Stochastic nature of series of waiting times

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Although fluctuations in the waiting time series have been studied for a long time, some important issues such as its long-range memory and its stochastic features in the presence of nonstationarity have so far remained unstudied. Here we find that the "waiting times" series for a given increment level have long-range correlations with Hurst exponents belonging to the interval $1/2 < H < 1$. We also study positive-negative level asymmetry of the waiting time distribution. We find that the logarithmic difference of waiting times series has a short-range correlation, and then we study its stochastic nature using the Markovian method and determine the corresponding Kramers-Moyal coefficients. As an example, we analyze the velocity fluctuations in high Reynolds number turbulence and determine the level dependence of Markov time scales, as well as the drift and diffusion coefficients. We show that the waiting time distributions exhibit power law tails, and we were able to model the distribution with a continuous time random walk.

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I. INTRODUCTION

Stochastic processes occur in many phenomena, ranging from various indicators of economic activities in the stock market to velocity fluctuations in turbulent flows and heartbeat dynamics, etc. [1]. There are several advanced methods to analyze such time series. The well-known methods are, for instance, detrended fluctuation analysis [2], detrended moving average [3], wavelet transform modulus maxima [4], rescaled range analysis [5], scaled windowed variance [6], Markovian method [7], detrended cross-correlation analysis [8], multifactor analysis of multiscaling [9], inverse statistics [10], etc. (see Refs. [1,7,11,12] for other methods).

Probability density function (PDF) of the first-passage times or waiting times (WT) has been studied extensively in the literature and has been used in the different fields such as physics, economic, biology, etc. For instance, knowing the mean first passage time is important in the study of the transport of biological molecules, such as DNA, RNA, and proteins, across nanoporous membranes, which is of fundamental importance to life processes. This transport is known as the translocation process [13,14]. The translocation and its distribution is important in gene therapy [15], drug delivery [16], and rapid DNA sequencing [17]. Another important application of the WT is in market risk analysis [18] (see also Refs. [19–21] for other applications).

Here we concentrate on the inverse statistic analysis and the Markovin method. In the inverse statistics we are interested in studying the waiting time distributions for different increment levels [22–26]. One of the main findings of inverse statistics in financial markets is discovery of asymmetry between the most likely time to profit and to loss. This gain-loss asymmetry is revealed by inverse statistics and closely related to empirically finding first passage time. Indeed, for financial indices it was found that while the maximum of the inverse statistics for a given positive return occurs at a specific time, the maximum of the inverse statistics for the same negative return appears much earlier [27–30].

In practice, the inverse statistics provide the distribution of waiting times needed to achieve a predefined increment level obtained from every time series. This distribution typically goes through a maximum at a time called the optimal time scale, which is the most likely waiting time for obtaining a given level.

Although the fluctuations in WT series have been known for a long time, however, some important issues, such as its long-range memory, its stochastic features have so far remained unstudied. To investigate its long-range memory we use the standard power spectrum or detrended fluctuation analysis (DFA) of WT series and study its exponents to detect the short- or long-range memory [2].

We would also like to address a question that for given fluctuating sequentially measured set of WT series, how does one find its dynamical equation, assess their underlying trends, and discover the characteristics of the fluctuations that generate the measured WT series in the statistical sense?

To answer this question we study the stochastic nature of WT series using the Markovian method [7]. The Markovian method is a robust statistical method which has been developed to explore an effective equation that can reproduce stochastic data with an accuracy comparable to the measured one [7,31–40]. As many early researchers have confirmed, one may utilize it to (1) reconstruct the original process with similar statistical properties and (2) understand the nature and properties of the stochastic process.

One of the main tasks in this paper is to quantify most relevant statistical properties such as characteristic time scales of series, drift, and diffusion coefficients. Here we use the Markovian method to analyze the WT fluctuations for high Reynolds number turbulence time series. Using this method we find the level dependence of Markov time scales, drift, and diffusion coefficients of process. We show that the "log returns" (logarithmic differences) of the WT series have short-range correlation and can be modeled by a Langevin dynamics with multiplicative noise. By fitting the observational data we have succeeded in finding the different Kramers-Moyal (KM) coefficients $D^{(n)}$ and have shown that the fourth-order coefficient tends to zero, whereas the first and second coefficients have well-defined limits. Then by addressing the implications dictated by theorem (Pawula theorem) we find a Fokker-Planck evolution operator. The Fokker-Planck description of probability measure is equivalent with the Langevin description of log-returns WT series.

The rest of this paper is organized as follows: In Sec. II we give a short review of inverse statistics and show that the WT series has long-range correlation for turbulence time series. We show that the WT distributions exhibit power law tails and model the distribution with continuous time random walk. In Sec. III we analyze the WT series using the Markovian method and find the Kramers-Moyal coefficients. Section IV closes with a discussion and conclusion of the present results.

II. LONG-RANGE CORRELATION IN "WAITING TIME" SERIES

Understanding intermittency effects in some stochastic processes and the associated multiscaling spectrum of exponents is one of the important problems in time series analysis. For a given process $x(t)$, the traditional way of describing the intermittency is to consider the scaling behavior of increments $\Delta x = x(t+\tau) - x(t)$ between two points of the time series, raise this difference to the moment q, and then study the variation with respect to the distance between the two points, also called structure functions, where the corresponding scaling exponents are called structure function exponents.

In inverse statistics it is proposed to invert the structure function equation, and consider instead averaged moments of the distance between two points, given a $x(t)$ difference between those points:

$$\langle [\Delta x(\tau)]^q \rangle \sim |\tau|^{\xi_q} \rightarrow \langle [\tau(\Delta x)]^q \rangle \sim |\Delta x|^{\delta_q}. \qquad (1)$$

For monofractal (nonintermittent or linear) time series one expects a trivial set of exponent δ_q, where the variation with the moment q is determined by one exponent. In the case of intermittent and singular data this would be completely different and δ_q will be nonlinear function of q. To demonstrate the method we use the data for velocity time series in free-jet turbulence with helium gas at 4.2 K (with Reynolds number $\mathrm{Re} \simeq 800\,000$ and about 10^8 data points).

Let us first study the memory in the WT series. Here we report that the WT series for a given increment level have long-range correlation with Hurst exponents $1/2 < H < 1$. Consider a given level γ, i.e., $\gamma = v(t+\tau) - v(t)$, where $v(t)$ is the velocity time series of turbulence. Now construct the series $\Theta = \{\tau_1, \tau_2, \ldots\}$, where τ_i are the waiting times to observe the level γ and are in the units of data lag. The Θ series are an implicit function of the level γ. We use the DFA method to extract the scaling exponent of the WT series fluctuations [2].

In Figs. 1 and 2 the WT series and its corresponding log returns for level 0.5σ (where σ is the variance of original velocity time series) are plotted. For the Θ series, we find that the Hurst exponent depends on γ and has values belonging to interval $1/2 < H < 1$. For instance, for the negative levels -0.5σ, -1.5σ, -2.5σ, -3.5σ, and -4.5σ, the obtained Hurst exponents are 0.74, 0.77, 0.81, 0.84, and 0.91, respectively. The error bars are about ± 0.02. For the positive levels 0.5σ, 1.5σ, 2.5σ, 3.5σ, and 4.5σ, we found the Hurst exponents to be 0.64, 0.64, 0.65, 0.68, and 0.71, respectively. This analysis shows that the WT series for negative increments has longer memory with respect to those with positive values, and this analysis shows that WT series has long-range correlation function for both positive and negative levels.

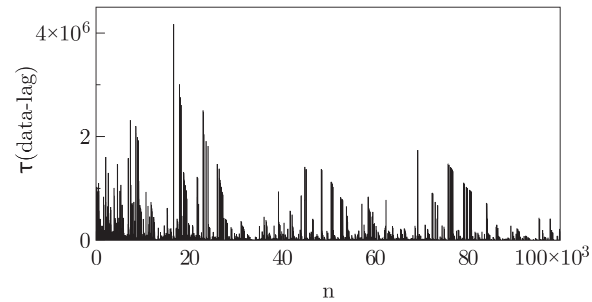

FIG. 1. Waiting time series for the level $\gamma = 0.5\sigma$. The units are in lag and n is the number of data for τ.

The probability distribution function (PDF) of the Θ series for different levels is given in Fig. 3 and can be modeled with a generalized γ function as [22]

$$P(\tau,\gamma) = \frac{\nu}{\Gamma(\alpha/\nu)} \frac{|\beta|^{2\alpha}}{|\tau + T_0|^{\alpha+1}} \exp\left\{ -\left(\frac{\beta^2}{\tau + T_0}\right)^{\nu} \right\}, \qquad (2)$$

where α, ν, β, and T_0 are constants which are depend on the level γ. In Fig. 3 we provide the fitting parameter $\delta = \alpha + 1$ for $\tau \gg T_0$ and obtain the exponent to be $\delta = 1.5 \pm 0.1$. The exponent is almost independent of the level γ. It appears that the waiting times PDFs have a fat right tail with power-law behavior in the interval with about two orders of magnitude. Also as shown in Fig. 3 the PDF tails for positive and negative levels are almost the same (considering their error bars).

We observe that the optimum time scale for small positive and negative levels such as $\pm 0.5\sigma$ and $\pm 1\sigma$ coincides with each others, whereas for the large levels such as $\pm 4\sigma$ the maximum of negative level occurs earlier than positive level. This phenomenon is similar to the gain-loss asymmetry in financial time series. Physically it stems from the fact that in turbulence the third order moment of increments (longitudinal) is negative, $\langle (\Delta_r v)^3 \rangle < 0$, which means that for given length scale r, the number of events with negative increments are more popular than positive values. Therefore in inverse

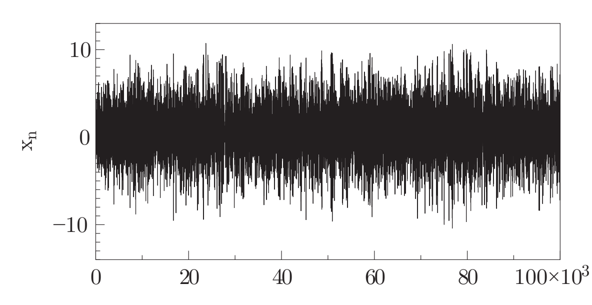

FIG. 2. Log returns of the waiting time series for level $\gamma = 0.5\sigma$, i.e., $x_n = \log(\tau_{n+1}/\tau_n)$.

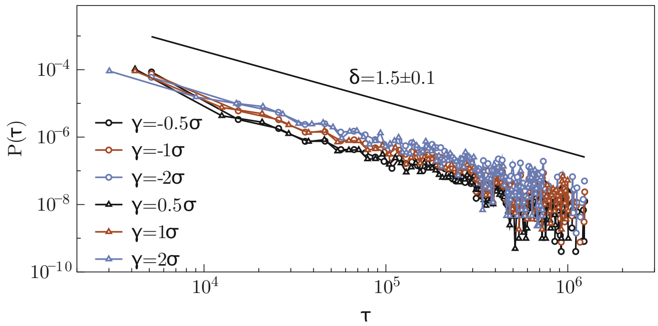

FIG. 3. (Color online) The probability distribution $P(\tau)$ of waiting time τ needed to reach levels $\gamma = \pm 0.5\sigma$, $\pm 1\sigma$, and $\pm 2\sigma$. Solid line is the fitted curve based on Eq. (2).

statistics as well a population for negative increment levels will be more popular, and then their maximum in a PDF occurs earlier. In Fig. 4 we plot the PDF of WT series for levels $\pm 4.5\sigma$, and it is evident that the maximum of the PDF for a negative level is about 200 data points earlier than those for the positive level.

The WT series also can be modeled by the continuous time random walk (CTRW) [43–47]. In this approach the survival time probability distribution $\psi(\tau)$ [which has the relation with $P(\tau,\gamma)$ as $P(\tau,\gamma) = -d\psi(\tau)/d\tau$] can be shown to behave as

$$\psi(\tau) = E_\beta(-\tau^\beta), \tag{3}$$

where $E_\beta(-t^\beta)$ is the Mittag-Leffler function of order β and $0 < \beta < 1$. The Mittag-Leffler function has asymptotic in the limit $\tau \to \infty$: $\psi(\tau) \sim \sin(\beta\pi)/\pi\ \Gamma(\beta)/\tau^\beta$.

Therefore the waiting time PDF, $P(\tau,\gamma)$, behaves as

$$P(\tau,\gamma) \sim \frac{\sin(\beta\pi)}{\pi}\frac{\Gamma(\beta+1)}{\tau^{\beta+1}}. \tag{4}$$

The WT PDF is well fitted by a power law function having an exponent $\beta = 0.5$; see Ref. [48] for more details.

III. MARKOV ANALYSIS OF THE "WAITING TIMES" SERIES

To obtain short-range correlated data from Θ series, we define logarithmic returns (log returns) series as $x_i = \log(\tau_{i+1}/\tau_i)$. We found that for given level γ the log-return series has short-range correlation, therefore it appears that can

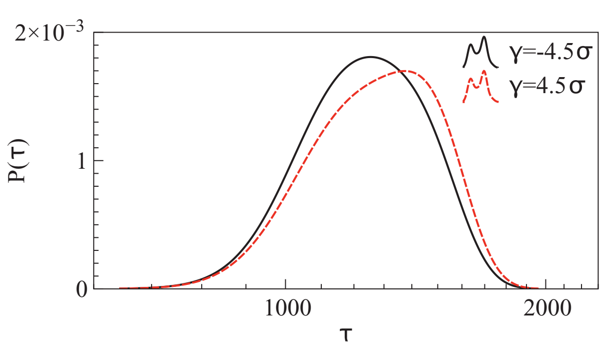

FIG. 4. (Color online) The probability distribution function $P(\tau)$ of waiting time τ needed to reach levels $\gamma = \pm 4.5\sigma$. The maximum of the PDF for the negative level occurs earlier than the positive level.

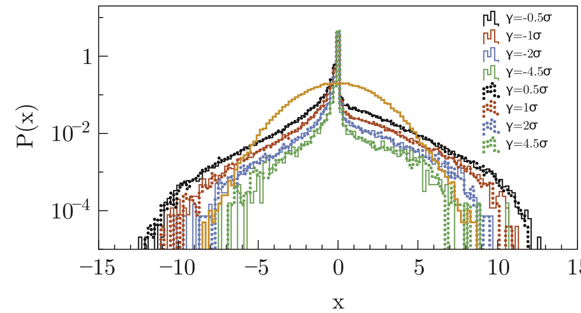

FIG. 5. (Color online) The probability distribution $P(x)$ of a log-return waiting times series for the levels $\gamma = \pm 0.5\sigma$, $\pm 1\sigma$, $\pm 2\sigma$, and $\pm 4.5\sigma$. The orange curve is a Gaussian PDF with variance 2 and is plotted for comparison.

be modeled by a Langevin equation [7]. It was straightforward to show, using three different methods, that the resulting series x_i are also stationary. (1) We computed the averages and variances of the series x_i in moving windows of increasing sizes to check that they are essentially invariant. (2) We computed the spectral densities $S(\omega)$ of the series x_i. The result, $S(\omega) \sim \omega^\beta$ with $\beta \simeq 0$, indicated the absence of long-range correlations in x_i. (3) We also analyzed the series x_i using the detrended fluctuation analysis [2] to further check that the series x_i are stationary. Thus, the series x_i are, at least to a good degree of approximation, stationary. The log return of WT series, depicted in Fig. 1, is plotted in Fig. 2. The PDF of the WT log returns for different levels γ are plotted in Fig. 5. The PDFs are positive skewed and have fat right tail (with respect to the Gaussian PDF) structure. This means that with high probability the waiting time τ_{i+1} is greater than τ_i. The variance Σ and skewness S of log returns for different levels γ are given in Table I.

Since long-range, nondecaying correlations are absent in $x(t)$, but short-range decaying correlations do usually exist, we first check whether the data follow a Markov chain [7]. If so, we measure the Markov time scale t_M, the minimum time interval over which the data can be considered as a Markov process.

Let us review the steps that lead to the development of a stochastic equation, based on the (stochastic WT) data set, which is then utilized to reconstruct the original data, as well as an equation that describes the phenomenon [7].

(1) As the first step we check whether the data follow a Markov chain and, if so, estimate the Markov time scale (MTS)

TABLE I. The variance Σ, maximum of x_i, and skewness S of log returns for different levels γ.

Level $\gamma[\sigma]$	Variance Σ	Skewness S	Max($\vert x_i \vert$)
−0.5	1.60	0.73	∼13
0.5	1.62	0.67	∼12
−1	1.23	1.12	∼11
1	1.21	1.12	∼11
−2	0.86	1.66	∼10
2	0.86	1.58	∼9
−4.5	0.58	2.68	∼11
4.5	0.59	2.44	∼9

(here Markov number) n_M. The MTS is the minimum time interval over which the data can be considered as a Markov process. There are several method to estimate the MTS [7]. The simplest way to estimate n_M for a stationary process is to use the least square method. For a Markov process one has

$$P(x_3,n_3|x_2,n_2;x_1,n_1) = P(x_3,n_3|x_2,n_2), \tag{5}$$

where x_k and n_k are the kth WT log return and its number in the series, respectively.

Therefore a process is Markov if Eq. (5) is satisfied for a *certain* time separation, with the n_M being $n_M = n_3 - n_2$. To measure the n_M, we have the following relation for any three-point PDF in terms of conditional probability functions: $P(x_3,n_3;x_2,n_2;x_1,n_1) = P(x_3,n_3|x_2,n_2;x_1,n_1)P(x_2,n_2;x_1,n_1)$. Using the Markov processes properties we obtain $P_{\text{Markov}}(x_3,n_3;x_2,n_2;x_1,n_1) = P(x_3,n_3|x_2,n_2)P(x_2,n_2;x_1,n_1)$. So we can compute the deviations of P_{Markov} from a three-point joint PDF, using the least square method:

$$\chi^2 = \int dx_3\, dx_2\, dx_1 \frac{[P(x_3,n_3;x_2,n_2;x_1,n_1) - P_{\text{Markov}}(x_3,n_3;x_2,n_2;x_1,n_1)]^2}{\sigma^2 + \sigma^2_{\text{Markov}}}, \tag{6}$$

where $\sigma^2 + \sigma^2_{\text{Markov}}$ are the variance of the terms in the nominator. Take $n_1 = 0$ and $n_2 = 2n_3$, then plot the reduced χ^2, $\chi^2_\nu = \frac{\chi^2}{\mathcal{N}}$ ($\mathcal{N}$ is the number of degrees of freedom), as a function of time scale $n_2 - n_1$. The n_M is that value of $n_3 - n_1$ at which χ^2_ν is minimum.

(2) Deriving an effective stochastic equation that describes the fluctuations of the quantity $x(n)$ constitutes the second step.

The Markovian nature of the log returns of WT fluctuations enables us to derive a Fokker-Planck equation (a truncated Kramers-Moyal equation) for the evolution of the PDF $p(x,n)$, in terms of number n. The Chapman-Kolmogorov (CK) equation, formulated in differential form, yields the following Kramers-Moyal (KM) expansion [41]:

$$\frac{\partial}{\partial n} p(x,n) = \sum_{k=1}^{\infty} \left(-\frac{\partial}{\partial x}\right)^k [D^{(k)}(x,n)p(x,n)], \tag{7}$$

where $D^{(n)}(x,n)$ are called the Kramers-Moyal coefficients. These coefficients can be estimated directly from the moments, $M^{(n)}$, and the conditional probability distributions as

$$D^{(k)}(x,n) = \frac{1}{k!} \lim_{\Delta n \to n_M} M^{(k)}, \tag{8}$$

$$M^{(k)} = \frac{1}{\Delta n} \int dx'(x'-x)^k p(x',n+\Delta n|x,t). \tag{9}$$

For a general stochastic process, all Kramers-Moyal coefficients are different from zero. According to the Pawula's theorem, however, the Kramers-Moyal expansion stops after the second term, provided that the fourth-order coefficient $D^{(4)}(x,n)$ vanishes. In that case, the Kramers-Moyal expansion reduces to a Fokker-Planck equation (also known as the backwards or second Kolmogorov equation) [41]

$$\frac{\partial}{\partial n} p(x,n) = \left\{ -\frac{\partial}{\partial x} D^{(1)}(x,n) + \frac{\partial^2}{\partial x^2} D^{(2)}(x,n) \right\} p(x,n). \tag{10}$$

Also the evolution equation for conditional probability density function is given by the above equation except that $p(x,n)$ is replaced by $p(x,n|x_1,n_1)$. Here $D^{(1)}$ is known as the drift term and $D^{(2)}$ as a diffusion term which represents the stochastic part. The Fokker-Planck equation describes the evolution of probability density function of a stochastic process generated by the Langevin equation (we use the Itô's definition) [41]

$$\frac{\partial}{\partial n} x(n) = D^{(1)}(x,n) + \sqrt{D^{(2)}(x,n)}\, f(n). \tag{11}$$

Here n is the data point number in log-return time series and derivative with respect to n means $x(n+1) - x(n)$. Also $f(n)$ is a random force, δ-correlated white noise in n with zero mean and Gaussian distribution, $\langle f(n) f(n') \rangle = 2\delta_{n,n'}$.

Using Eqs. (8) and (9), for collected data sets, we calculate drift, $D^{(1)}$, and diffusion, $D^{(2)}$, coefficients, shown in Figs. 6 and 7. It turns out that the drift coefficient $D^{(1)}$ is a linear function in x, whereas the diffusion coefficient $D^{(2)}$ is a fourth-order polynomial. For large values of x, our estimations become poor, and the uncertainty increases. To plot different drift coefficients $D^{(1)}$ (and $D^{(2)}$) for different levels in one figure we normalize the x data to their maximum, i.e., $x \equiv \frac{x}{\max|x|}$. The values of $\max|x|$ for different levels are given in Table I.

We computed the fourth-order coefficient $D^{(4)}$ and found that, $D^{(4)} \simeq 10^{-2} D^{(2)}$. Furthermore, it becomes clear that we are able to separate the deterministic and the noisy components

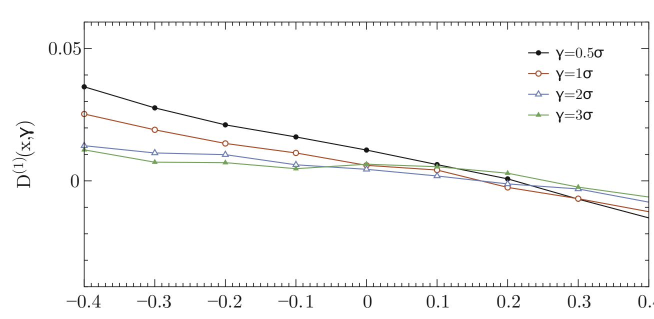

FIG. 6. (Color online) Drift coefficient $D^{(1)}(x,\gamma)$ for different level γ.

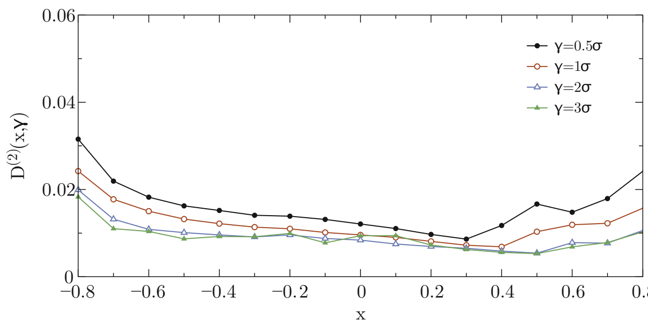

FIG. 7. (Color online) Diffusion coefficient $D^{(2)}(x,\gamma)$ for different level γ.

of the fluctuations in terms of the coefficients $D^{(1)}$ and $D^{(2)}$ [33,42].

We find that the MTS n_M for log returns is almost independent from the level γ and is about 10 data lag. The analysis of the data yields the following approximates for the drift and diffusion coefficients for x:

$$D^{(1)}(x) = a(\gamma)x, \quad D^{(2)}(x) = b(\gamma) + c(\gamma)x^2 + d(\gamma)x^4, \quad (12)$$

where coefficients a, b, c, and d for positive levels are given by

$$a(\gamma) = -0.003 - 0.125\gamma + 0.095\gamma^2 - 0.028\gamma^3 + 0.004\gamma^4,$$
$$b(\gamma) = 0.006 + 0.018\gamma + 0.017\gamma^2 - 0.007\gamma^3 - 0.001\gamma^4,$$
$$c(\gamma) = -0.023 - 0.031\gamma - 0.009\gamma^2 - 0.004\gamma^3 + 0.002\gamma^4,$$
$$d(\gamma) = 0.074 - 0.027\gamma - 0.010\gamma^2 + 0.014\gamma^3 - 0.003\gamma^4.$$

Also the coefficients a, b, c, and d for negative levels are given by

$$a(\gamma) = -0.059 - 0.033\gamma + 0.045\gamma^2 - 0.017\gamma^3 + 0.003\gamma^4,$$
$$b(\gamma) = -0.008 + 0.051\gamma - 0.044\gamma^2 + 0.016\gamma^3 - 0.003\gamma^4,$$
$$c(\gamma) = 0.025 - 0.078\gamma + 0.056\gamma^2 - 0.018\gamma^3 + 0.003\gamma^4,$$
$$d(\gamma) = -0.070 + 0.285\gamma - 0.226\gamma^2 + 0.078\gamma^3 - 0.012\gamma^4.$$

Moreover, if we analyze different parts of the time series separately, we find (1) almost the same Markov time scale n_M for different parts of the time series, but with some differences in the numerical values of the drift and diffusion coefficients, and (2) that the drift and diffusion coefficients for different parts of the time series have the same *functional forms*, but with *different coefficients* in equations.

IV. CONCLUSION

The "waiting time" series often represent nonstationary series that are very difficult to analyze. In this paper we analysed the log-returns of WT series using the Markovian method. The method is based on (1) constructing a stationary series based on the successive WT series, (2) checking whether the new series follows the properties of a Markov process, and (3), if so, analyzing the series based on the Markov processes and the Kramers-Moyal expansion. In many cases, such as the present study, the expansion terminates after the second-order term, hence yielding a Fokker-Planck equation, which is equivalent to a Langevin equation. We present the results for the waiting time series in inverse statistics of high Reylond turbulence data; however, the method is general and applicable to a large class of nonstationary processes.

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