unit disc. Show that $f(z) = \sqrt{h(z^2) - w}$ is well-defined and conformal in the exterior of unit disc and its power series expansion at ∞ has the form $f(z) = z + \frac{a_0 - w}{2} z^{-1} + \ldots$. Use the Area Formula as in the discussion of Bieberbach-de Brange Theorem for exterior domains to show that $|w - a_0| \leq 2$. (*To be rigorous, you need to say that for any $r > 1$, the restriction of f to the exterior of the disc of radius r extends to an element of $W^{1,2}_{\mathrm{loc}}(\mathbb{C})$ in order for the Area Formula to be applicable. You may ignore this subtle point.*)
- **Exercise 2** - With the notation and hypothesis of Exercise 1, and assume in addition that h is a homeomorphism of $\mathbb{C}$ onto itself. Show that the image of the unit is contained in the disc of radius 2 centered at a_0. (*This may be regarded as the analogue of Koebe's $\frac{1}{4}$ Theorem at infinity.*)
- **Exercise 3** Let f be a conformal mapping of a domain $U \subset \mathbb{C}$ onto $V \subset \mathbb{C}$ and $\zeta \in U$. Show that

$$\frac{1}{4}|f'(\zeta)|\text{dist.}(\zeta, \partial U) \leq \text{dist.}(f(\zeta), \partial V) \leq |f'(\zeta)|\text{dist.}(\zeta, \partial U).$$

  (*Use the function $\frac{f(\zeta + z\delta) - f(\zeta)}{f'(\zeta)\delta}$, where $\delta = \text{dist.}(\zeta, \partial U)$, Koebe's Theorem and Schwarz' Lemma applied to f^{-1}.*)
- **Exercise 4** - Let $f : \mathbb{D} \to \mathbb{C}$ be a conformal mapand let $\zeta \in \mathbb{D}$. Show that

$$(1 - \zeta|^2)\frac{|f''(\zeta)|}{|f'(\zeta)|} \leq 6.$$

  (*Set*

$$\varphi(z) = \frac{f(\frac{z+\zeta}{1+\bar{\zeta}z}) - f(\zeta)}{(1 - |\zeta|^2)f'(\zeta)}.$$

  Then $\varphi(0) = 0$ and $\varphi'(0) = 1$. Apply the inequality $|a_2| \leq 2$ in the Bieberbach-de Brange Theorem.)
- **Exercise 5** - Let M be a Riemann surface whose universal cover is the unit disc $f : \mathbb{D} \to M$. Let $\rho(z)|dz|$ denote the induced Poincaré metric on M on a local coordinate system with z_0 corresponding to the center 0 of disc. Use the Schwarz Lemma to show

$$\rho(z_0) = \frac{1}{|f'(0)|}.$$