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Coherent backscattering of electromagnetic waves in random media

A. SHEIKHAN[1], P. MAASS[2] and M. REZA RAHIMI TABAR[2,3]

[1] *National Institute for Theoretical Physics - Private Bag X1, 7602 Matieland, South Africa*
[2] *Fachbereich Physik, Universität Osnabrück - Barbarastraße 7, 49076 Osnabrück, Germany, EU*
[3] *Department of Physics, Sharif University of Technology - 11365-9161, Tehran, Iran*

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Abstract – The single and multiple scattering regimes of electromagnetic waves in a disordered system with fluctuating permittivity are studied by numerical simulations of Maxwell's equations. For an array of emitters and receivers in front of a medium with randomly varying dielectric constant, we calculate the backscattering matrix from the signal responses at all receiver points j to electromagnetic pulses generated at each emitter point i. We show that the statistical properties of the backscattering matrix are in agreement with the recent experimental results for ultrasonic waves (AUBRY A. and DERODE A., *Phys. Rev. Lett.*, **102** (2009) 084301) and light (POPOFF S. M. *et al.*, *Phys. Rev. Lett.*, **104** (2010) 100601). In the multiple scattering regime the singular value distribution of the backscattering matrix obeys the quarter-circle law.

Wave propagation in heterogeneous media is of both fundamental scientific and practical interest and has been studied for a long time [1]. A well-known approach for studying single and multiple scattering of waves is the analysis of backscattering phenomena. Various groups have reported the observation of coherent backscattering of light [2–6] and sound [7–10], which manifests itself as a peak in the backscattered intensity and originates from the constructive interference between pairs of waves, where one of the waves is multiply scattered along a path and the other wave is multiply scattered along the corresponding time-reversed path. In condensed-matter physics, the coherent backscattering leads to weak localization of electrons in metals with impurities [11,12]. The long-range Coulomb interaction between the electrons, however, makes the analysis of the localization problem very difficult [13]. Classical waves, by contrast, do not interact with one another and therefore provide an ideal probe for studying localization phenomena. Due to multiple scattering, the propagation of intensity in the scattering medium becomes diffusive with a diffusion coefficient D. If the diffusive length scale $\sqrt{Dt}$ at time t is much smaller than the distance R between source and scattering medium, $\sqrt{Dt} \ll R$, the width of the backscattering peak decreases as $\lambda/\sqrt{Dt}$ [14,15], where λ is the wavelength. For long times, $R \ll \sqrt{Dt}$, the backscattered intensity exhibits a narrow peak (coherent contribution) which lies on top of a wide arc (incoherent contribution). The width of the narrow peak is independent of time and of the order λ/R [16,17], while the width of the arc increases as $\sqrt{Dt}$.

In recent experimental studies of coherent backscattering with ultrasonic waves [18], an array of emitters and receivers placed in front of the scattering medium has been used, where an incident pulse is emitted from one array element i and the backscattered signal is recorded in all elements j, as sketched in fig. 1. The responses of the elements j to the emitted signals in the element i are given by the complex elements K_{ij} of the backscattering matrix K. For a disordered medium, K can be considered as a random matrix and its statistical properties have been analyzed in recent experiments with ultrasonic waves and light [18–20]. Factorization of the symmetric and non-Hermitian matrix K by singular value decomposition yields $K = USV^\dagger$, where U and V are unitary matrices and S is a diagonal matrix with the non-negative singular values s as diagonal elements. The distribution $\rho(s)$ of the singular values is different for single and multiple scattering regimes. In the multiple scattering regime, according to the random matrix theory, it was expected to obey

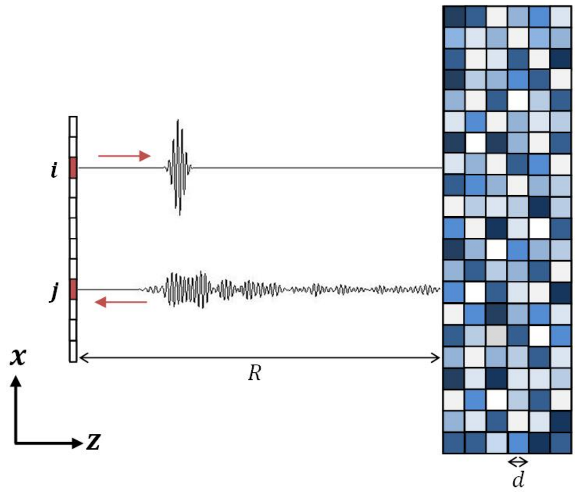


Fig. 1: (Color online) The setup for the simulation. An array of antennas is placed in front of the medium and the medium is made of parallel bars perpendicular to the page with different dielectric constant (different colors).

the the quarter-circle law. In the single scattering regime, $\rho(s)$ was shown to follow a Hankel distribution [18].

Here we address the question whether the features observed in recent experiments can be seen in numerical simulations of Maxwell's equations. To this end, we consider a two-dimensional setup resembling the one used in the experiments with ultrasonic waves [18] and determine the coherent backscattering from a medium with randomly varying dielectric function $\epsilon(\mathbf{x})$. The experimental set-up is a collection of rods with the same physical properties. The rods are located randomly in the x-z plane with their axes perpendicular to this plane. To mimic this system we considered bars on a regular grid with random dielectric constants ϵ drawn from a uniform distribution with mean value ϵ_0 and variance W. We find that the spatial and temporal variation of the backscattered intensity display the features found in experiment. The distribution of the backscattering matrix elements K_{ij} is nearly Gaussian (for the case studied here, where $\epsilon(\mathbf{x})$ exhibits no long-range correlations), and the distribution of singular values of K obeys the quarter-circle law.

Figure 1 shows the two-dimensional set-up considered in our modeling approach: A chain of antennas i is placed along the x-direction with a distance R in the z-direction from the scattering medium. The optical properties of the scattering medium are characterized by a fixed magnetic permeability $\mu=1$ and a statistically homogeneous random dielectric function $\epsilon(x,z)=\bar{\epsilon}+\eta(x,z)$, where $\bar{\epsilon}=\langle\epsilon(x,z)\rangle$ denotes its mean value, and the fluctuating part $\eta(x,z)$ is a uniformly distributed white noise with correlation function $\langle\eta(x_1,z_1)\eta(x_2,z_2)\rangle=W^2\delta(x_1-x_2)\delta(z_1-z_2)$. For the emitted pulse in elements i, we consider two different polarizations of the electromagnetic field, corresponding to incident electric fields with either perpendicular or parallel orientation to the scattering medium. The pulse $\psi_i(x,t)$ emitted from array's antenna i has central frequency ω and is Gaussian in time with width σ_t and Gaussian in the center of the antenna i with width σ_x:

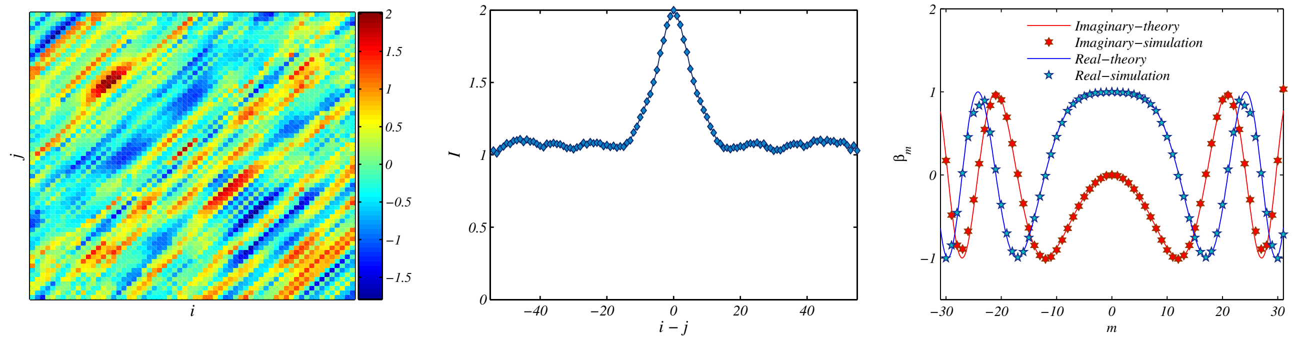
$$\psi_i(x,t)=\sin(\omega t)\exp\left\{-\frac{(t-t_0)^2}{\sigma_t^2}\right\}\exp\left\{-\frac{(x-i)^2}{\sigma_x^2}\right\}. \qquad (1)$$

The field $\Psi_i(x,t)$ is the electric or the magnetic field.

To simulate the multiple and single scattering processes, we considered a set-up as shown in fig. 1 and locate a bar with random dielectric constant in each lattice point. Then we solve the Maxwell equations by the finite-difference time-domain (FDTD) method [21–23]. To apply the FDTD method for the Maxwell equations one needs to write the discrete form of the equations in a staggered mesh (Yee's mesh) [21]. The random part of the dielectric constant is rescaled as $\eta'=\eta/\bar{\epsilon}$ and is non-correlated and uniformly distributed in the interval $[-\sigma,+\sigma]$. The permittivity should be positive, so there is a restriction for the amplitude of disorder, *i.e.* $|\sigma|<1$ (the details can be found in [22]). The variance W^2 is related to σ as $W^2=\frac{\sigma^2}{3d^2}$. Also the lattice constant considered in the dimensionless Maxwell equations is $d=1$, the time step is $\mathrm{d}t=d/8=1/8$. The whole parameters reported here are in rescaled space-time units. The whole system is a $500d\times1000d$ grid with disordered medium of size $250d\times1000d$ at a distance $z=R=250d$ from the array. The simulation stops before the boundaries affect the backscattered waves. There are 64 antennas in the array at $z=0$ that send and receive the wave.

We start with the situation where the incident electric field is parallel to the rods. Let $h_{ij}(t)$ be the backscattered signal detected in antenna j when the initial pulse is emitted into the medium from antenna i. Because of the reciprocity the impulse response matrix $H(t)$ is symmetric. To study the process in different time scales, $H(t)$ is truncated into short time windows. For different time windows (Δt), we have $k_{ij}(T,t)=h_{ij}(T-t)[\Theta(t+\Delta t/2)-\Theta(t-\Delta t/2)]$, where Θ is the Heaviside function. The backscattering matrix $K(T,f)$ at time T and frequency f is calculated by the discrete Fourier transform of the matrix $K(T,t)$. Calculating this matrix is the first step in the study of the coherent backscattering and then investigation of the single and multiple scattering. The backscattered intensity $I_{i-j}=\langle K_{ij}K^*_{ij}\rangle_{T,f}$ is calculated by averaging over time and frequency where the difference $i-j$ is fixed.

The simulations are done for two different intensities of the disorder and the backscattered wave has different behaviors. Figure 2(a) is the imaginary part of the matrix K in early times for a medium with $\sigma=0.3$, and wave frequency $\omega=1.5$. In the early times the wave has propagated by few numbers of scatters and it is considered as a single scattering regime. Its rippled structure is completely consistent with the experimental results for ultrasounds [18]. In fig. 2(b) the intensity of the backscattered wave is shown for the same system as (a) which is averaged



(a) (b) (c)

Fig. 2: (Color online) Single scattering from the medium with $\sigma = 0.3$ and wave frequency $\omega = 1.5$. (a) The backscattered matrix K averaged over early times and whole frequencies. (b) The intensity of backscattered wave as a function of distance between source and receivers. (c) The imaginary and real part of β in theory and simulation.

for a long time interval. The initial wave is sent from the antenna i through the medium and the backscattered wave is detected at the receiver j. It can be seen that the intensity of the backscattered wave at the source is twice the other receivers at a distance.

Let us start with the single scattering regime. In optics Fresnel's diffraction or near-field diffraction occurs when the waves propagate through the medium in the near field or earlier times (when the wave has not propagated from the further scatters). If we know the electric field at the surface $(x', y', 0)$, then the electric field in each point of the space (x, y, z) is given by

$$E\ (x,y,z) = \frac{z}{i\lambda}\iint E(x',y',0)\frac{e^{ikr}}{r^2}\mathrm{d}x'\mathrm{d}y' \approx \frac{e^{ikz}}{i\lambda z}\iint E(x',y',0)e^{\frac{ik}{2z}[(x-x')^2+(y-y')^2]}\mathrm{d}x'dy'. \tag{2}$$

The second term is calculated by Fresnel's approximation. In the near-field backscattering the elements k_{ij} of backscattering matrix can be written as [18]

$$k_{ij}\left(T, f = \frac{ck}{2\pi}\right) \propto \frac{e^{2ikR}}{R}\sum_{d=1}^{N_d} A_d \exp\left[\frac{ik(x_i - X_d)^2}{4R}\right]\exp\left[\frac{ik(x_j - X_d)^2}{4R}\right], \tag{3}$$

where $2R = cT$ and x_i is the coordinate of the i-th antenna in the array and X_d is the transverse position of the d-th scatterer which is placed approximately at distance R from the array or contributes in the backscattered wave at time T. The amplitude A_d depends on the dielectric constant of the d-th scatterer and shows its reflectivity. Here the dielectric constant is random and so the amplitude A_d is also random and k_{ij} can be written as a multiplication of deterministic and random terms:

$$k_{ij} \propto \frac{e^{2ikR}}{R}\exp\left[\frac{ik(x_i - x_j)^2}{8R}\right] \times \sum_{d=1}^{N_d} A_d \exp\left(\frac{ik(x_i + x_j - 2X_d)^2}{8R}\right). \tag{4}$$

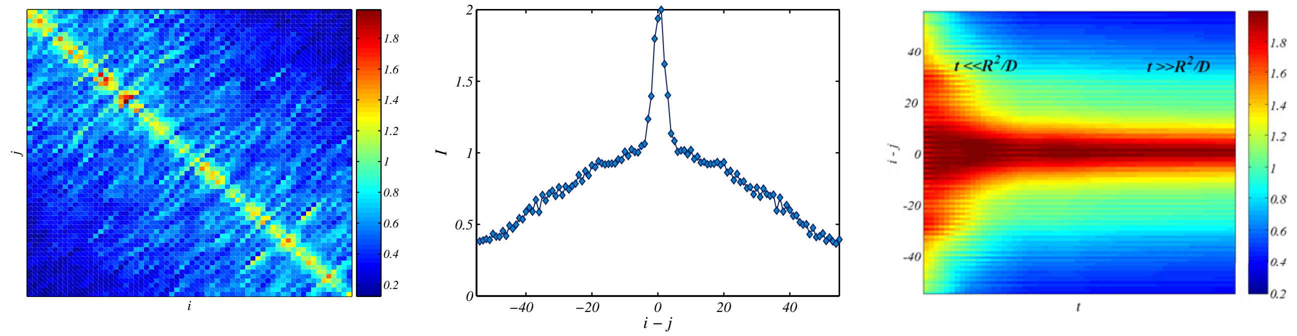
We note that the parameter β_m is deterministic [18]

$$\beta_m = \frac{k_{(i-m)(i+m)}}{k_{ii}} = \exp\left[\frac{ikm^2}{2R}\right]. \tag{5}$$

Figure 2(c) shows the real and imaginary part of β for the same system as in fig. 2(a) and (b) and for early times. It is in good agreement with the theoretical prediction in (5).

In fig. 3(a) the backscattered intensity matrix $K^2 = \langle K_{ij}K_{ij}^*\rangle_{T,f}$ from the disordered medium is plotted. The intensity of disorder is $\sigma = 0.9$ and the frequency of the propagated wave is $\omega = 1.5$. The matrix is averaged over late times and frequencies. Figure 3(b) is the backscattered intensity $I_{i-j} = \langle K_{ij}K_{ij}^*\rangle_{T,f}$ from the same media as in fig. 3(a). It has the same information as in fig. 2(a). Comparing to fig. 2(b), it can be seen that the backscattered intensity has different behavior for different intensity of disorder. In fig. 3(c) the backscattered intensity is calculated for different times and for early times it can be seen that the peak's width is bigger than for late times. For short times the wave has just propagated through a few layers of scatterers and is considered as a single scattering regime with a wide peak in the backscattered intensity. However for long times the wave has scattered back by too many rods and is considered as a multiple scattering regime with a narrower peak in the backscattered intensity.

As mentioned above we have considered two polarizations in which the electric field is parallel to the rods or is perpendicular to them. One can show that the backscattered intensity matrix for the magnetic field has



(a) (b) (c)

Fig. 3: (Color online) Multiple scattering from the medium with $\sigma = 0.9$ and wave frequency $\omega = 1.5$. (a) The backscattered intensity matrix K^2 averaged over time and frequency. (b) The intensity of backscattered wave for long times as a function of distance between source and receivers. (c) The backscattered intensity (same as (b)) in different times.

a similar behavior, when we take the magnetic field to be parallel to the rods. The probability distribution function of the matrix elements for the multiple scattering regime is calculated for both real and imaginary parts and over the whole frequencies and time intervals. The elements of each scattering matrix at a specific time and frequency are normalized to the variance of matrix elements. As is shown in the inset of fig. 4 it is not well fitted by the Gaussian distribution. There is a deviation from the Gaussian distribution in the tails of the distribution. However by considering only the elements that are not close to the diagonal of the scattering matrix the distribution becomes closer to the Gaussian distribution. The distribution is almost Gaussian for the single scattering regime. A Gaussian distribution function with the same variance is plotted to have a comparison.

In the multiple scattering regime the distribution of singular values $\rho(s)$ will be a standard parameter to study. This distribution is different for different regimes and in multiple scattering it is close to the quarter-circle law. The $N \times N$ matrix whose complex elements are zero-mean, independent and identically distributed variables with variance $1/N$ has quarter-circle singular value distribution $\rho(s) = \frac{1}{\pi}\sqrt{4 - s^2}$ [24]. The singular value decomposition of the $N \times N$ matrix $K(T, f)$ provides N singular values s_i for specific time T and frequency f. In order to compare the results with the random matrix theory we should normalize all s's as follows: $\widetilde{s}_i = s_i[\sum_{j=1}^{N} s_j^2/N]^{-\frac{1}{2}}$, where $\rho(s)$ is the distribution of different sets of normalized s's for different times and frequencies. Figure 4 (main panel) shows the singular value distribution for high intensity of disorder, $\sigma = 0.9$, high frequency, $\omega = 1.5$, whose backscattered intensity is shown in fig. 3(b). Also the quarter-circle distribution is plotted for comparison. The same result is calculated by the weaker scatterer ($\sigma = 0.3$) but for the late times.

To relax the white-noise type randomness we have repeated the numerical simulations with a Gaussian *short-range*–correlated disorder. To generate the short-range correlations we used the Kernel method [25] with some

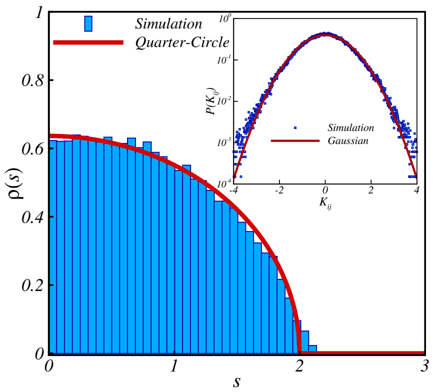


Fig. 4: (Color online) The distribution of singular values of the backscattered matrix K is plotted for the random medium with $\sigma = 0.9$ (same as fig. 3) and for frequency $\omega = 1.5$. The bars shows the result of the simulation and the solid line is the quarter-circle law for the symmetric random matrices. The inset shows the distribution of the elements of the backscatterd matrix K_{ij} for the same system which is close to the Gaussian distribution.

finite width (for instance $h = 3d$, and $h = 6d$, where d is the mesh size and h is correlation length of the disorder) (see fig. 5). We found that the distribution is not exactly a quarter-circle law. However, if we consider each second array element (both in the columns and the rows) in the scattering matrix (for correlation lengths $h = 3d$, and $h = 6d$), then the distribution follows again the quarter-circle law. This shows that the effect of correlations is reflected in the scattering matrix but this effect can be removed by considering elements with a larger spacing corresponding to a coarse-graining procedure. By increasing the correlation length the medium becomes almost transparent, corresponding to the single scattering regime.

In summary, the single and multiple scattering regimes of electromagnetic waves in media with random permittivity are studied for two different types of polarization (one of the electric or magnetic fields is parallel to the rods

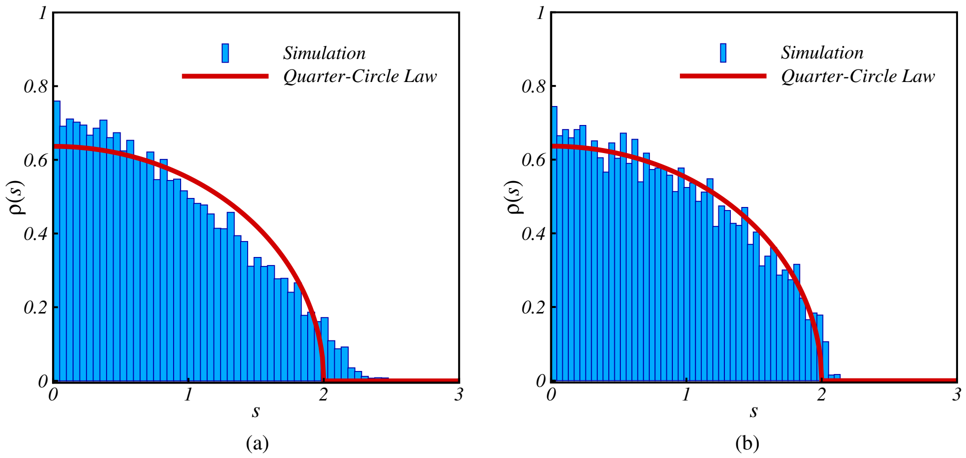


Fig. 5: (Color online) Singular value distribution of (a) the backscattered matrix K (b) the matrix obtained by removing the neighboring elements of K to eliminate inter-element correlations. The simulation is for random media with Gaussian distribution and short-range correlation (variance of the Gaussian disorder is 0.3 and the correlation length is $6d$) and the incident wave frequency is $\omega = 1.5$.

with random dielectric constants). We showed that the statistical properties of the backscattering matrix from non-correlated random media are in agreement with the observations in recent experimental results for ultrasonic waves and light. The matrix's elements are correlated for short-range–correlated disorder and also for short times (both polarizations) and these correlations are limited to adjacent elements and by considering only one of the two elements, the distribution of singular values becomes the same as quarter-circle law.

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