Localizaton of Elastic Waves in Heterogeneous Media with Off-Diagonal Disorder and Long-Range Correlations

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Using the Martin-Siggia-Rose method, we study propagation of acoustic waves in strongly heterogeneous media which are characterized by a broad distribution of the elastic constants. Gaussian-white distributed elastic constants, as well as those with long-range correlations with nondecaying power-law correlation functions, are considered. The study is motivated in part by recent discovery that the elastic moduli of rock at large length scales may be characterized by long-range power-law correlation functions.

Depending on the disorder, the renormalization group (RG) flows exhibit a transition to localized regime in any dimension. We have numerically checked the RG results using the transfer-matrix method and direct numerical simulations for one- and two-dimensional systems, respectively.

Understanding how waves propagate in heterogeneous media is fundamental to such important problems as earthquakes, underground nuclear explosions, the morphology of oil and gas reservoirs, oceanography, and medical and materials sciences [1]. For example, seismic wave propagation and reflection are used not only to estimate the moduli of rock at large length scales may be characterized by long-range power-law correlation functions.

Scattering, caused by the heterogeneities of the medium, may cause their localization. Unlike electrons, however, the classical waves do not interact with one another and, therefore, propagation of such waves in heterogeneous materials provides [7] an ideal model for studying the classical Anderson localization [8–12] in strongly disordered media. This is the focus of this Letter. We study localization of acoustic waves in strongly heterogeneous media, and formulate a field-theoretic method to investigate the problem in the media that are characterized by a broad distribution of the elastic constants. Localization of acoustic waves was previously studied by several groups [13]. In particular, Baluni and Willemesen [14] studied the propagation of acoustic waves in a 1D layered system. However, these studies [13,14] did not consider the type of systems that we consider in the present Letter, which represent the continuum limit of an acoustic system with an off-diagonal disorder.

Our approach is based on the method first introduced by Martin, Siggia, and Rose [15] for analyzing dynamical critical phenomena. We calculate to one-loop order the beta functions [8,15] for both spatially delta-correlated and power-law correlated disorder in the elastic constants, and show that in any case there is a disorder-induced transition from delocalized to localized states for any $d$. In addition to its general importance, our study is motivated in part by the recent discovery [16] that the distribution of the elastic moduli of heterogeneous rock contains long-range correlations characterized by a nondecaying power-law correlation function. However, our results are completely general and apply to any material in which the local elastic constants are distributed by the type of distributions we consider.
Wave propagation in a medium with a distribution of elastic constants is described by the following equation (for simplicity we consider the scalar wave equation):

\[ \frac{\partial^2}{\partial t^2} \psi(x, t) - \nabla \cdot [\lambda(x) \nabla \psi(x, t)] = 0, \quad (1) \]

where \( \psi(x, t) \) is the wave amplitude, and \( \lambda(x) = C(x)/m \) is the ratio of the elastic stiffness \( C(x) \) and the mean density \( m \) of the medium. We then write \( \lambda \) as

\[ \lambda(x) = \lambda_0 + \eta(x), \quad (2) \]

where \( \lambda_0 = \langle \lambda(x) \rangle \). We assume \( \eta(x) \) to be a Gaussian random process with a zero mean and the covariance

\[ \langle \eta(x) \eta(x') \rangle = 2K(|x - x'|) \]

\[ = 2D_0 \delta^d(x - x') + 2D_\rho |x - x'|^{2d - d}, \quad (3) \]

in which \( D_0 \) and \( D_\rho \) represent the strength of the disorder

\[
S_c(\psi_l, \psi_R, \tilde{\psi}_l, \chi, \chi^*) = \int dx dx' \left[ i \tilde{\psi}_l(x') \left( \nabla^2 + \frac{\omega^2}{\lambda_0} \right) \psi_l(x) + i \tilde{\psi}_R(x') \left( \nabla^2 + \frac{\omega^2}{\lambda_0} \right) \psi_R(x) + \chi^*(x') \left( \nabla^2 + \frac{\omega^2}{\lambda_0} \right) \chi(x) \right] \\
\times \delta(x - x') + (i \nabla \tilde{\psi}_l \nabla \psi_l + i \nabla \tilde{\psi}_R \nabla \psi_R + \nabla \chi \nabla \chi) \frac{K(x - x')}{\lambda_0^2} (i \nabla \tilde{\psi}_l \nabla \psi_l + i \nabla \tilde{\psi}_R \nabla \psi_R + \nabla \chi \nabla \chi). \quad (5)
\]

Here, \( \tilde{\psi}_l(x), \tilde{\psi}_R(x), \chi, \) and \( \chi^* \) are the auxiliary and Grassmanian fields of the field-theoretic formulation, respectively, and \( g_0 = D_0/\lambda_0^2 \) and \( g_\rho = D_\rho/\lambda_0^2 \) are two coupling constants. Thus, we carry out a renormalization group (RG) analysis in the critical limit, \( \omega^2/\lambda_0 \to 0 \), to derive, to one-loop order, the beta functions \([7,15]\) that govern \( g_0 \) and \( g_\rho \) under the transformation. The results are given by

\[ \beta(\tilde{g}_0) = \frac{\partial \tilde{g}_0}{\partial \ln l} = -d \tilde{g}_0 + 8 \tilde{g}_0^2 + 10 \tilde{g}_0^3 + 20 \tilde{g}_0 \tilde{g}_\rho, \quad (6) \]

\[ \beta(\tilde{g}_\rho) = \frac{\partial \tilde{g}_\rho}{\partial \ln l} = (2d - d) \tilde{g}_\rho + 12 \tilde{g}_0 \tilde{g}_\rho + 16 \tilde{g}_\rho, \quad (7) \]

where \( l > 1 \) is the rescaling parameter, and \( \tilde{g}_0 \) and \( \tilde{g}_\rho \) are given by

\[ \tilde{g}_0 = k_d \left[ \frac{d + 5}{2d(d + 2)} \right] g_0, \quad (8) \]

\[ \tilde{g}_\rho = k_d \left[ \frac{d + 5}{2d(d + 2)} \right] g_\rho, \quad (9) \]

with \( k_d = S_d/(2\pi^d) \), and \( S_d \) being the surface area of the \( d \)-dimensional unit sphere. Examining the RG flows, Eqs. (6) and (7), reveals that, depending on \( \rho \), there are two distinct regimes.

(i) For \( 0 < \rho < d/2 \) there are three sets of fixed points: The trivial Gaussian fixed point \( g_0^* = g_\rho^* = 0 \), which is stable, and two nontrivial fixed points and eigendirections. One is \( \{g_0^* = d/8, g_\rho^* = 0\} \), while the other set is given by

due to the delta-correlated and power-law correlated parts of the disorder. Previously, Souillard and co-workers \([17]\) studied wave propagation in disordered fractal media, which is characterized by a decaying power-law correlation function. Their study is not, however, related to our work.

Consider a wave component \( \psi(x, \omega) \) with angular frequency \( \omega \), which is obtained by taking the temporal Fourier transform of Eq. (1) which yields the following equation for propagation of a wave component in a disordered medium:

\[ \nabla^2 \psi(x, \omega) + \frac{\omega^2}{\lambda_0} \psi(x, \omega) + \nabla \cdot \left[ \frac{\eta(x)}{\lambda_0} \nabla \psi(x, \omega) \right] = 0. \quad (4) \]

Since \( \eta(x) \) is a Gaussian variable, we obtain a Martin-Siggia-Rose effective action \( S_c \) for the probability density functional of the wave function \( \psi(x, \omega) \), given by

\[
g_0^* = -\frac{4}{41} \left[ d + \frac{5}{16} (2d - d) \right] - \frac{4}{41} \times \sqrt{d + \frac{5}{16} (2d - d) + \frac{205}{256} (2d - d)^2}, \\
g_\rho^* = \frac{3}{4} g_0^* + \frac{1}{16} (d - 2d),
\]

which is stable in one eigendirection but unstable in the other eigendirection. The corresponding RG flow diagram is shown in Fig. 1. Therefore, for \( 0 < \rho < d/2 \) the RG calculation indicates that the system with uncorrelated disorder is unstable against long-range correlated disorder toward a new fixed point in the coupling constants space, for which there is a phase transition from delocalized to localized states with increasing the disorder intensity.

(ii) For \( \rho > d/2 \) there are two fixed points: the Gaussian fixed point which is stable on the \( g_0 \) axis but unstable on the \( g_\rho \) axis, and the nontrivial fixed point, \( \{g_0^* = d/8, g_\rho^* = 0\} \), which is unstable in all directions; see Fig. 2. Thus, while the power-law correlated disorder is relevant, no new fixed point exists to one-loop order and, therefore, the long-wavelength behavior of the system is determined by the long-range component of the disorder. That is, for \( \rho > d/2 \) the waves are localized for any \( d \). In addition, in both cases (i) and (ii) the system undergoes a disorder-induced transition when only the uncorrelated disorder is present. These results are general so long as \( D_\rho > 0 \). For \( D_\rho < 0 \) the above phase space is valid for \( \rho > \frac{d}{2} \).

To test these predictions, we carried out numerical simulations of the problem in both 1D and 2D systems.
In 1D disordered systems the waves are localized when the
wave functions are of the form $\psi(x) = f(x) \times \exp(-|x - x_0|/\xi)$, where $f(x)$ is a stochastic function that de-

dpends on the particular realization of the chain, and $\xi$ is the localization length. The simplest 1D system
that exhibits wave localization is [18] a 15 m long steel
wire with a 0.178 mm diameter, suspended vertically. The
tension in the wire is maintained with a weight attached at
wire with an electromechanical actuator at one end of the
wire. It was shown [18] that, even for small deviations (less
than 1%) from periodicity, the diagonal disorder (e.g.,
variations in the resonance frequencies of the oscillators)
produces localization (in agreement with Furstenberg’s
theorem [19]), while variations (up to 13%) in the sizes
of the masses (off-diagonal disorder) result in localization
lengths that are much larger than the size of the system.
This is in agreement with our theoretical prediction.

To study the 1D system, we used the transfer-matrix
(TM) method [12]. Discretizing Eq. (1) and writing down
the result for site $n$ of a linear chain yields
$(\omega + \lambda_n)\psi_n +
\lambda\psi_{n+1} - (\lambda_{n+1} - \lambda_n)\psi_{n+1} = 0$ or, in the recursive
form, $M_n(\psi_{n+2}, \psi_{n+1})^T = (\psi_{n+1}, \psi_n)^T$, with $T$ represent-
ing the transpose operation, and

$$M_n = \begin{pmatrix}
\frac{\omega^2 - \lambda_n + \lambda_{n-1}}{\lambda_n} & \frac{\lambda_{n-2}}{\lambda_n} \\
\frac{\lambda_{n-1}}{\lambda_n} & 0
\end{pmatrix}$$

$\xi(\omega)$ is then defined by $\xi(\omega)^{-1} = \lim_{N \to \infty} N^{-1}|\psi_N/\psi_0|$, where $N$ is the chain’s length. For every realization of
the disorder we computed $\psi_N$ and, hence, $\xi(\omega)$, using $\psi_0 =
\psi_1 = 1/\sqrt{2}$, and averaging $\xi$ over a large ensemble of
realizations for a fixed $N$ and frequency $\omega$. The procedure
was repeated for several values of $N$ and $\omega$. The extended
states correspond to having $\lim_{N \to \infty} \xi/N = \text{const} > 1$. As $\xi$ is a function of $\omega$ also, we chose $\omega = 2\pi\sqrt{\lambda_0/N}$, which is
the system’s smallest mode. Our RG analysis indicates
that this mode most likely passes through the chain.

The TM computations indicate that the coupling con-
stant $g(N)$ follows finite-size scaling, $g(N) = g_0 + 1/N$, with $g_0 = g(N \to \infty) \approx 0.117$. Moreover, as $\xi \to N$, one
has $N \approx (g - g_0)^{-\nu}$, with $\nu$ being the localization ex-
ponent. Our RG results indicate [20] that in $d$ dimensions,
$\nu = 1/d$, which agrees with the TM calculations that yield
$\nu = 1$. Figure 3 shows the results for $\xi(\omega)$ which confirm
the RG predictions.

We also solved Eq. (1) in 2D using the finite-difference
(FD) method with second-order discretization for both the
space and time variables. Such approximations are accept-
able as we work in the limit of low frequencies or long
wavelengths. For short wavelengths we should use higher-
order discretizations for the spatial variables [20]. A $L_x \times
L_y$ grid was used with $L_x = 8000$ and $L_y = 400$. The
midpoint displacement method [21] was used to distribute
in the grid $\lambda(x)$, the gridblock-scale elastic constant, with
the power-law correlation function. Also simulated was the
case in which $\lambda(x)$ was randomly and uniformly distrib-
uted, with the same variance as that of the power-law case.
Periodic boundary conditions were imposed in the lateral direction, which did not distort the nature of the wave propagation, as large systems were used. By inserting a wave source on a line on one side of the grid, the wave equation was solved numerically. The decay in the wave’s amplitude is caused by scattering from heterogeneities of the system generated by the distribution of \( \lambda(x) \). The solution’s accuracy was checked by considering the stability criterion and the source’s wavelength [22], and using higher-order FD discretizations. The amplitude decay was computed by collecting the numerical results at 80 grid points along the direction of wave propagation, distributed evenly in the grid. The results were averaged over 32 realizations of the system.

Figure 4 presents the decay in the wave amplitude through the uniformly random medium, and that of one with a nondecaying power-law correlation function for \( \lambda(x) \), with \( \rho = 1.3 \) and 1.8. The wave amplitudes for the correlated cases decline much faster than those in the uniformly random medium. In particular, for \( \rho = 1.3 \), which corresponds to negative correlations (that is, a large local elastic modulus is likely to be neighbor to a small one, and vice versa), the amplitude decreases rather sharply, hence confirming the RG predictions.

These results, which contradict the generally accepted view that off-diagonal disorder has a much weaker effect on localization than the diagonal disorder, have important practical implications [23]. For example, in order for seismic records to contain meaningful information on the geology and content of a natural porous formation of linear size \( L \), the localization length \( \xi \) must be larger than \( L \). Otherwise, the seismic records would provide the information only up to \( \xi \) (which, itself, is a function of \( d \), \( \rho \), and other parameters), but not at larger length scales.

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[20] The details will be given elsewhere.