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Logarithmic Correlation Functions in Two Dimensional Turbulence

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Abstract

We consider the correlation functions of two-dimensional turbulence in the presence and absence of a three-dimensional perturbation, by means of conformal field theory. In the persence of three dimensional perturbation, we show that in the strong coupling limit of a small scale random force, there is some logarithmic factor in the correlation functions of velocity stream functions. We show that the logarithmic conformal field theory $c_{8,1}$ describes the 2D- turbulence both in the absence and in the presence of the perturbation. We obtain the following energy spectrum $E(k) \sim k^{-5.125} \ln(k)$ for perturbed 2D - turbulence and $E(k) \sim k^{-5} \ln(k)$ for unperturbed turbulence. Recent numerical simulation and experimental results confirm our prediction.

1 Introduction

Polyakov has shown [1] that the exponent of the energy spectrum of 2D-turbulence can be found by means of conformal field theory. He shows that the energy spectrum behaves as $k^{4\Delta_\phi+1}$, where Δ_ϕ is the dimension of the velocity stream function. This spectrum is different from the $k^{-3}\ln^{\frac{-1}{3}}(kL)$ law proposed by Kraichnan [2]. Experimental and numerical simulation results seem to be even more controversial. Nearly all of the experiments have concentrated on the case of decaying turbulence, which depends strongly on the initial conditions [3]. These experiments predict the energy spectrum to be initially proportional to k^{-3}, with the exponent changing with time. The case of decaying turbulence has been considered recently by many authors (see [4] and references therein) in the context of confomal field theory. Stationary experiments of the two-dimensional turbulence has been considered in [5-7], which show that there is a strong deviation from a k^{-3} spectrum. Borue [8] has performed direct numerical simulation of the 2D-Navier-Stokes equations with a white noise in time, and with non-zero correlation in momentum space at some characteristic scale k_f ($k_f \sim \frac{1}{L}$, where L is the scale of system). The main results of [8] are as follows: both the stirring force and dissipation lead to a correction to k^{-3} law, and near $k \sim k_f$ the energy spectrum behaves as $E(k) \sim k^{-3}\ln(kL)^{\frac{-1}{3}}$ which is different from the one-loop result of Kraichnan [2]. However Falkovich and Lebedov [9] have derived Kraichnan's spectrum by using Quasi-Lagrangian (QL) variables. They predict the correlation of vorticity to be $< \omega^n(r_1)\omega^n(r_2) > \sim \ln^{\frac{2n}{3}}(\frac{L}{|r_1-r_2|})$, which for $n = 1$ gives the Kraichnan spectrum. Here we wish to address the question: how can one find an energy spectrum with a logarithmic factor in the context of conformal turbulence. Recently we have considered the existence of such logarithmic factors with integer power, in the energy spectrum of turbulent 2D - magnetohydrodynamics [10]. In ref.[10] it was shown that, when the Alf‘ven effect (i.e. equipartition of energy between velocity and magnetic modes [11]) is taken into account one is naturally lead to consider, conformal field theories which have logarithmic terms in their correlation functions. There has been considerable interest in logarithmic conformal field theories (LCFT) [10-21,27]. In these theories there exist atleast two field with equal conformal dimensions, such theories admit logarithmic correlation functions [12]. Recently Moriconi [22] has considered the problem of conformal turbulence taking into account the influence of three-dimensional effects. He has considered a quasi two-dimensional fluid which is perturbed by a small scale noise representing the effect of the additional degree of freedom perpendicular to the plane of motion. Another scheme of perturbation was considered in [23]. Here we consider the perturbed conformal turbulence proposed in [22] and in the context of quasi-two-dimensional turbulence, we first, show that there are some constraints similar to the condition imposed by the Alf‘ven effect. Then we show that these constraints guarantee an energy spectrum with a logarithmic factor, and present a solution.

The paper is organized as follows; in section 2 we give a brief summary of perturbed conformal turbulence and the implication of logarithmic terms in correlation functions of velocity stream function. In section 3 we consider the strong coupling limit of small scale external random force perpendicular to plane of motion, and find the constraints for reducing the number of candidate CFT models. We find a solution within the $c_{p,1}$ series and derive the energy spectrum and show that the model $c_{8,1}$ describe both perturbed and unperturbed 2D-turbulence.

2 Quasi Two - Dimentional Turbulence

There are interesting experiments [24], which have investigated the 2D turbulent fluid, where there was a fluctuating grid responsible for the perturbation of the two-dinensional motion of the fluid. The important observation is that the fluid should be described in terms of two-dimensional equations containing not only the large scale forces but also a small-scale random perturbation along the direction perpendicular to the direction of motion. Two-dimensional Navier-Stokes equations take the form,

$$\partial_t v_\alpha + v_\beta \partial_\beta v_\alpha = \nu \partial^2 v_\alpha + f_\alpha^{(1)} + g f_\alpha^{(2)} - \partial_\alpha P \tag{1}$$

where v is the velocity field, and ν is the viscosity and $f_\alpha^{(1)}(x,t)$ and $f_\alpha^{(2)}(x,t)$ are the stirring forces defined at large scales L and medium scales $a << y << L$, respectively, a is the dissipation scale. An appropriate correlation for $f_\alpha^{(1)}$ is

$$< f_\alpha^{(1)}(\mathbf{x},t) f_\alpha^{(2)}(\mathbf{x}',t') >= \delta_{\alpha\beta} k(0)(1 - \delta((\mathbf{x}-\mathbf{x}')^2 - L^2))\delta(t-t') \tag{2}$$

where $\alpha, \beta = 1, 2$, $f_3^{(1)} = 0$ and $f_1^{(2)} = f_2^{(2)} = 0, f_3^{(2)} \neq 0$. The dimensionless constant g shows a coupling with the three-dimensional modes of the fluid. As pointed out in ref.[20,24], when we have some external noise along the direction perpendicular to the direction of motion one has to be careful of the compressibility condition for velocity field in two dimensions. When considering the three-dimensional velocity field, compressibility condition is $\partial_i v_i = 0$, where $i = 1, 2, 3$. If we project this constriant to the two-dimensional plane it follows that $\partial_\alpha v_\alpha = O(g)$, $\alpha = 1, 2$. Therefore to take this point into account, the velocity field may be written as

$$v_\alpha = \epsilon_{\beta\alpha} \partial_\beta \psi + g \partial_\alpha \phi \tag{3}$$

Where ψ and ϕ are the velocity stream function and the velocity potential, respectively. The divergence of two dimentional velocity field is $\rho = g\partial^2\phi$. Following [22], we expand ψ and ϕ in powers of g in the following forms:

$$\psi = \sum_{n=0}^{\infty} g^n \psi^{(n)} \qquad \omega = \sum_{n=0}^{\infty} g^n \omega^{(n)} \tag{4}$$

$$\phi = \sum_{n=0}^{\infty} g^n \phi^{(n)} \qquad \rho = \sum_{n=0}^{\infty} g^{n+1} \rho^{(n)} \tag{5}$$

By taking curl of both sides of eq.(1) and substituting eqs.(4) and (5), one can find exact infinite chain of equations for $\omega^{(n)}$ and $\rho^{(n)}$ as follows.

$$\partial_t\omega^{(n)} \quad + \textstyle\sum_{p=0}^{n} \epsilon_{\alpha\beta}\partial_\alpha\psi^{(p)}\partial_\beta\partial^2\psi^{(n-p)} + \sum_{p=0}^{n-1}\left[\partial_\beta\phi^{(p)}\partial_\beta\partial^2\psi^{(n-p-1)} + \partial^2\phi^{(p)}\partial^2\psi^{(n-p-1)}\right]$$
$$= \nu\partial^2\omega^{(n)} + \epsilon_{\alpha\beta}\partial_\alpha f_\beta^{(2)}\delta_{n,1}\ , \tag{6}$$

$$\partial_t\omega^{(0)} \qquad + \epsilon_{\alpha\beta}\partial_\alpha\psi^{(0)}\partial_\beta\partial^2\psi^{(0)} = \nu\partial^2\omega^{(0)} + \epsilon_{\alpha\beta}\partial_\alpha f_\beta^{(1)} \tag{7}$$

$$\partial_t\rho^{(n)} + \textstyle\sum_{p=0}^{n-1}\left[\partial_\alpha\partial_\beta\phi^{(p)}\partial_\alpha\partial_\beta\phi^{(n-p-1)} + \partial_\alpha\phi^{(p)}\partial_\alpha\partial^2\phi^{(n-p-1)}\right]$$
$$+ \textstyle\sum_{p=0}^{n}\left[2\epsilon_{\alpha\beta}\partial_\beta\partial_\sigma\phi^{(p)}\partial_\alpha\partial_\sigma\psi^{(n-p)} + \epsilon_{\alpha\beta}\partial_\alpha\psi^{(n-p)}\partial_\beta\partial^2\phi^{(p)}\right]$$
$$+ \textstyle\sum_{p=0}^{n+1}\left[\partial_\alpha\partial_\beta\psi^{(p)}\partial_\alpha\partial_\beta\psi^{(n-p+1)} - \partial^2\psi^{(p)}\partial^2\psi^{(n-p+1)}\right] = \nu\partial^2\rho^{(n)} \tag{8}$$

$$\partial_t\rho^{(0)} + 2\partial_\alpha\partial_\beta\psi^{(0)}\partial_\alpha\partial_\beta\psi^{(1)} + 2\epsilon_{\alpha\beta}\partial_\beta\partial_\sigma\phi^{(0)}\partial_\alpha\partial_\sigma\psi^{(0)} + \epsilon_{\alpha\beta}\partial_\alpha\psi^{(0)}\partial_\beta\partial^2\phi^{(0)}$$
$$-2\partial^2\psi^{(0)}\partial^2\psi^{(1)} = \nu\partial^2\rho^{(0)} + \partial_\alpha f_\alpha^{(2)} \tag{9}$$

Eq.(7) is identical to the case of an unperturbed (i.e. $g = 0$) two-dimensional turbulent fluid. This means that the field $\psi^{(0)}$ will be related to an enstrophy or energy cascade, even in the presence of three dimensional fluctuations. In other words the enstrophy and energy cascade conditions do not change in the presence of 3D- perturbation. We will consider this important point in the end of this next section. The basic assumption of Moriconi [22], is that not only $\psi^{(0)}$ but also the other components in the power expansions of ψ and ϕ are primary operators which belong to some conformal field theory. There is an important point here: noting that g is a dimensionless coupling constant eqs. (4) and (5) tell us all the $\psi^{(n)}$'s and $\phi^{(n)}$'s have the same scaling dimension. To avoid this difficulty it has been suggested [22] that there is a hidden scale l in the problem which may be related to the intermittency effect. Therefore expansions in the eqs. (4) and (5) change to

$$\psi = \Sigma f_n l^{2\Delta_{\psi^{(n)}}} g^n \psi^{(n)} \tag{10}$$

$$\phi = \Sigma f_n' l^{2\Delta_{\phi^{(n)}}} g^n \phi^{(n)}$$

where l is some scale proportional to $\sim \nu^\alpha < \omega^2 >^\beta < v^2 >^\gamma$ where $2\alpha + 2\gamma = 1$ and $\alpha + 2\beta + 2\gamma = 0$. If one accepts this prescription still there is some ambiguity in the determination of the energy spectrum exponents, as one can select all of the exponents $4\Delta_{\phi^{(n)}} + 1$ and $4\Delta_{\psi^{(n)}} + 1$ as the exponents of the energy spectrum. In this situation we have to use a CFT with an infinite number of primary fields. However if we restrict ourselves to the strong coupling limit, a finite number of primary fields suffices as can be seen from eq.(13) below.

Furthermore there is the possiblity of some fields $\psi^{(n)}$ and $\phi^{(n)}$ in the expansions (10) having the same scaling dimension this leads to logarithmic correlators. According to Gurarie [12] if the operator product expansion (OPE) of some fields in CFT model possess at least two operators with the same dimension, one naturally gets logarithmic correlation functions. In other words the operators with the same scaling dimension form the basis of the Jordan cell for L_0[12]. Here we assume that two or more operators form the basis of the Jordan cell for operator L_0 i.e. they have equal dimensions, then:

$$
\begin{aligned}
L_0\Psi^{(n)} &= \Delta_{\Psi^{(n)}}\Psi^{(n)} + \Psi^{(n-1)} \qquad n > 0 \\
L_0\Psi^{(0)} &= \Delta_{\Psi^{(0)}}\Psi^{(0)}
\end{aligned}
\tag{11}
$$

where $\Psi^{(n)}$ may be any of $\psi^{(n)}$ or $\phi^{(n)}$. Therefore the standard OPE [12] takes the following form:

$$
\Phi^{(n_1)}(z)\Phi^{(n_2)}(0) = z^{2(\Delta_{\Psi^{(m-n)}} - \Delta_{\Psi^{(n_2)}} - \Delta_{\Psi^{(n_1)}})}(\Sigma_n \ln^n(z)\Psi^{(m-n)} + decendents) \tag{12}
$$

This argument shows that, we have to consider logarithmic conformal field theories as candidates for describing such systems. To find the explicit form of the energy spectrum, we concentrate on the strong coupling limit.

3 Strong Coupling Limit and The Logarithmic Correlation

As discussed in [22], by only considering $\psi^{(0)}$ and $\psi^{(1)}$ and $\phi^{(0)}$, we can completely describe the strong coupling limit of eqs.(1). Noting that if the constant flux condition is not satisfied by the pair of fields $\psi^{(0)}$ and $\phi^{(0)}$, then there exists no further soultions for the model under consideration. Therefore it is enough to consider those models for which the field $\psi^{(0)}$ satisfies the nonperturbative constraint and $\psi^{(1)}$ and $\phi^{(0)}$ satisfy the constraints associated with the three-dimensional effect. Taking this into account , the expansions of ψ

and ϕ can be written as

$$\begin{aligned}\psi &= \psi^{(0)} + f_a(g)\psi^{(1)} \\ \phi &= f_b(g)\phi^{(0)}\end{aligned} \tag{13}$$

where $f_a(0) = f_b(0) = 0$. Following [22] we consider three limits for $f_{a,b}$

$$\begin{aligned} &a) \quad g \to 0 \qquad f_{a,b} \to 0 \\ &b) \quad g \to 0 \qquad f_{a,b} \sim 1 \\ &c) \quad g \to \infty \qquad f_{a,b} \to diverges \end{aligned} \tag{14}$$

the case (c) may be defined as the strong coupling regime. It is clear that in the case of the strong coupling, the contribution of $\psi^{(0)}$ field can be discarded. However in the presence of any perturbation the constant enstrophy cascade condition depend only on $\psi^{(0)}$, which as mentioned follows fron eq.(7). It has been shown in [22], that eq.(14-a) dose not give any correction to the power law spectrum. Careful consideration shows that, the other two cases lead to a logarithmic factor in the energy spectrum. In the strong coupling limit, where the inertial range exponent derived from $\psi^{(0)}$ may be discarded and the exponent can be determined by considering the the dimension of $\psi^{(1)}$ and $\phi^{(0)}$. Here we can apply the same argument as that of [11]. Applying the dynamical renormalization group one can show that by the existence of a critical dynamical index of eqs. (6), (7) and (9) for $\psi^{(1)}$ and $\psi^{(0)}$ and $\phi^{(0)}$, leads to condition that

$$\Delta_{\psi^{(0)}} = \Delta_{\psi^{(1)}} = \Delta_{\phi^{(0)}} \tag{15}$$

This means that in the steady state there is some equipartition of energy between different components of the velocity field [11].

However in the strong copling limit the contribution of $\psi^{(0)}$ field can be discarded therefore we have two fields (i.e $\psi^{(1)}$ and $\phi^{(0)}$) in some CFT model and one naturally obtains a logarithmic factor in the energy spectrum. The main idea is as follows.

the operator product expansion of two fields A and B, which have two fields $\phi^{(0)}$ and $\psi^{(1)}$ of equal dimensions in their fusion rule, has a logarithmic term:

$$A(z)B(0) = z^{h_\phi - h_A - h_B}\{\psi^{(1)}(0) + \ldots + \log z(\phi^{(0)}(0) + \ldots)\} \tag{16}$$

to see this it is sufficient to look at the four-point function :

$$< A(z_1)B(z_2)A(z_3)B(z_4) >\sim \frac{1}{(z_1 - z_3)^{h_A}} \frac{1}{(z_2 - z_4)^{h_B}} \frac{1}{[x(1-x)]^{h_A + h_B - h_\phi}} F(x) \tag{17}$$

Where the cross ratio x is given by :

$$x = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)} \tag{18}$$

In degenerate minimal models $F(x)$ satisfies a second order linear differential equation. Therefore a solution for $F(x)$ can be found in terms of a series expansion :

$$F(x) = x^\alpha \sum a_n x^n \tag{19}$$

It can be easily shown that the existence of two fields with equal dimensions in OPE of A and B is equivalent to the secular equation for α having coincident roots [9], in which case two independent solutions can be constructed according to :

$$\sum b_n x^n + \log x \sum a_n x^n \tag{20}$$

Now consistency of equations (16) and (20) requires :

$$< A(z_1)B(z_2)\psi(z_3) >=< A(z_1)B(z_2)\phi(z_3) > \{\log \frac{(z_1 - z_2)}{(z_1 - z_3)(z_2 - z_3)} + \lambda\} \tag{21}$$

$$< \psi(z)\psi(0) >\sim \frac{1}{z^{2h_\psi}}[\log z + \lambda^{'}] \tag{22}$$

$$< \psi(z)\phi(0) >\sim \frac{1}{z^{2h_\phi}} \tag{23}$$

where λ and $\lambda^{'}$ are constants. The IR-problem of such system has been discussed in [9]. The energy spectrum of this type of correlations has following:

$$E(k) \sim k^{-4|\Delta_{\phi^{(0)}}|+1} \ln(kL) \tag{24}$$

Let us rewrite the constant enstrophy condition, in order to find a logarithmic CFT for 2D - turbulence. Consider the fusion of field $\psi^{(0)}$ with itself:

$$\psi^{(0)} \times \psi^{(0)} = \chi + \cdots \tag{25}$$

such that χ is the field with minimum conformal dimension , on the right hand side. Then the constant enstrophy condition implies:

$$\Delta_{\psi^{(0)}} + \Delta_\chi = -3 \qquad and \qquad \Delta_\chi > 2\Delta_{\psi^{(0)}} \tag{26}$$

According to [22] field $\psi^{(0)}$ will be related to an enstrophy cascade, even in the presence of three - dimensional effect. A possible candidate may exist within the $c_{p,1}$ series [19,25]. The central charge for this series is $c = 13 - 6(p + p^{-1})$. This series is special since it has $c_{eff} = 1$. These CFT‘s posess $3p - 1$ highest weight representation with conformal dimensions:

$$h_{ps} = \frac{(p-s)^2 - (p-1)^2}{4p} \qquad 1 \leq s \leq 3p - 1 \tag{27}$$

of these $2(p-1)$ have pair wise equal dimensions. Two field ϕ_s and $\phi_{s'}$ have equal and negative weights provided that $s + s' = 2p, (s \neq 1, 2p - 1)$. Let us adopt such a pair as candidates for $\psi^{(1)}$ and $\phi^{(0)}$. We can now look for the candidate values of s such that eq.(26)

is satisfied. The only solution is given by $p = 8$ with $c = -\frac{286}{8}$, where the set of fields with negative dimensions are:

$$(-\frac{3}{2}, -\frac{3}{2}, -\frac{3}{4}, -\frac{3}{4}, -\frac{5}{4}, -\frac{5}{4}, -\frac{13}{32}, -\frac{13}{32}, -\frac{33}{32}, -\frac{33}{32}, -\frac{45}{32}, -\frac{45}{32}, -\frac{49}{32}, -\frac{49}{32}) \tag{28}$$

With $\Delta_{\psi^{(0)}} = -\frac{3}{2}$. Therefore for unperturbed turbulence we have:

$$E(k) \sim k^{-5} \ln(k) \tag{29}$$

For perturbed turbulence $\phi^{(0)}$ and $\psi^{(1)}$ can be assigned as any pair from this LCFT. Different choices lead to different exponents for the energy spectrum, these are:

$$(-2, -4, -0.625, -4.025, -5.125, -3.125) \tag{30}$$

therefore the best exponent to fit the experimental data [22,7] is -5.125 which corresponds to conformal weights:

$$\Delta_{\phi_{(0)}} = \Delta_{\psi^{(1)}} = -\frac{49}{32} \tag{31}$$

The energy spectrum thus is given by:

$$E(k) \sim k^{-5.125} \ln(k) \tag{32}$$

which confirms witn the numerical analysis by Borue [8], and experimental date [22,7]. Furthermore we can relax the conditions and ask which types of conformal field theory may be used for modeling of 2D - turbulence , provided we assume the condition

$$\Delta_{\phi^{(0)}} \simeq \Delta_{\psi^{(1)}} \tag{33}$$

as well as the cascades of constant enstrophy and the constant energy. In [9], it has been shown that the standard minimal models, do not have logarithmic correlators becuase for (p

, q) coprime, all the primary fields in minimal models have different dimensions. However almost logarithmic behavior is obtained when two primary fields have almost equal conformal dimensions. Where we have $|\Delta\phi^{(0)} - \Delta\psi^{(1)}| = \epsilon$, and ϵ satisfies [10]:

$$\epsilon \leq \frac{1}{5/2 \log R_e} \tag{34}$$

provided z lies in inertial range:

$$a << z << R \tag{35}$$

where R_e is the Reynold's number of system, a and R are the dissipation and the large scales of the system repectively. The table of CFT models are nearly consistent with eq.(25) is given in reference [22]. Eq.(34) is the relation of the dimensions of fields $\psi^{(1)}$ and $\phi^{(0)}$ and the Reynold's number of system.

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