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LJUSTERNIK-SCHNIRELMANN THEORY AND CONLEY INDEX A NONCOMPACT VERSION*

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Abstract. In this paper we use a noncompact version of Conley index theory to obtain a Ljusternik-Schnirelmann type result in critical point theory: Let X be a complete Finsler manifold, $f \in C^1(X, \mathbb{R})$ which satisfies Palais-Smale condition and φ^t be the flow relative to a pseudo-gradient vector field for f. If $I \subset X$ is a (c)-invariant set with $f(I)$ bounded, then f has at least $\nu_H(h(I)) - 1$ critical points in I where ν_H is the (homotopy) Ljusternik-Schnirelmann category and $h(I)$ is the Conley index of I.

1. Introduction. Conley's homotopy index was first constructed for isolated invariant sets of continuous flows on locally compact metric spaces [2]. The compactness assumption is crucial both in the existence and uniqueness of the Conley index. An important result in Conley index theory is the generalized Morse inequalities which introduced Conley's work as a generalization of Morse theory. The compactness assumption is also crucial in basic results of Morse theory. Indeed every noncompact boundaryless manifold admits a smooth function without critical points. In order to generalize these theories to the noncompact case, we must assume some compactness property of the flow. This has been done in [1] and [7] in two different ways both leading to the generalized Morse inequalities. In classical critical point theory, the compactness assumption is replaced by the Palais-Smale condition:
(P-S) Let X be a Banach manifold and $f \in C^1(X)$. We say that f satisfies Palais-Smale condition if any sequence $\{x_n\}$ such that $f(x_n)$ is bounded and $||Df(x_n)|| \longrightarrow 0$ possesses a convergent subsequence.

DEFINITION. *A Finsler structure on the tangent bundle of a Banach manifold X is a continuous function $||.|| : T(X) \longrightarrow [0, +\infty)$ such that*

(a) For every $x \in X$, the restriction $||.||_x = ||.|||_{T_x(X)}$ is an equivalent norm on the tangent space $T_x(X)$,
(b) For each $x_0 \in X$ and $k > 1$, there is a trivializing neighborhood U of x_0 in which

$$\frac{1}{k}||.||_x \leq ||.||_{x_0} \leq k||.||_x.$$

A C^1-Banach manifold X together with a Finsler structure on its tangent bundle $T(X)$ is called a Finsler manifold.

Now suppose that X is a complete Finsler manifold and $f \in C^1(X)$. In [4], Palais proved that f admits a pseudo-gradient vector field i.e. a map $F : X \longrightarrow T(X)$ such that

(i) The equation $\dot{x} = F(x)$ has a unique solution for every initial point $x_0 \in X$,
(ii) $\langle Df(x), F(x)\rangle \geq \alpha(||Df(x)||)$ where α is a strictly increasing continous function with $\alpha(0) = 0$,

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