1 Complex Geometry

1.1 Almost complex structures and Hermitian metrics

Let M be a real manifold and $J = \{J_x\}$ a smooth family of endomorphisms of tangent spaces $\mathcal{T}_xM$ such that $J_x^2 = -I$ for all $x \in M$. Since the minimal polynomial of J is $\lambda^2 + 1 = 0$, eigenvalues of J_x are $\pm i$ and since each J_x is a real linear transformation, the eigenvalues occur in pairs $\pm i$. In particular, the dimension of M is even which we write as $2m$. Such a pair (M, J) is called an *almost complex manifold.* It is clear that complex manifolds are almost complex by looking at them as real manifolds and the operator J induced by multiplication by i on each tangent space.

The complexifications $\mathcal{T}_xM \otimes \mathbb{C}$ of the tangent spaces split under the action of J into two m-dimensional complex subspaces according to eigenvalues i and $-i$ which we denote respectively by $\mathcal{T}_x^{1,0}M$ and $\mathcal{T}_x^{0,1}M$. At each point $x \in M$ the fibres of these bundles are vectors of the forms

$$\xi - iJ\xi, \qquad \xi + iJ\xi, \quad \text{for } \xi \in \mathcal{T}_xM, \quad \text{respectively.}$$

Dually we have the decomposition of the complexification of the cotangent bundle into $\mathcal{T}_xM \otimes \mathbb{C} = \mathcal{T}^{\star\ 1,0}M \oplus \mathcal{T}_x^{\star\ 0,1}M$ and the fibres are covectors of the form

$$\phi + iJ\phi, \qquad \phi - iJ\phi, \quad \text{for } \phi \in \mathcal{T}_x^\star M, \quad \text{respectively.}$$

This decomposition extends to higher tensor powers. For example, the l^{th} exterior power of the complexification of the cotangent bundle decomposes into a direct sum

$$\Lambda^l \mathcal{T}_x^\star M \otimes \mathbb{C} \;=\; \sum_{p+q=l} \mathcal{T}^{\star\ p,q}M,$$

where $\mathcal{T}_x^{\star\ p,q}M$ is obtained from the wedge product of p copies of $\mathcal{T}_x^{\star\ 1,0}M$ and q copies of $\mathcal{T}_x^{\star\ 0,1}M$.

On a complex manifold M the operator d has a decomposition into $d = \partial + \bar{\partial}$. For example if f is function on M and $\{z_1, \ldots, z_m\}$ are complex coordinates then

$$\partial f = \sum_{k=1}^{m} \frac{\partial f}{\partial z_k} dz_k, \quad \bar{\partial} f = \sum_{k=1}^{m} \frac{\partial f}{\partial \bar{z}_k} d\bar{z}_k,$$

where for the decomposition into real and imaginary parts $z_k = x_k + iy_k$, $\frac{\partial}{\partial z_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k}\right)$ and $\frac{\partial}{\partial \bar{z}_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} + i\frac{\partial}{\partial y_k}\right)$. The extension to forms is in the obvious fashion. Therefore if $\Gamma(M, \mathcal{T}^{\star p,q}M)$ denotes the space of sections of (p,q)-forms on a complex manifold M then

$$d\Gamma(M, \mathcal{T}^{\star p,q}M) \subset \Gamma(M, \mathcal{T}^{\star p+1,q}M) \oplus \Gamma(M, \mathcal{T}^{\star p,q+1}M). \tag{1.1}$$

On an almost complex manifold we can only assert

$$d\Gamma(M, \mathcal{T}^{\star p,q}M) \subset \sum_{j+k=p+q+1} \Gamma(M, \mathcal{T}^{\star j,k}M).$$

In fact we have the fundamental result

Theorem 1.1 *A necessary and sufficient condition for an almost complex structure to be the underlying one for a complex structure is*

$$d\Gamma(M, \mathcal{T}^{\star 1,0}M) \subset \Gamma(M, \mathcal{T}^{\star 2,0}M) \oplus \Gamma(M, \mathcal{T}^{\star 1,1}M), \quad \text{or}$$
$$d\Gamma(M, \mathcal{T}^{\star 0,1}M) \subset \Gamma(M, \mathcal{T}^{\star 1,1}M) \oplus \Gamma(M, \mathcal{T}^{\star 0,2}M).$$

The proof of this theorem (due to Newlander and Nirenberg) will not discussed at this point and it involves subtle points of analysis. An almost complex structure satisfying the requirements of Theorem 1.1 is called *integrable.*

Remark 1.1 In practise, rather than directly finding the endomorphisms J_z, $z \in U$, defining the almost complex structure, one defines a pair of projections $P_{1,0}$ and $P_{0,1}$ on the space $\mathcal{A}^1$ of C^∞ complex valued 1-forms on M such that

1. $P_{1,0}$ and $P_{0,1}$ are linear relative to the $\mathcal{A}^\circ$-module structure of $\mathcal{A}^1$, where $\mathcal{A}^\circ$ is the space of complex-valued C^∞ functions.
2. $P_{1,0} + P_{0,1} = \text{Id.}$, and $P_{1,0}P_{0,1} = P_{0,1}P_{1,0} = 0$.
3. For $\eta \in \mathcal{A}^1$, $P_{0,1}(\bar{\eta}) = \overline{P_{1,0}(\eta)}$.

It is clear that the annihilators of $\text{Im}(P_{1,0})$ and $\text{Im}(P_{0,1})$ give a decomposition of the complexified tangent spaces into m-dimensional complex subspaces on which the almost complex structure operator acts as $\pm i$. The integrability condition of Theorem 1.1 is that $d\eta$, for a $(0,1)$-form η has no $(2,0)$-component and from this it follows that for a (p,q)-form η, $d\eta$ has only $(p+1,q)$ and $(p,q+1)$ components. This enables one to define operators ∂ and $\bar{\partial}$ such that $d = \partial + \bar{\partial}$ and transfer the machinery for solving $\bar{\partial}u = \eta$ to the almost complex case. Let $\eta_1, \ldots, \eta_m$ be 1-forms spanning $\text{Im}(P_{1,0})$ as an $\mathcal{A}^\circ$-module. Then the integrability criterion of Theorem 1.1 translates into $\mathcal{J}$ being closed under d where $\mathcal{J}$ is the ideal generated by $\eta_1, \ldots, \eta_m$. The essential part of the proof of Theorem 1.1 is to establish existence of functions $u_1, \ldots, u_m$ such that $du_1, \ldots, du_m$ span $\text{Im}(P_{1,0})$ as an $\mathcal{A}^\circ$-module. It is in this form that Theorem 1.1 is often used. ∎

Let M be an almost complex manifold with the almost complex structure defined by a tensor J. By an Hermitian metric h on M we mean the smooth assignment of Hermitian inner products relative to the complex structure defined by J on each tangent space. Therefore

$$h(J\xi, J\eta) = h(\xi, \eta), \tag{1.2}$$

where $\xi, \eta \in \mathcal{T}_x M$ and all $x \in M$. Consequently, the transformation J is a proper orthogonal transformation in each tangent space, and there are local frames $e_1, \ldots, e_{2m}$ and coframes $\omega_1, \ldots, \omega_{2m}$ for the tangent and cotangent spaces of M relative to which

$$h = \omega_1^2 + \ldots + \omega_{2m}^2, \quad J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

The matrix representation of J is relative to $e_1, \ldots, e_{2m}$, and its transpose describes its action on the coframe.

The metric h, regarded as a Riemannian metric, admits of a (unique) Levi-Civita connection (ω_{ij}). In general, the corresponding operator of covariant differentiation does not commute with the action of J and our immediate goal is to investigate when this condition holds. In view of the defining relation $d\omega_j + \sum \omega_{jk} \wedge \omega_k = 0$ for the Levi-Civita connection, the condition of commutativiy of J (or its transpose) with the covariant derivative reduces to the commutativity of the matrices (ω_{jk}) and J. Now a matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ commutes with J if and only if it is of the form

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \tag{1.3}$$

To put this commutativity condition in a more useful form technically, we note that an Hermitian metric decomposes into real and imaginary parts, with its real part symmetric and imaginary part skew-symmetric. For example for the standard Hermitian inner product on $\mathbb{C}^m$ we have

$$< z, w >= \sum_{j=1}^{m} (x_j u_j + y_j v_j) + i \sum_{j=1}^{m} (x_j v_j - y_j u_j).$$

It follows easily that for the Hermitian metric $h = \sum \omega_j^2$ (Hermitian relative to J) the imaginary part of h is

$$\varphi = \sum_{j=1}^{m} \omega_j \wedge \omega_{j+m}.$$

We refer to φ as the *Kähler form* associated to the Hermitian metric h. We have

Proposition 1.1 *Let (M, J) be an almost complex manifold and h an Hermitian metric on M. The covariant derivative relative to the associated Levi-Civita connection commutes with J if and only if the associated Kähler form is closed.*

Proof - This is a straightforward computation. In fact calculating the coefficients of $\omega_j \wedge \omega_k$, $\omega_j \wedge \omega_{k+m}$ and $\omega_{j+m} \wedge \omega_{k+m}$ in the 3-form $d\varphi$ using the defining relation $d\omega_j + \sum \omega_{jk} \wedge \omega_k = 0$ one sees easily that $d\varphi = 0$ is equivalent to the skew-symmetric matrix (ω_{jk}) being of the form $\begin{pmatrix} \eta & \theta \\ -\theta & \eta \end{pmatrix}$. ■

An (almost) complex manifold (M, J) with an Hermitian metric h satisfying the condition $d\varphi = 0$ of Proposition 1.1 is called an (almost) *Kähler manifold*, and the corresponding Hermitian metric a *Kähler metric.* Note that the condition of commutativity of J and the covariant derivative is equivalent to parallelism of J , i.e., $D(J) = 0$. In fact, for a vector field ξ we have

$$D(J\xi) = D(J)\xi + JD\xi,$$

from which the claim follows.

Example 1.1 Kähler manifolds exist in abundance. Consider $\mathbb{C}^m$ with the standard Hermitian $<,>$. Clearly this is a Kähler metric. Let Λ be a lattice in $\mathbb{C}^m$ so that the quotient space is $\mathbb{C}^m/\Lambda$ is a complex torus. In view of the translation invariance of $<,>$, there is an induced Kähler metric on the complex torus $\mathbb{C}^m/\Lambda$. A more sophisticated example is the Fubini-Study metric on the complex projective space. More generally, regarding the Grassmann manifold $\mathsf{G}_{k,n-k}$ (k-linear subspaces of $\mathbb{C}^n$) as the homogeneous space G/K with $G = U(n)$ and $K = U(k) \times U(n-k)$, there is a unique, up to multiplication by a constant, G-invariant Hermitian metric on it. Identifying the the tangent space at eK with the space $\mathcal{M}$ of skew-hermitian matrices of the form

$$\begin{pmatrix} 0 & S \\ -\bar{S}' & 0 \end{pmatrix},$$

G-invariant Hemitian metrics on $\mathsf{G}_{k,n-k}$ are in natural bijection with $\mathrm{Ad}(K)$-invariant Hermitian inner products on $\mathcal{M}$. There is a unique, up to multiplication by a constant, such inner product and is given by $< \xi, \eta >= -\mathrm{Tr}(\xi\eta)$, where $\xi, \eta \in \mathcal{M}$. The Kähler form φ is a G-invariant 2-form and $d\varphi = 0$. In fact, note $[\mathcal{M}, \mathcal{M}] \subset \mathcal{K}$, where $\mathcal{K}$ is the set of skew-hermitian matrices of the form $\begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix}$ (i.e., the Lie algebra of $U(k) \times U(n-k)$. The expression $d\varphi(\xi_1, \xi_2, \xi_3)$, where ξ_j's are left invariant vector fields on G, consists of sums of terms of the form $\xi_i\varphi(\xi_k,\xi_l)$ and $\varphi([\xi_i,\xi_j],\xi_k)$. The former term vanishes since by invariance of φ, $\varphi(\xi_k,\xi_l)$ is a constant, and the latter term vanishes since $[\xi_i,\xi_j]\in\mathcal{K}$. The Hermitian metric in question is in fact Kähler. ■

It is an elementary but useful observation that the imaginary part of an Hermitian form determines it. In fact if a skew symmetric non-degenerate bilinear form φ is given then the Hermitian metric is given by

$$h(\xi,\eta)=\varphi(J\xi,\eta)+i\varphi(\xi,\eta).$$

It is quite useful and enlightening to obtain local coordinate expressions for the Fubini-Study metric on $\mathbb{C}\mathcal{P}(n)$.

Example 1.2 Let $[z_\circ,\dots z_n]$ denote homogeneous coordinates for $\mathbb{C}\mathcal{P}(n)$, and set

$$\varphi=-4i\partial\bar\partial\log(|z_\circ|^2+\dots+|z_n|^2).$$

Then φ is a real 2-form on $\mathbb{C}\mathcal{P}(n)$ (regarded as a real manifold) since it is invariant under the multiplication of z_j's by a non-zero complex number λ. It is clearly invariant under the action of the unitary group $U(n+1)$ and is non-degenerate. Therefore φ is the Kähler form of the Hermitian metric associated to it, namely, $ds^2(\xi,\eta)=h(\xi,\eta)=\varphi(i\xi,\eta)+i\varphi(\xi,\eta)$. To obtain local expressions, let U_α be the open subset defined by $z_\alpha\neq 0$. Set $u_{j,\alpha}=\frac{z_j}{z_\alpha}$, then $u_{j,\alpha}$'s with $j\neq\alpha$ form local coordinates on U_j and we set $u_{\alpha,\alpha}=1$. Define

$$f_\alpha=\sum_{j=0}^{n}|u_{j,\alpha}|^2.$$

In view of the transformation property $f_\alpha=f_\beta|u_{\beta,\alpha}|^2$ on $U_\alpha\cap U_\beta$ we have

$$\partial\bar\partial\log f_\alpha=\partial\bar\partial\log f_\beta,\qquad \text{on}\quad U_\alpha\cap U_\beta,$$

so that the expression $-4i\partial\bar\partial f_\alpha$ is the local expression on U_α of a globally defined 2-form on $\mathbb{C}\mathcal{P}(n)$. Clearly it is the 2-form φ (associated to h) defined above. The corresponding metric has the expression

$$ds^2=\frac{\left(1+\sum_{\alpha=1}^n|u_\alpha|^2\right)\left(\sum_{\alpha=1}^n du_\alpha d\bar u_\alpha\right)-\left(\sum_{\alpha=1}^n\bar u_\alpha du_\alpha\right)\left(\sum_{\alpha=1}^n u_\alpha d\bar u_\alpha\right)}{\left(1+\sum_{\alpha=1}^n|u_\alpha|^2\right)^2},$$

on $U_\circ$. ■

Example 1.3 The local description of the $U(n+1)$-invariant metric can be extended to the complex Grassmann manifold $\mathsf{G}_{k,n-k}$. We will indicate briefly how this is done. A linear k-dimensional subspace of $\mathbb{C}^n$can be described (non-uniquely) by a set of k column vectors

$$Z = \begin{pmatrix} z_1^1 & z_1^2 & \dots & z_1^k \\ z_2^1 & z_2^2 & \dots & z_2^k \\ \vdots & \vdots & \ddots & \vdots \\ z_n^1 & z_n^2 & \dots & z_n^k \end{pmatrix}$$

Then it is easily verified that

$$\varphi = -4i\partial\bar{\partial} \log \det(Z^\star Z),$$

where $Z^\star$ denotes the conjugate transpose of Z, is a well-defined real non-degenerate $(1,1)$-form on on $\mathsf{G}_{k,n-k}$. It is clearly invariant under the action of $U(n)$, and the associated Hermitian form is the required Kähler metric. ∎

Remark 1.2 The local expressions for the Kähler metric as $\partial\bar{\partial}f$ for projective spaces and Grassmann manifolds is not accidental. In fact, since $d\varphi = 0$ and φ is a $(1,1)$-form, $\partial\varphi = 0$ and $\bar{\partial}\varphi = 0$. The former equation implies $\varphi = \partial\eta$ (locally) for a $(0,1)$-form η. Now

$$0 = \bar{\partial}\varphi = \bar{\partial}\partial\eta = -\partial\bar{\partial}\eta.$$

Therefore $\bar{\partial}\eta = \partial u$ for a $(0,1)$-form u. But the left hand side is a $(0,2)$-form while the right hand side is a $(1,1)$-form. Hence $\bar{\partial}\eta = 0$ and $\eta = \bar{\partial}h$ and $\varphi = \partial\bar{\partial}h$ locally. ♡

So far we have shown that complex tori and complex projective and Grassmann manifolds are examples of compact Kähler manifolds. The following simple observation provides many more examples of Kähler manifolds:

Lemma 1.1 *Let (M, J) be an almost Kähler manifold with hermitian metric h, and $N \subset M$ a submanifold on which J induces an almost complex structure. Then (N, J) is an almost Kähler manifold.*

Proof - The hypothesis means that the tangent spaces to N are invariant under the action of J. Since the metric is given by $h(\xi, \eta) = \varphi(J\xi, \eta) + i\varphi(\xi, \eta)$, the metric h induces a hermitian metric h_N on N, and the restriction φ_N of φ to N is the imaginary part of h_N. Clearly $d\varphi_N = 0$ and (N, J) with the metric h_N is almost Kähler. ∎

Lemma 1.1 in particular implies that all complex submanifolds of complex projective spaces are Kähler. However there are many Kähler manifolds which are not complex submanifolds of any complex projective space. The determination of when a Kähler manifold is a submanifold of a complex projective space (in other words, it is algebraic) is a deep problem which will be discussed later.

The volume element associated to a Riemannian metric played an important role in differential geometry of real manifolds. From the moving frame expression of the volume element of a Riemannian metric we immediately obtain the following:

Lemma 1.2 *Let M an almost Kähler manifold of real dimension $2m$ relative to the Hermitian metric h. Then the volume element associated to h is the m-fold wedge product of φ with itself, that is $\varphi \wedge \ldots \wedge \varphi$.*

1.2 Holomorphic Vector Bundles

Let $\pi : E \to M$ be a smooth complex vector bundle on a manifold M, and let l be the (complex) dimension of a fibre. By an *Hermitian metric* h on E we mean the smooth assignment of Hermitian inner products on the fibres E_z for all $z \in M$. A complex vector bundle together with an Hermitian metric is called an *Hermitian vector bundle.* The bundle is locally trivialized by the choice of a frame $e_1^\alpha, \ldots, e_l^\alpha$ on U_α where $\mathcal{U} = \{U_\alpha\}$ is a sufficiently fine covering of M. Then on U_α the Hermitian metric h is a smooth mapping h_α from U_α into the space of $l \times l$ Hermitian matrices. Furthermore the family $\{h_\alpha\}$ satisfies the transformation property

$$h_\alpha = \varphi_{\alpha\beta}^\star h_\beta \varphi_{\alpha\beta},$$

where * denotes the conjugate transpose of the matrix and $\varphi_{\alpha\beta}$ is the matrix valued function on $U_\alpha \cap U_\beta$ describing the change of frames.

In order to develop the differential geometry of vector bundles it is necessary to generalize the notion of connection from (co)tangent bundle of a manifold to that of a (complex) vector bundle. The key property of the connection that we need to adapt to the new situation is that the connection matrix ω (in the Riemannian case) enabled us to calculate, in principle, the (exterior) derivative of the basic 1-forms $\omega_1, \ldots, \omega_m$ which constituted local frames for the cotangent bundle. Instead of basic 1-forms here we have local frames for the bundle $s_1^\alpha, \ldots, s_l^\alpha$ for the vector bundle $\pi : E \to M$ for each open set U_α of (sufficiently fine) covering $\mathcal{U} = \{U_\alpha\}$. Now suppose we are given $l \times l$ matrices of 1-forms (ω_{jk}^α) defined on the open sets U_α. Since exterior differentiation of sections of a vector bundle has no meaning in general we define the *covariant derivative* of the frames $s_1^\alpha, \ldots, s_l^\alpha$ as

$$Ds_k^\alpha = \sum_{j=1}^{l} \omega_{jk} s_j^\alpha$$

Furthermore for any local section $s^\alpha = \sum \phi_k s_k^\alpha$, where ϕ_k's are complex valued smooth functions, we extend the definition of D by the requirement of the Leibnitz rule

$$Ds^\alpha = \sum_{k=1}^{l} (d\phi_k) s_k^\alpha + \sum_{k=1}^{l} \phi_k D s_k^\alpha.$$

With no condition on (ω_{jk}^α) there is no reason for Ds^α to be well-defined. Let $A : U \to GL(l, \mathbb{C})$ be a smooth mapping describing the change of frame on an open set U (e.g. $U = U_\alpha \cap U_\beta$). In fact assume the frames s_k^α and s_k^β are related by

$$s_k^\alpha = \sum_{j=1}^{l} A_{jk} s_j^\beta.$$

Applying the covariant differentiation operator D and substituting (ω_{jk}^{α}) on the left hand side and (ω_{jk}^{β}) on the right hand side we immediately obtain the $l \times l$ matrix equation

$$(\omega_{jk}^{\alpha}) = A^{-1}(\omega_{jk}^{\beta})A \; + \; A^{-1}dA. \tag{1.4}$$

It is also immediate that if the equation (1.4) is satisfied then the operator D of covariant derivative is well-defined. *Henceforth by a* **connection** *on a complex vector bundle we mean either the operator D or equivalently a collection of locally defined matrices of 1-forms (ω_{jk}^{α}) such that (1.4) is fulfilled.* It is customary and sometimes useful to regard the connection as defined on the corresponding bundle of frames $P_E \rightarrow M$ since the matrices (ω_{jk}^{α}) depend on the choice of the frame.

While the (Levi-Civita) connection enabled one to differentiate tensors on a Riemannian manifold, the key geometric object was the curvature form Ω associated to the Levi-Civita connection, and was defined as the failure of the validity of the structure equations for Euclidean space, namely, $\Omega = d\omega + \omega \wedge \omega$. A similar definition is possible in the case of a vector bundle. In fact, observe that as in the Riemannian case we have (we write ω^{α} for the matrix (ω_{jk}^{α}))

$$\begin{aligned} d\omega^{\alpha} + \omega^{\alpha} \wedge \omega^{\alpha} \;\; &= \;\; -A^{-1}dA \wedge A^{-1}\omega^{\beta}A + A^{-1}d\omega^{\beta}A \\ &\quad -A^{-1} \wedge \omega^{\beta} \wedge dA - A^{-1}(dA)A^{-1} \wedge dA \\ &\quad +A^{-1}\omega^{\beta}A \wedge A^{-1}\omega^{\beta}A + A^{-1}\omega^{\beta}A \wedge A^{-1}dA \\ &\quad +A^{-1}dA \wedge A^{-1}\omega^{\beta}A + A^{-1}dA \wedge A^{-1}dA \\ &= A^{-1}\Big(d\omega^{\beta} + \omega^{\beta} \wedge \omega^{\beta}\Big)A. \end{aligned}$$

Therefore if we emulate the Riemannian case and define the curvature matrix (relative to a fixed frame on U_{α}) as $\Omega^{\alpha} = d\omega^{\alpha} + \omega^{\alpha} \wedge \omega^{\alpha}$, then the matrices Ω^{α} of 2-forms have similarly the (nice) transformation property

$$\Omega^{\alpha} = A^{-1}\Omega^{\beta}A. \tag{1.5}$$

The entries of the matrices Ω^{α} are 2-forms and therefore they commute (under wedge product). Therefore it makes sense to think of symmetric functions of the characteristic roots of the matrices Ω^{α} and these symmetric functions are expressible as polynomials in the entries of the matrices Ω^{α} and are independent of the choice frame on U_{α}. In particular these symmetric functions are globally defined forms on the manifold M and since they are independent of the choice of frame, it is reasonable to surmise that they contain essential geometric information about the vector bundle $E \rightarrow M$. This idea is central to the development of complex geometry of vector bundles and will be exploited in the sequel.

There is a different way of looking at the connection and curvature which is useful. Let $\mathcal{A}^r(E)$ (resp. $\mathcal{A}^{p,q}(E)$) denote the bundle obtained by the tensor product of $E \to M$ and $\Lambda^r \mathcal{T}^\star M$ (resp. $\mathcal{T}^{\star p,q} M$) and $\Gamma(E, M)$ or simply $\Gamma(E)$ the space of sections of $E \to M$. Then elements of $\Gamma(\mathcal{A}^r \mathcal{T}^\star M)$ (resp. $\Gamma(\mathcal{A}^{p,q} M)$) are finite sums of terms of the form $\omega \otimes s$ or simply $\omega.s$ where ω is an r-form (resp. (p,q)-form). Then just as in the case of sections of $E \to M$ a connection extends to mapping

$$D : \Gamma(\mathcal{A}^r(E)) \longrightarrow \mathcal{A}^{r+1}(E),$$

by the signed Leibnitz rule (that is, $D(\omega.s) = d\omega.s + (-1)^{\deg \omega} \omega \wedge Ds$). For M a complex manifold we furthermore have

$$D : \Gamma(\mathcal{A}^{p,q}(E)) \longrightarrow \Gamma(\mathcal{A}^{p+1,q} M) \oplus \Gamma(\mathcal{A}^{p,q+1}(E)). \tag{1.6}$$

In particular, $D : \Gamma(E) \longrightarrow \mathcal{A}^1(E)$ and

$$D \circ D : \Gamma(E) \longrightarrow \Gamma(\mathcal{A}^2(E)).$$

Let $R = D \circ D$. Regarding $\Gamma(E)$ and $\Gamma(\mathcal{A}^2(E)$ as modules over the ring $\mathcal{A}$ of smooth complex valued functions on M, we note that R is a map of $\mathcal{A}$-modules. In fact,

$$R(fs) = D(df.s + f.Ds) = (-df.Ds + df.Ds + f.D \circ Ds) = fR(s).$$

Let $s_1^\alpha, \ldots, s_l^\alpha$ be a local frame on an open set U_α. Then

$$\begin{aligned} Rs_k^\alpha &= D(\sum_{j=1}^{l} \omega_{jk}^\alpha s_j^\alpha) \\ &= \sum_{j=1}^{l} \left(d\omega_{jk}^\alpha . s_j^\alpha - \sum_{n=1}^{l} \omega_{jk}^\alpha \wedge \omega_{nj}^\alpha . s_n^\alpha \right) \\ &= \sum_{n=1}^{l} (d\omega^\alpha + \omega^\alpha \wedge \omega^\alpha)_{nk} . s_n^\alpha . \end{aligned}$$

Therefore $R = D \circ D$ is simply the curvature matrix as defined earlier, which can also be expressed by saying that R (or curvature) is a 2-form on M with values in $\mathrm{End}(E)$. The fact that explicit differentiation does not appear in $R = D \circ D$ explains the validity of the relation $R(fs) = fR(s)$.

An important property of the Levi-Civita connection in Riemannian geometry is the fact the metric tensor is parallel relative to the connection. This means $D(g) = 0$ or equivalently

$dg(\xi, \eta) = g(D\xi, \eta) + g(\xi, D\eta)$. In analogy with the Riemannian case we call a connection (on a Hermitian vector bundle) *Hermitian* if

$$dh(\xi, \eta) = h(D\xi, \eta) + h(\xi, D\eta),$$

where h denotes the Hermitian inner product on the vector bundle. We have

Lemma 1.3 *For an Hermitian connection, the connection form and the curvature matrix are skew-Hermitian matrices when expressed relative to an orthonormal frame.*

Proof - Let $e_1, \ldots, e_l$ be an orthonormal frame, then $dh(e_j, e_k) = \delta_{jk}$ and consequently

$$h(\omega_{kj}e_k, e_k) + h(e_j, \omega_{jk}e_j) = 0,$$

from which the skew-Hermitian character of the connection matrix follows. That the curvature matrix is skew-Hermitian now follows from that of ω the defining equation $\Omega = d\omega + \omega \wedge \omega$ of curvature. ■

So far we have made no use of the holomorphic structure. By (1.6) we can decompose the operator of covariant differentiation D into $D = D' + D''$ and then the curvature R decomposes into three components

$$R = D' \circ D' + (D' \circ D'' + D'' \circ D') + D'' \circ D''.$$

On a holomorphic vector bundle we have connections D with a particularly simple D''-part. But first note that for (local) sections s of a holomorphic bundle $E \to M$, $\bar{\partial}s$ is well-defined and is a section of $\mathcal{T}^{*0,1} \otimes E \to M$ (or element of $\mathcal{A}^{0,1}(E)$. We have

Lemma 1.4 *Let $E \to M$ be a holomorphic vector bundle and M be a complex manifold. Then there is a connection D on $P_E \to M$ such that $D'' = \bar{\partial}$.*

Remark 1.3 The statement $D'' = \bar{\partial}$ means that relative to a (local) holomorphic frame the connection matrix (ω_{jk}) is a matrix of $(1,0)$-forms. Thus if $s_1, \ldots, s_l$ is a local holomorphic frame, then for $s = \sum f_j s_j$ we have

$$Ds = \sum_{j,k} f_j \omega_{kj} s_k + \sum_j \partial f_j . s_j + \sum_j \bar{\partial} f_j . s_j .$$

The first two sums are $(1,0)$-forms (the term $D's$) and the third sum consists of $(0,1)$-forms (the term $D''s$). ♡

Proof - Let $\mathcal{U} = \{U_\alpha\}$ be a locally finite covering of M such $E_{|U_\alpha} \to U_\alpha$ is holomorphically trivial for every α, and fix a holomorphic trivializing frame $s_1^\alpha, \ldots, s_l^\alpha$ on each U_α. Let $\{\phi_\alpha\}$ be a partition of unity, set $D^\alpha s_k^\alpha = 0$ and define the connection D as

$$D = \sum_\alpha \phi_\alpha D^\alpha.$$

Let $\rho^{\beta\alpha} : U_\alpha \cap U_\beta \to GL(l, \mathbb{C})$ be holomorphic transition functions for the bundle $E \to M$ relative to the above holomorphic trivializations, and then $\rho^{\alpha\beta}$ is the inverse of $\rho^{\beta\alpha}$. Thus $s_k^\alpha = \sum_j \rho_{jk}^{\beta\alpha} s_j^\beta$, and

$$\begin{aligned} Ds_k^\alpha &= \sum_{\beta,j} \phi_\beta \partial(\rho_{jk}^{\beta\alpha}) s_j^\beta \\ &= \sum_{n=1}^{l} \sum_{\beta,j} \phi_\beta \partial(\rho_{jk}^{\beta\alpha}) \rho_{nj}^{\alpha\beta} s_n^\alpha. \end{aligned}$$

Note that in view of the holomorphy condition only the $(1,0)$ component $\partial\rho^{\beta\alpha} = d\rho^{\beta\alpha}$ appears. From this it follows that the D''-component of Ds of a local section $s = \sum_k f_k s_k^\alpha$ is simply $\sum_k \bar\partial(f_k) s_k^\alpha$ which proves the lemma. ∎

If in addition we have an Hermitian metric on the holomorphic vector bundle $E \to M$ then we can make a unique choice for the connection, which may be regarded as the holomorphic analogue of the Levi-Civita connection, as shown by the following:

Proposition 1.2 *Given an Hermitian metric on the holomorphic vector bundle $E \to M$ there is a unique Hermitian connection D such that $D'' = \bar\partial$.*

Proof - First we prove uniqueness. Let $\mathcal{U} = \{U_\alpha\}$ be a covering of M holomorphically trivializing the bundle $E \to M$ and $s_1^\alpha, \ldots, s_l^\alpha$ be a holomorphic frame on U_α. The matrix of the Hermitian inner product relative to this frame is (h_{jk}^α) where

$$h_{jk}^\alpha = h(s_j^\alpha, s_k^\alpha).$$

Now $dh_{jk}^\alpha = h^\alpha(Ds_j^\alpha, s_k^\alpha) + h^\alpha(s_j^\alpha, Ds_k^\alpha)$ and $Ds_j^\alpha = D's_j^\alpha \in \mathcal{A}^{1,0}(E)$ since $D'' = \bar\partial$ and $\{s_j^\alpha\}$ is a local holomorphic frame. Therefore (using the convention that $h(.,.)$ is $\mathbb{C}$-linear in the second and anti-linear in the first variable)

$$\begin{aligned} \partial h_{jk}^\alpha &= \sum_{n=1}^{l} h^\alpha(s_j^\alpha, \omega_{nk}^\alpha s_n) \\ &= \sum_{n=1}^{l} h_{jn}^\alpha \omega_{nk}^\alpha. \end{aligned}$$

That is, $(h^\alpha)^{-1}(\partial h^\alpha) = \omega^\alpha$ which proves the uniqueness. The existence follows from the fact that the matrices of $(1,0)$-forms $\omega^\alpha = (h^\alpha)^{-1}(\partial h^\alpha)$ and ω^β satisfy the transformation property (1.5) characterizing a connection and give the desired connection. ∎

The connection uniquely specified by Proposition 1.2 is called the *Chern connection* for the holomorphic vector bundle $E \to M$ equipped with an Hermitian metric.

Corollary 1.1 *Let $E \to M$ be a holomorphic complex vector bundle with an Hermitian metric h, then the curvature form Ω of the Chern connection is a matrix of* $(1,1)$*-forms.*

Proof - The curvature matrix $\Omega \leftrightarrow D \circ D$ has no $(0,2)$-component since $D'' = \bar{\partial}$. Furthermore, the Hermitian character of the connection implies that Ω is skew-hermitian relative to h, i.e.,

$$\Omega^* h + h\Omega = 0.$$

Therefore Ω has no $(2,0)$-component either. ∎

The following theorem is an important tool in the differential geometry of complex vector bundles and its proof is an enlightening application of some of the concepts we have introduced:

Theorem 1.2 *Let M be a complex manifold and $E \to M$ a C^∞ complex vector bundle with a connection D such that $D'' \circ D'' = 0$. Then there is a unique complex structure on E making $E \to M$ a holomorphic vector bundle such that $D'' = \bar{\partial}$.*

Proof - Let $\mathcal{U} = \{U_\alpha\}$ be an open covering of M locally trivializing the bundle $\pi : E \to M$. We describe complex structures locally on opens sets $U = U_\alpha$ such that their compatibility is immediate. Let $U = U_\alpha, E_{|U} \simeq U \times \mathbb{C}^l$, $z_1, \ldots, z_m$ complex coordinates in U and $w_1, \ldots, w_l$ standard coordinates of the fibre $\mathbb{C}^l$. To obtain the required complex structure on $U \times \mathbb{C}^l$ we exhibit an almost complex structure on $E_{|U}$ and verify the integrability criterion of Theorem (1.1. In view of Remark 1.1 we exhibit a family of $(1,0)$-forms on E and verify that the ideal generated by these forms is d-closed which establishes the existence of the required complex structure. Consider the $m + l$ 1-forms

$$dz_a, \quad dw_i - \sum_{j=1}^{l} w_j \omega''_{ji}, \quad a = 1, \ldots, m, \quad i = 1, \ldots l, \tag{1.7}$$

where ω''_{ij} is the $(0,1)$-part of the connection matrix. The 1-forms in (1.7) are linearly independent over $\mathcal{A}^\circ$ and we want to verify that the ideal generated by them is d-closed. We have

$$
\begin{aligned}
d(dw_i - \sum_{j=1}^{l} w_j \omega''_{ji}) &= -\sum dw_j \wedge d\omega''_{ji} - \sum w_j \wedge d\omega''_{ji} \\
&\equiv -\sum w_k \Big(d\omega''_{ki} + \sum \omega''_{kj} \wedge \omega''_{ji} \Big) \quad \bmod \mathcal{J} \\
&\equiv -\sum w_k \partial \omega''_{ki} \quad \bmod \mathcal{J},
\end{aligned}
$$

where for the first $\equiv$ we substituted from $dw_i - \sum_{j=1}^{l} w_j \omega''_{ji} = 0$ and the second follows from the hypothesis $D'' \circ D'' = 0$. Now the $(1,1)$-form $\partial \omega''_{ij}$ is in the ideal generated by differentials dz_a's, and therefore $\mathcal{J}$ is d-closed which gives a complex structure on $\pi^{-1}(U) = E_{|U}$. Of course this calculation was local and to obtain a global holomorphic structure on E we must show that the subspaces defined by the forms (1.7) can be glued together to make sense globally. To do so write w_j's as a row vector w and let $\omega = (\omega_{jk})$. For coordinates $u = (u_1, \ldots, u_l)$ defined relative to another frame related to first by an invertible matrix A (of functions) let $\tilde{\omega}$ be the corresponding connection form. Then $w = uA$ and $\omega = A^{-1}\tilde{\omega}A + A^{-1}dA$ and we obtain

$$
\begin{aligned}
dw - w\omega &= du.A + udA - uA.A^{-1}\tilde{\omega}A - uA.A^{-1}dA \\
&= (du - u\tilde{\omega})A.
\end{aligned}
$$

Since ω'_{jk} and $\tilde{\omega}'_{jk}$ are in the ideal generated by dz_a's, the subspaces defined by (1.7) are globally defined and we have endowed E with a global holomorphic structure. The fact that the projection π (locally given by $(z, w) \to z$) is holomorphic, i.e., $E \to M$ is a holomorphic vector bundle follows from the fact dz_a's are in the ideal $\mathcal{J}$. To show that $D'' = \bar{\partial}$ it suffices to prove the following:

- For every local section $s : U \to E_{|U}$ such that $D''s = 0$ and every $(1,0)$-form η on $E_{|U}$, the pull-back $s^\star(\eta)$ is a $(1,0)$-form on U.

Now writing the section s as $s(z) = (z, w)$, the condition $D''s = 0$ becomes

$$
\bar{\partial} w_i - \sum_{j=1}^{l} w_j \omega''_{ji} = 0. \tag{1.8}
$$

Therefore for the pull-back by s of $(1,0)$ forms we obtain

$$
s^\star(dz_a) = dz_a, \quad s^\star(dw_i - \sum w_j \omega''_{ji}) = \partial w_i
$$

in view of (1.8). Therefore condition • is satisfied. Since the uniqueness is clear, the proof of the Theorem is complete. ∎

Example 1.4 Consider a Kähler manifold of (complex) dimension 2 and let (ω_{jk}) be the Levi-Civita connection relative to an orthonormal coframe $\omega_1, \ldots, \omega_4$ for $T^\star M$ regarded as a real manifold. A basis for $\mathcal{T}^{\star\, 1,0}M$ is given by the complex 1-forms $\psi_1 = \omega_1 + i\omega_3$ and $\psi_2 = \omega_2 + i\omega_4$. Then the defining equation $d\omega_j + \sum \omega_{jk} \wedge \omega_k = 0$ for the Levi-Civita connection translates into the equations

$$d\psi_1 - i\omega_{13} \wedge \psi_1 + (\omega_{12} - i\omega_{14}) \wedge \psi_2 = 0; \quad d\psi_2 - (\omega_{12} + i\omega_{14}) \wedge \psi_1 - i\omega_{24} \wedge \psi_2 = 0.$$

Thus if we write the Levi-Civita connection as the 4×4 matrix $\begin{pmatrix} \theta & -\eta \\ \eta & \theta \end{pmatrix}$, then $\psi = \theta + i\eta$ is a skew-hermitian matrix, and we obtain the complex form of the defining equation for the Levi-Civita connection of a Kähler metric, namely,

$$d\psi_j + \sum_k \psi_{jk} \wedge \psi_k = 0. \tag{1.9}$$

Now let A be a $GL(2, \mathbb{C})$-valued function such that $\Phi = A\psi$ is a vector function the components of which give a local holomorphic frame for $T^{\star\, 1,0}M \to M$. The equation (1.9) gives

$$\partial \Phi_j = d\Phi_j = -\sum_k \Psi_{jk} \wedge \Phi_k, \tag{1.10}$$

where $(\Psi_{jk}) = A^{-1}(\psi_{jk})A + A^{-1}dA$. Since $\partial\Phi_j$ and Φ_k are (holomorphic) $(2,0)$ and $(1,0)$ forms respectively, it is seen easily that (Ψ_{jk}) is a matrix of of $(1,0)$ forms. Therefore the D''-component of the Levi-Civita connection D is $\bar{\partial}$. This argument made little use of the Levi-Civita connection and the conclusion $D'' = \bar{\partial}$ is true much more generally. It should be noted that while for simplicity of notation and explicit calculation we considered the case $\dim_{\mathbb{C}} M = 2$, the argument is valid in all dimensions essentially verbatim. ∎