My Research Perspectives in Arithmetic Geometry

Arash Rastegar

September 20, 2005

As predicted by Poincare in 'Science and Hypothesis', 'arithmetic' always has to lay on different branches of mathematics, to get new ideas for further development. The research perspectives I propose here, are motivated by my background and previous research, and are supposed to indicate my general directions in forthcoming research. Following Poincare, each research subject is formed by an attempt to link two branches in mathematics, in order to contribute to at least one of the two. In this proposal, always the arithmetic part is the part which takes the benefit.

I will try to explain the general background of each problem and give motivations for importance of the corresponding research direction together with a prediction of expected progress. I hope that the expository style of this proposal does not hurt the scientific value of its predictions. This proposal is addressed to experts in arithmetic geometry.

A. Modular forms and motivic fundamental groups

Contributions of motivic fundamental groups to the theory of modular forms.

Background. The theory of modular forms is the crossing point of many branches of mathematics; namely, algebraic number theory, analytic number theory, complex analysis, algebraic geometry, arithmetic geometry, Lie groups and Lie algebras, representation theory, harmonic analysis, etc. There are two important and central features in this theory. Modular forms despite being analytic objects are believed to have motivic origin, which is supported by many results and conjectures in arithmetic geometry. Secondly, modular forms are very friendly with L-functions. Both of these links contribute greatly to arithmetic problems.

Motivic fundamental groups are Lie algebras associated to profinite fundamental group of an algebaric curve by Deligne. These objects have contributed to many important problems in arithmetic geometry; namely, Grothendieck's anabelian conjectures, theory of motives, Diophantine geometry, etc. Motivic fundamental groups are unipotent Lie algebras, and the main feature of these objects is their non-linearity. They are constructed to compensate linearity of various cohomology theories.

Motivation. Using motivic fundamental groups of modular curves, one could translate the language of Galois representations to the language of Lie algebras. The advantage is that one avoids working with an infinitely generated profinite group.

Motivic fundamental groups can absorb many arithmetic aspects of modularm forms. Hecke operators act on this object. There exists a Lie version of Schlessinger criteria for deformation of such Lie algebras, which is developed by Pridham. These objects also keep some non-linear information which would be their advantage against cohomology groups. Therefore, they have flexibility to provide a more geometric proof of the Shimura-Tanyama conjecture.

Expected progress. Action of Hecke operators on modular forms, association of Galois representations, and Hodge theory data on various cohomologies of modular curves should be easy to translate to the language of motivic fundamental groups. Some calculations in deformation theory of Galois representations can be performed in the motivic fundamental group language. It may contribute to some class number problems.

B. Vojta's conjectures and arithmetic fractals

Contribution of arithmetic fractals to Vojta's value distribution philosophy.

Background. Vojta introduced a dictionary between value distribution theory of Nevanlinna and Diophantine approximation theory of Roth. Vojta's philosophy is that this dictionary should continue to hold in the higher dimensional case. This philosophy helps him to formulate qualitative conjectures in arithmetic geometry which covers almost every important conjecture in the field; namely, ABC-conjecture, Mordell's conjecture, some of Lang's conjectures, etc. The perspective given by Arakelov theory is central in Vojta's conjectures.

Arithmetic fractals are self-similar objects whose ambient spaces are algebraic varieties, and similarity maps are given by endomorphisms of these varieties. They provide a language in which one can unite several results and conjectures in arithmetic geometry; namely, Manin-Mumford's conjecture proved by Raynaud, some of Lang's conjectures, Andre-Oort conjecture, etc. Arithmetic fractals provide a dynamical picture to Diophantine problems.

Motivation. Self-similarity being a natural formulation of arithmetic problems, imposes a more general formulation of higher dimensional Vojta conjectures; not in terms of estimates but in terms of the geometry beyond the conjectures. This certainly contributes to Vojta's philosophy, and also there could be new implications of the fractal version of Vojta's conjectures which may give us a better perspective in Diophantine problems. On the other hand, utilizing the concept self-similarity may be also effective in proving some of Vojta's conjectures. It also contributes to the Nevanlinna side.

Expected progress. Fractal formulation of Vojta's conjectures is very available. This could already bring in special cases which are interesting in their own right. Self-similarity could also contribute to better understanding and ultimately to a simpler formulation of Vojta's conjectures.

C. Langlands correspondence and deformation theory

Contributions of deformation theory of algebraic structures to Langlands program.

Background. Atrin's theory of L-functions which related Abelian Galois theory to L-functions of Dirichlet characters motivated Artin's conjectures in non-abelian Galois theory. It was the insight of Langlands which connected this picture with the similar L-function formalism for modular forms which was developed by Hecke. This insight gave a new life to the theory of modular forms and also to infinite dimensional representation theory. By Langlands correnspondence, we mean a correspondence in special case, between certain n-dimensional Galois representations and certain automorphism representations of GL_n over the ring of adeles. This is the most important and influential research program in number theory.

Deformation theory of algebras was developed in the middle of previous century by Gerstenhaber. Importance and influence of non-commutative geometry philosophy has revived the theory again. Generalizations of this theory go even as far as algebras over operads. Derived deformation theory is a central theme in deformation theory which relates deformation of differential graded Lie algebras to several deformation problems in geometry. One of the successful tricks is 'hidden smoothness' philosophy which claims that one can desingularize moduli spaces by translating their moduli problem to the language of differential graded Lie algebras.

Motivation. Quantization of Langlands correspondence is not a new idea. But this is different from an attempt in linear approximation of Langlands correspondence in the language of Lie algberas. Glaois representations can be translated to the language of Lie algebras as explained in section A. One expects that the concept of first derivative of an automorphism funcation make sense as an ℓ-adic function on a quotient of a Lie algebra corresponding to GL_n over the ring of adeles. This way, one may be able to get a linear approximation to Langlands correspondence which has the advantage that, the objects under correspondence are introduced in the common language of Lie algebras.

Expected progress. One could get ideas from Local Langlands correspondence which is formulated in $\wp$-adic families. One can start from a Lie algebra approximation of Local langlands correspondence which is proved to hold by Harris and Taylor, and then try to generalize that to the Global Langlands correspondence which is still conjectural. May be the special case of Shimura-Tanyama conjecture proved by Wiles could also bring some insight to the theory.

D. Arithmetic geometry and dynamical systems

Dynamical formulation of problems in arithmetic geometry.

Background. Arithmetic geometry could be defined as geometric approach to arithmetic Diophantine problems; namely, Mordell conjecture, Siegel's theorem, Lang's hyperbolicity conjectures, Arakelov theory, minimal models of abelian varieties etc. or as arithmetic formulation of problems in algebraic geometry; namely, Grothendieck's anabelian conjectures, arithmetic theory of modular forms, Weil conjectures, Shimura varieties, Grothendieck's theory of dessains etc. This field is located in the very middle of algebraic geometry and various aspects of number theory, and it contributes equally to both sides. The main feature, as mentioned by Poincare, is the dependence of arithmetic on different geometric concepts in order to get nourishment for further development.

Dynamical systems is a dynamical approach to geometric objects which could be discrete or continuous. Discrete meaning that, forward orbit of an endomorphism of a geometric object is under focus. Continuous meaning that, there is flow on a manifold and its qualitative dynamics is under study. The theory of real dynamical systems is very analytic and could be hardly modeled with arithmetic. The theory of complex dynamical systems is more algebraic, although the techniques of proving theorems are as analytic as the former.

Motivation. The theory of arithmetic dynamics of algebraic dynamical systems was initiated by Northcott. He studies the number of periodic points of an algebraic endomorphism of a projective space which are defined over a number field. This lead him to the notion of height functions in arithmetic. Kawaguchi and Fakhruddin also have some results in the case of finitely generated fields. Moriwaki theory of height functions provides a stronger tool to handle the case of finitely generated fields. Arithmetic fractals, which was discussed in section B, provide a more general framework in which one could study the multi-dynamics of algebraic functions.

Expected progress. Study of the multi-dynamics of an algebraic variety using cohomogical invariants which could be easily associated to this formalism seems to be accessible. Such a theory, being a geometric theory should be easily extended to dynamics of correspondences on an algebraic variety. In case one obtains quantitative results in these directions, there are immediate applications in arithmetic geometry; namely, Hecke orbit in the moduli apace of pricipally polarized abelian varieties, Andre-Oort conjecture, generalized Lang's conjecture on forward orbit of a subvariety, etc.