Skills of
MATHEMATICAL THINKING
in undergraduate level

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The goal of this paper is to suggest a package of basic mathematical skills which forms an appropriate background for an undergraduate student in mathematics. Having such a package available, one could decide about the mathematical content of undergraduate courses designed for math-majors in a way that they provide an appropriate atmosphere for development of the above-mentioned skills. One would be able to afford a reasonable suggestion for this package, only if one has a clear mind about the role of mathematics education in educating students and about the importance of mathematics to human civilization.

Perspectives towards mathematics education

It is not easy to find all aspects of mathematics and mathematics education influencing mathematicians and the human civilization, but it is possible to list a few important aspects coming to mind at first sight [1],[2]:

- Mathematics education has an important role in development of mental abilities.
- Mathematics education is an efficient tool in developing the culture of scientific curiosity.
- We use mathematics in solving everyday problems.
- Development of the science of mathematics and our understanding of nature are correlated.
- Doing mathematics has an important role in learning and development of mathematics.
- Development of tools and Technology is correlated with development of mathematics.
- We use mathematics in studying, designing, and evaluation of systems.
- We use mathematical modeling in solving everyday problems.
- Group thinking and learning is more efficient than individual learning.
- Mathematics is a web of connected ideas, concepts and skills.

Now, we shall try to understand how each of these perspectives influences the manners in which we teach mathematics in university.

Mathematics education has an important role in development of mental abilities.

- We shall teach our students the skills of communication in words and pictures.
- Strategic thinking should be regarded as one of the main skills of problem solving.
- Students should be taught to critically view the process of problem solving from outside.
- Logical thinking has different levels of skillfulness which should be considered in teaching mathematics.
- Creative thinking should be encouraged.
- Imagination of students should be enriched.
- Abstract thinking has different levels of sophistication which should be obtained one by one.
- Symbolical thinking should be practiced regularly.

- Students should have a well-defined stream of thought during the process of problem solving.

Mathematics education is an efficient tool in developing the culture of scientific curiosity.

- Critical character should be encouraged among students.
- Students should be given role models in accepting critical opinions.
- Skills of using available information should be practiced.
- Curiosity and asking good questions should be taught to students.
- We shall teach students how to rigorously describe abstract mathematical constructions.
- Comparison with results of other experts should be an everyday practice.
- The skills of making logical assumptions should be taught in mathematics classes.
- Development of new theories, or at least new formulations of old theories should be a serious concern of the educators.

We use mathematics in solving everyday problems.

- Students should be able to find the common mathematical structures appearing in everyday problems.
- The ability to design a new problem is an important skill of a creative student.
- Students should judge about everyday issues using scientific methods.
- Students should practice developing new mathematics in order to solve new problems.
- Mathematical modeling is the way we translate a problem to the language of mathematics.

Development of the science of mathematics and our understanding of nature are correlated.

- Getting mathematical ideas from nature should be practiced.
- Study of the nature should be a concern for students.
- Control of nature by mathematics should be practiced by students.
- Our mind like nature chooses the simplest ways.

Doing mathematics has an important role in learning and development of mathematics.

- Logical assumptions by students should be based on experience.
- Internalization is a fruit of doing mathematics.
- Experience does not replace rigorous arguments of students.
- Students should analyze their experiences and compare them with others'.

Development of tools and Technology is correlated with development of mathematics.

- Limitations of Technology limit students in applying technology.
- Mathematical models affect development of technology by students.
- We shall utilizing technology in education of students.

We use mathematics in studying, designing, and evaluation of systems.

- Viewing natural and social phenomena as mathematical systems is an important tool in gaining a mathematical understanding of the world.
- Division of systems to subsystems should be practiced by students.
- We shall teach students how to discover similarities of different systems.
- Students should be able to summarize a system in a simpler system.

- Analysis of systems should be taught in mathematics classes.
- Mathematical modeling should be regarded as a tool for studying systems.
- Students should be capable of changing and controlling existing systems.

We use mathematical modeling in solving everyday problems.

- Students should be taught how to use old models in similar problems.
- Limitations of models should be detected by students.
- Getting ideas from models should be practices by students.
- Students shall try to find the simplest models which solve a problem.

Group thinking and learning is more efficient than individual learning.

- Problems are solved more easily in groups.
- Comparing different views should be an everyday practice of students.
- Group work develops personal abilities of students.
- Morals of group discussion should be taught in classes.

Mathematics is a web of connected ideas, concepts and skills.

- Students should be used to solving a problem with different ideas.
- Atlas of concepts and skills of a problem should be drawn by students.
- Webs can help students to discover new ideas.
- Students should regard mathematics as a tree growing both from roots and branches.

Now, it's time to suggest a package of important skills of thinking in geometry, algebra, analysis and combinatorics which should be gained by undergraduate students in mathematics. Glancing through the implications of our perspectives towards mathematics education one can list a few fundamental skills. In each paragraph, we shall introduce a few examples and the way we could go about supporting these skills:

Some skills of mathematical thinking

1. The skill of mathematical intuition

Geometric imagination- Solving two dimensional and three dimensional puzzles are practices which support the **three dimensional intuition**. Trying to understand three dimensional image of a four dimensional cube rotating along an axis of symmetry is a practice supporting **higher dimensional intuition**. Combinatorial geometry develops **combinatorial geometric intuition**. Constructing a new geometric object with predetermined properties, gives rise to **creative imagination**, and geometric understanding of algebraic stacks and moduli spaces develops **abstract imagination**.

Algebraic intuition- Proving algebraic inequalities supports the **computational intuition**. Studying group representations is a practice supporting the **symmetric imagination**. Comparing number fields and function fields develops **systematic intuition**. Constructing new ring with predetermined properties, gives rise to **constructive intuition**. Recognizing if two algebraic objects are isomorphic develops **similarity intuition**.

Analytical intuition- Trying to bound infinite sums or normalizing infinite products strengthens the skill of **estimation intuition**. Comparing the growth of functions and sequences gives rise to **growth intuition**. Solving functional and differential equations develops **global analytical intuition**. Comparing the geometric distinction between differentiable and double differentiable functions gives rise to **smooth intuition**. Trying to prove that L-functions are holomorphic functions develops **holomorphic intuition**. Probabilistic approach to analytical problems develops **probabilistic intuition**.

Combinatorial intuition- Counting the number of elements of a finite set on which a finite group acts, develops **counting intuition**. Discrete optimization problems coming from computer science supports **optimization intuition**. Associating discrete invariants to geometric objects like in knot theory develops **categorization intuition**. Solving problems in many different ways develops **strategic intuition**.

2. The skill of making arguments

Geometric arguments- In intersection theory, we practice **set theoretic arguments**. In differential geometry we practice **local geometric arguments** like those involving curvature, and we deal with **global geometric arguments** by integrating local data over the whole space. We practice **superposition** in physics and use **algebraic coordinatization** to replace geometric arguments by algebraic ones.

Algebraic arguments- Playing with symbols, following the rule of algebra supports **symbolic arguments**. Trying to prove that algebraic structures are isomorphic using algebraic invariants supports **categorization arguments**. Studying an algebraic structure using morphisms to other simpler algebraic structures develops skillfulness in **simplifying arguments**. Arguments which reduce the problem to localizations support **local algebraic arguments**. Arguments which induce global information to completions support **global algebraic arguments**.

Analytical arguments- Bounding an infinite sum from above and below develops **estimation arguments**. Translating a problem to the language of functions on a space supports the skill of using **functional arguments**. Defining local intersection number of analytical varieties using Wierstrass preparation theorem supports **local analytical arguments**. Trying to prove functional equations for L-functions develops **global analytical arguments**.

Combinatorial arguments- One can even get isomorphism between rings using **counting arguments**. In many arguments, we recognize different cases and treat each case differently. These are called **case arguments**. In some combinatorial problem we introduce artificial structure, which help combinatorial computations, like the notion of artificial orthogonality. These support the skill of making **artificial arguments**.

3. The skill of describing mathematical objects

Geometric description- One can give a **global description** of a manifold by introducing the universal cover together with its group of symmetries and give a **local description** by introducing charts, and **algebraically describe** an algebraic variety by introducing algebraic equations. One can introduce a compactification of the moduli space of principally polarized abelian varieties by **combinatorial description** of added strata and one can give more **abstract descriptions** by introducing more abstract formulations like the stack formalism for schemes or moduli description of a geometric object.

Algebraic description- A **global description** of algebraic objects can be given by equations. One can describe an algebraic structure by introducing **generators and relations** or by giving a **free resolution** or **decomposition to simple objects.** One can give a **local description** for an algebraic structure by giving information about its localizations. One can also **describe indirectly** an algebraic structure as algebraic invariants associated to geometric, combinatorial or analytic objects. One can give more **abstract descriptions** by introducing more abstract formulations like introducing an object by its universal properties.

Analytical description- Taylor series give **local description** for functions. One can give a **global description** for functions by introducing equations or by considering functions as dual objects. One can give more **abstract descriptions** of functions by introducing a categorical description of them.

Combinatorial description- Any description in terms of finitely many numbers or finitely many basic objects is a **finite description.** Sometimes continuous objects could be approximated by discrete objects, and hence we get a **discrete description.** Sometimes our description of an object is only a **locally finite description** and hence only locally combinatorial.

4. The skill of making assumptions

Geometric assumptions- One can make **local geometric assumptions** like assuming a manifold being of negative curvature, or make **global assumptions** like assuming a manifold to have finite volume, or make **algebraic assumptions** like assuming a variety to be projective. One can make assumptions on the Hodge diamond of a complex Kahler manifold which is a **combinatorial assumption** and one can also make assumptions about the morphisms of an object to other objects, which is an example of more **abstract assumptions**.

Algebraic assumptions- Many relevant assumptions on algebraic objects are **assumptions of geometric origin**, for example a ring being a complete intersection. **Symbolic assumptions** on generators are another type which have algebraic or combinatorial origin, like study of one relator groups. Sometimes one considers assumptions on localizations of a ring. These are **local assumptions**. There are many **global assumptions** which are equivalent to assuming local assumptions everywhere, like flatness of rings.

Analytical assumptions- Convergence of power series of numbers, functions or integrals are examples of **convergence assumptions**. Functional equations are examples of **algebraic assumptions**. Analytical continuation links L-function to **geometric assumptions**. Limiting variation of variables are examples of **limiting assumptions**. Smoothness assumptions are examples of **infinitesimal assumptions**.

Combinatorial assumptions- Every **finite assumption** are somehow combinatorial assumptions. **Discreteness assumptions** could also be combinatorial assumptions. **Finiteness assumptions** are always combinatorial assumptions.

5. The skill of recognition of common structures

Geometric structures- One can recognize **set theoretical geometric structures** like the geometry of quadric line complex. One can recognize **local structures** like finding and classifying singularities. One can recognize **global structures**, like proving existence and study of complex structures on Riemann surfaces. One can find **underlying algebraic structures** like Chow's theorem. **Combinatorial structures** could also be recognized in geometric objects, like Hodge structures of Kahler manifolds or triangulation of manifolds.

Algebraic structures- Structures over finite fields and finitely generated structures are examples of **finite algebraic structures.** Recognition of symmetry groups of objects and structures is an example of **symmetry structures**. Finding algebraic equations for objects is an example of **algebrization structures**. K-groups and homology groups are examples of **generated algebraic structures**. Recognition of equivalence of categories is recognition of **abstract structures**.

Analytical structures- L-functions are examples of **generated analytical structures**. Proving existence of complex structures is an example of **global analytical structures**. One can recognize or distinguish **local analytical structures**, like deciding if two singularities are of the same type.

Combinatorial structures- Graphs are structures that could be considered as common **geometric structures** in combinatorial objects. Matrices are examples of **algebraic structures** appearing in combinatorial objects. Common **abstract**

structures could also be recognized in combinatorial structures like equivalence of finite categories.

6. The skill of mathematical modeling in mathematics

Geometric modeling- We encounter most of these skills in physics. One can construct **solid models** like in mechanics of material points or mechanics of a solid body. One can construct **fluid models** like in fluid mechanics or in dynamical systems. One can also introduce **probabilistic models** like in statistical mechanics and in quantum mechanics.

Algebraic modeling- One can construct **number models**, like real and complex numbers in number fields and functions in function fields and matrices in algebraic groups. One can construct **numerical system models** like rings, group-rings and fields. Coordinates are examples of **algebrization models**. Morphisms between categories are examples of **abstract modeling**.

Analytical modeling- Generating functions and L-functions are examples of **power series models** and these are special kinds of **function models**. Integrations and analytical morphisms are examples of **simplifying models**. Fourier and Legendre transforms are examples of **transformation models**. Writing differential equations are examples of constructing **equation models** for analytical objects.

Combinatorial modeling- Combinatorial models are algebraic or geometric. One can construct **network models** like in graph theory. One can construct **matrix models** like in Latin squares. One can construct **discrete models** like in finite element method. One can associate **finite models** like in reducing curves over finite fields.

7. The skill of constructing mathematical structures

Geometric constructions- One can construct **set theoretical structures** with predetermined properties, like in combinatorics. One can construct **local structures** like blowing up a manifold in a point. One can also construct **global structure** like defining the formalism of differential forms on manifolds. Algebraic coordinatization leads to construction of **algebraic structures** and convex polygons associated to toric varieties leads to construction of **combinatorial structures**. One can consider a geometric structure representing a functor as a **representing structure**.

Algebraic constructions- Generating an algebraic system by a few relations is an example of a **combinatorial structure**. On can associate algebraic invariants to many mathematical objects, which is an example of **invariant structures**. Construction of an algebraic object satisfying universal properties is an example of **universal structures**. In particular, an algebraic object representing a functor could be regarded as a **representing structure**. Different kinds of morphisms between algebraic objects are **mapping constructions**. Translating to the language of K-theory is an example of **global constructions** and reducing a problem to local rings is an example of **local structures**.

Analytical constructions- Objects which are obtained by a limiting process are **limiting constructions**. In particular, when we construct an object by integration of infinitesimal objects, this is an example of **integrating constructions**. Construction of multiplication on a vector space, to produce a Hilbert space, is an example of **global constructions**. Translating to the language of Lie-algebras in deformation theory is a special form of **local structures**.

Combinatorial constructions- Representing data by a formula is an example of **constructions by formulas.** Specifying objects and their relations individually is a form of **detailed construction.** Characterizing a combinatorial construction by assuming combinatorial properties, is a kind of **construction by combinatorial properties**.

8. The skill of performing calculations

<u>**Geometric calculations-**</u> One can perform calculations in geometry after **translating geometric facts to algebraic objects**, like calculations in basic manifold theory. **Reducing the problem to limiting cases** happens when we localize a problem around a singularity. We **reduce the problem to extremes** like in Morse theory. We also **translate between different representations of geometric objects** to perform certain calculations in new formalisms like in algebraic topology when we introduce several cohomology theories. Sometimes **more abstract formulations** help us to compute like category theory which is an appropriate ambient space for certain calculations.

<u>**Algebraic calculations-**</u> Sometimes we **work with generators and relations** to calculate in algebraic structures, like in calculus of commutators. Sometimes we **use universal properties** to characterize algebraic structures, like deformation theory of Galois representations. We **make new algebraic structures** and **construct maps between defferent structures** to perform calculations, like in algebraic K-theory. We **use self-similarty** in algebraic structures to understand them better or make interesting examples, like in study of transcendence and examples in self-similar groups. To solve a problem, sometimes we **translate to the formalism of geometric objects** like in geometric group theory and translate **to the formalism linear algebra** like cohomology of groups.

<u>**Analytical calculations-**</u> We use **integration** to perform infinite sums and **zeta functions** to perform infinite products. We use **estimation** in order to control the behavior of variable depending on others, like in numerical analysis. We translate to the language of function spaces and **use formalism of functions** to perform some analytical calculations like in Hilbert-spaces structures induced on sections of some bundles. We **approximate** analytical structures with simpler ones like in Lie-algebras. In particular, we use **differentiation**, which is a kind of linear approximation in order to simplify calculations, like in differential geometry. We use conceptual themes of computational techniques, like producing orthogonality relations or the circle method.

<u>**Combinatorial calculations-**</u> **Counting** is the most effective way in combinatorial calculations and **using symmetry** is the most importanct technique to simplify counting. We **translate to the language of graphs** in order to get a geometric picture for our calculations. We **use algebraic formalisms** over finite commutative rings like in Weil conjectures, and **analytical expansions** like in generating functions. Sometimes we **translate to the formalism of algebra**, to make calculations easier, like in algebraic combinatorics.

9. The skill of mathematical modeling of everyday problems

<u>**Geometric modeling-**</u> **Linear modeling**, **algebraic modeling** and **exponential and logarithmic modeling** are examples of modeling the growth of natural phenomena. One could also introduce **mechanical or combinatorial models**. Geometric models are usually **analogy machines**.

<u>**Algebraic modelling-**</u> In physics we find **algebraic equations** to explain every natural phenomena. **Linear modeling**, **algebraic modeling or exponential and logarithmic modeling** could also be considered as algebraic modeling if we insist on using the algebraic formulas not the geometric pictures.

<u>**Analytical modelling-**</u> We use **differential equations** to explain variation in lots of different conetexts. **Deformation theory** of algebras and more generally deformation theory of operads, are more abstract modes for variation structures. **Singularity theory** and **catastrophy theory** are aspects of this modelization which could also be studied using techniques from algebra and geometry.

<u>Combinatorial modelling-</u> Finite element method is the most profound technique to approximate a natural phenomena by a finite structure. There are more elementary formalisms like **graph theory** or **matrix theory** which could be used to modelize and store combinatorial data.

10. The skill of doing mathematics in different parallel categories

<u>Geometric categories-</u> One can study a geometric object in different categories like in **topological category**, **smooth category**, **algebraic category**, or more abstract categories like **category of schemes or stacks**. One could reduce the problem to **finite categories** by for example reducing the object to an object over finite fields.

<u>Algebraic categories-</u> One can study algebraic phenomena in **commutative category** or **non-commutative category**. Also could use the **formalism of sheaves and schemes** to make it geometric. There are **formalisms in category theory** where you can translate algebraic phenomenas to those languages.

<u>Analytical categories-</u> To encode analytical data one can use **zeros and poles of zeta functions. Representations of Lie-groups and Lie algebras** are also used as a different variant where analytical phenomena could be studies. Most of the analytical phenomena are computational and at most one could translate them and use them for different objects and similar mathematical situations.

<u>Combinatorial categories-</u> Calculations related to combinatorial objects could be done in different formalisms, like **simple counting, graph theory, algebraic formulas, generating functions**. But there are always identical for trivial reasons.

For finite structures, usually there are not many different categories in which you could discuss the problem in a way that a major advantage is recognized in some particular formalizms. Because the roots and relations between ideas are usually clear.

11. The skill of using technology

Students could **draw** many geometric objects by computer. In order to perform this task one shall often be able to algebrize the geometric structure so that one can communicate with computer. One can also perform **algebraic and geometric calculations** by computer. In order to perform this task one shall be able to introduce an algorithmic version of the process of making calculations so that one can communicate with computer in this regard. Students could also go about **producing software** fit to specific geometric problems.

12. The skill of recording the history of concepts and skills

One shall be able to learn about history of concepts like the concept of a geometric space which changes through Euclidean geometry, spherical and hyperbolic geometries, space-time geometry, manifold geometry, and non-commutative geometry! One shall also be able to record personal history of concepts and skills ranging from the very beginning of formation of mathematical personality to the process of solving a particular problem.

Suggestions for undergraduate program

Having the above list of basic skills of mathematical thinking, one shall introduce a few topics which could cover so many diverse skills in a limited mathematical content. Of course, it is not possible to suggest an undergraduate program based on basic concepts everyone being found of it, and this is why we chose the language of basic skills to talk about the undergraduate curriculum. But it is important to put forward an example, only to prove the possibility of uniting and gathering all the basic skills under the same flag.

If we wanted to concentrate on conceptual complexities, we could suggest that courses in the first year of undergraduate studies of students should deal with basics in geometry, algebra, analysis, and combinatorics. During the second year, courses which are combination of a pair of the above should be taught, like algebraic geometry and differential topology, and through the third year triple combinations should be chosen for the topics, for example a course on analysis on manifolds. For the forth year, all of the above topics should be combined, like a course on quantum physics or a course on Arakelov theory.

Finally, a course on history of evolution of mathematical concepts, which explores roots and branches of all the mathematics which undergraduates will become familiar with, is suggested.

It seems to us, that these courses could be taught in a way that they cover all the basic skills we mentioned in this paper. Of course, in order to make sure of the truth of this claim, one shall develop a detailed curriculum for mathematics in the undergraduate level. This research is under progress.

References

[1] Bakhshalizade S., Davoodi K., Hajibabaee J., Rastegar A.: Curriculum for mathematics in elementary and middle school, Organization for research and planning, ministry of education.

[2] Bijanzade M., Iranmanesh A., Rabiee H., Rastegar A., Rezaei M., Rostami H.: Curriculum for mathematics in high-school, Organization for research and planning, ministry of education.