The $O(n)$ Model in the $n \to 0$ Limit (self-avoiding-walks)
and
Logarithmic Conformal Field Theory

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Abstract

We consider the $O(n)$ theory in the $n \to 0$ limit. We show that the theory is described by logarithmic conformal field theory, and that the correlation functions have logarithmic singularities. The explicit forms of the two-, three- and four-point correlation functions of the scaling fields and the corresponding logarithmic partners are derived.

I. INTRODUCTION

The n-vector model with its Landau-Ginzburg-Willson Hamiltonian, which has $O(n)$ symmetry, may be used to study physical properties of many critical systems. For example, in the limit $n = 1$ we obtain Ising-like systems which describe liquid-vapour transitions in the classical and critical binary fluids. The Helium superfluid transition corresponds to the limit $n = 2$. Only for the case of $n = 3$ does the experimental information come from truly ferromagnetic systems. The limit $n \to 0$ describes the statistical properties of self-avoiding walks (SAWs), which describe the universal properties of linear polymers, i.e., long nonintersecting chains, in a dilute solution. These properties can be computed by such techniques as the transfer-matrix method [1], series expansions [2], and the scanning Monte Carlo method [3]. More generally, deep insight into the problem of computing the statistical properties of linear polymers and the SAWs can be obtained from the $O(n)$ model in the $n \to 0$ limit, a discovery first made by de Gennes [4, 5].

Recently, it has been shown by Cardy that the $O(n)$ model in the $n \to 0$ limit has a logarithmic conformal field structure, with its correlation function having logarithmic singularity [7]. The logarithmic conformal field theories (LCFT) [6,7] are extensions of the conventional conformal field theories (CFT) [8-12], which have emerged in recent years in a number of interesting physical problems, such as the WZNW models [13-18], supergroups and super-symmetric field theories [19-25], Haldane-Rezzayi state in the fractional quantum Hall effect [26-30], multi-fractality [31], two-dimensional turbulence [32-34], gravitaitonally-dressed theories [35], polymer and abelian sandpiles [36-40], string theory [41-44] and the D-brane recoil [45-55], Ads/CFT correspondence [56-67], Seiberg-Witten solution to SUSY Yang-Mills theory [68] and disordered systems [69-80]. Moreover, such related issues as the Null vectors, Characters, partition functions, fusion rules, Modular Invariance, C-theorem, LCFT's with boundary and operator product expansions have been discussed in Refs. [81-119].

The LCFT are characterized by the fact that their dilatation operators L_0 are not diagonalized and admit a Jordan cell structure. The non-trivial mixing between these operators leads to logarithmic singularities in their correlation functions. It has been shown in Ref. [8] that the correlator of two fields in such field theories has a logarithmic singularity as follows:

$$\langle \psi(r_1)\psi(r_2) \rangle \sim |r_1 - r_2|^{-2\Delta_\psi} \ln |r_1 - r_2| + \dots \tag{1}$$

In this paper, we consider the correlation functions of the scaling fields of the $O(n)$ model and derive explicit expressions for the two-, three- and four-point correlation functions. In Section 2, we derive the scaling fields and their scaling exponents. In Section 3, we derive the Jordan cell structure of the theory in the $n \to 0$ limit, while Section 4 presents the derivation of the two-, three- and four-point correlation functions of the scaling fields and their logarithmic partners. The details of calculations are presented in the Appendices.

II. SCALING FIELDS IN THE $O(n)$ MODEL

The Landau-Ginzburg description of the $O(n)$ model starts with the effective Hamiltonian:

$$H = \int \frac{1}{2} \sum_a : [(\nabla \phi_a)^2 + m^2 \phi_a^2] : + g \sum_{a,b} : \phi_a^2 \phi_b^2 : d^d r, \tag{2}$$

where $a, b = 1, 2, \cdots, n$, and g is the coupling constant of the perturbation, $\sum_{a,b} : \phi_a^2 \phi_b^2 :$, to the free model. Suppose that under a scaling transformation $\vec{r} \to \vec{r} = \lambda \vec{r}$, ϕ_a behaves as, $\phi_a \to \phi_a' = \lambda^{-x_{\phi_a}^0} \phi_a$, where $x_{\phi_a}^0$ is called the scaling dimension of the field ϕ_a. Invariance of the Hamiltonian under scaling requires the coupling g to have the scaling dimension, $y_g = d - 4x_{\phi_a}^0$. It would be relevant at the pure fixed point if, $d > 4x_{\phi_a}^0$ [120-121]. If y_g is small, it is possible to develop a perturbative renormalization group (RG) equation in powers of these variables, which can then yield the fixed points.

To develop the RG equation for a typical coupling constant g_i, we need the operator product expansion (OPE) coefficients. The general form of the *beta*-function for coupling g_i is given by [6]:

$$\beta_{g_i} \equiv \frac{dg_i}{dl} = y_{g_i} g_i - \sum_{jk} C_{ijk} g_j g_k + \cdots, \tag{3}$$

where $l > 1$ is a re-scaling parameter. To derive the OPE coefficients we note that, when $m^2 = g = 0$, we obtain the Gaussian model, i.e., $H = \frac{1}{2} \int \sum_a : (\nabla \phi_a)^2 : d^d r$. In the Gaussian model, the various components ϕ_a are decoupled, so that the two-point correlation function has the following form:

$$\langle \phi_a(r_i) \phi_b(r_j) \rangle = \frac{\delta_{ab}}{r_{ij}^{d-2}}, \tag{4}$$

where, $r_{ij} = |\vec{r}_i - \vec{r}_j|$. Considering $\Phi = \sum_{a,b} : \phi_a^2 \phi_b^2 :$ as the perturbative term in $O(n)$ Hamiltonian, and using Wick's theorem, one can evaluate the OPE of the field Φ with itself as:

$$\begin{aligned} \Phi \cdot \Phi &= (\sum_{a,b} : \phi_a^2 \phi_b^2 :) \cdot (\sum_{c,d} : \phi_c^2 \phi_d^2 :), \\ &= 24n^2 + 96nE + \xi_\Phi \Phi + \cdots , \end{aligned} \tag{5}$$

where, $E = \sum_c : \phi_c^2 :$, $n = \sum_{ab} \delta_{ab}$ and $\xi_\Phi = (8n + 64)$ (see Appendix A). Therefore, using Eq.(3), one obtains the following RG equation for g:

$$\beta_g = y_g g - \xi_\Phi g^2 + \cdots . \tag{6}$$

To check that the field E is a scaling operator, we consider the OPE of E with a scaling dimension $x_E(n)$ and Φ:

$$\begin{aligned} E \cdot \Phi &= (\sum_c : \phi_c^2 :) . (\sum_{a,b} : \phi_a^2 \phi_b^2 :), \\ &= 8\Phi + \xi_E E + \cdots , \end{aligned} \tag{7}$$

where, $\xi_E = (4n + 8)$. Therefore, this form of the OPE for $E \cdot \Phi$ shows that the field E is a scaling operator.

As noted by Cardy [7], there is also another scaling field in the theory, $\widetilde{E}_{ab}$, with a scaling dimension $x_{\widetilde{E}}(n)$ which is given explicitly by the following expression in terms of the fields ϕ_a: $\widetilde{E}_{ab} =: \phi_a \phi_b : -\frac{\delta_{ab}}{n} \sum_c : \phi_c^2 :$. Its OPE with Φ is given by the following expression:

$$\begin{aligned} \widetilde{E}_{ab} . \Phi &= (: \phi_a \phi_b : -\frac{\delta_{ab}}{n} \sum_c : \phi_c^2 :) . (\sum_{d,e} : \phi_d^2 \phi_e^2 :), \\ &= -\frac{8}{n} \delta_{ab} \Phi + \xi_{\widetilde{E}} \widetilde{E}_{ab} + \cdots , \end{aligned} \tag{8}$$

with $\xi_{\widetilde{E}} = 8$.

To derive the scaling dimensions of the fields E and $\widetilde{E}_{ab}$, we perturb the Gaussian Hamiltonian by these fields with the coupling t_1 and t_2, respectively. Then, it is clear that their scaling dimensions are:

$$\begin{aligned} y_{t_1}^0 &= d - x_E^0, \\ y_{t_2}^0 &= d - x_{\widetilde{E}}^0. \end{aligned} \tag{9}$$

Then, according to Eq.(3), the RG equations for the couplings are given by:

$$\begin{aligned} \beta_{t_1} &= y_{t_1} t_1 - 2\xi_E g t_1 + \cdots , \\ \beta_{t_2} &= y_{t_2} t_2 - 2\xi_{\widetilde{E}} g t_2 + \cdots \end{aligned} \tag{10}$$

Because the fixed points and the RG eigenvalues correspond to the "zeros" and derivatives of the RG *beta* function at the fixed points, respectively, one has:

$$g^* = \frac{y_g^0}{\xi_\Phi}. \tag{11}$$

Using Eqs.(9), we obtain:

$$\begin{aligned} y_{t_1} &= \left. \frac{\partial \beta_{t_1}}{\partial t_1} \right|_0 , \\ &= y_{t_1}^0 - 2y_g^0 \frac{\xi_E}{\xi_\Phi}, \end{aligned} \tag{12}$$

and, similarly:

$$y_{t_2} = y_{t_2}^0 - 2y_g^0 \frac{\xi_{\widetilde{E}}}{\xi_\Phi}. \tag{13}$$

Substituting the values of ξ's, $y_{t_1}^0$, and $y_{t_2}^0$, and denoting $y_{t_1} = d - x_E$ and $y_{t_2} = d - x_{\widetilde{E}}$, we obtain the following scaling dimensions for E and $\widetilde{E}$:

$$x_E(n) = x_E^0 - \frac{2(4n+8)}{(8n+64)}y_g^0 + \cdots,$$
$$x_{\widetilde{E}}(n) = x_{\widetilde{E}}^0 - \frac{16}{(8n+64)}y_g^0 + \cdots. \tag{14}$$

One can then derive the OPE of the field $: \phi_a \phi_b :$ with the scaling dimension $x_{\phi\phi}(n)$ and Φ. If this is done, one obtains:

$$: \phi_a \phi_b : .\Phi = 4\delta_{ab}E + \xi_{\widetilde{E}} : \phi_a \phi_b : + \cdots, \tag{15}$$

Therefore, it is clear that $: \phi_a \phi_b :$ and $\widetilde{E}$ have the same scaling dimension.

III.   SCALING DIMENSIONS IN THE $n \to 0$ LIMIT

It is evident that in the $n \to 0$ limit, Eqs.(14) reduce to the following:

$$x_E(0) = x_E^0 - \frac{1}{4}y_g^0 + O(y_g^0)^2,$$
$$x_{\phi\phi}(0) = x_{\widetilde{E}}(0) = x_{\widetilde{E}}^0 - \frac{1}{4}y_g^0 + O(y_g^0)^2. \tag{16}$$

Therefore, in the $n \to 0$ limit, there are three fields with the same scaling dimensions. It is well-known in CFT that two scaling fields with the same conformal weights may constitute a Jordan cell of rank 2. Such theories are what are referred to as the logarithmic conformal field theories (LCFT). The correlation functions in LCFT may have logarithmic as well as power-law terms. Before calculating these correlation functions, we can show that $E(r)$ and $: \phi_a \phi_b(r) :$ constitute a Jordan cell in the $n \to 0$ limits. Since $E(r)$ and $\widetilde{E}(r)$ are two scaling fields in the $O(n)$ model with scaling dimensions $x_E(n)$ and $x_{\widetilde{E}}(n)$, we have:

$$E(\lambda r) = \lambda^{-x_E(n)} E(r),$$
$$\widetilde{E}(\lambda r) = \lambda^{-x_{\widetilde{E}}(n)} \widetilde{E}(r). \tag{17}$$

We are interested in the scaling behavior of $: \phi_a \phi_b(r) :$ in the $n \to 0$ limit. First, we write $: \phi_a \phi_b(r) :$ in terms of $E(r)$ and $\widetilde{E}_{ab}(r)$ as:

$$: \phi_a \phi_b(r) := \widetilde{E}_{ab}(r) + \frac{\delta_{ab}}{n}E(r), \tag{18}$$

and, then, using Eqs.(17), we find that:

$$: \phi_a \phi_b(\lambda r) := \lambda^{-x_E(n)}[\lambda^{x_E(n)-x_{\widetilde{E}}(n)}\widetilde{E}_{ab}(r) + \frac{\delta_{ab}}{n}E(r)]. \tag{19}$$

Since, $x_E(0) = x_{\widetilde{E}}(0)$, then:

$$\lambda^{x_E(n)-x_{\widetilde{E}}(n)} = 1 + nu \ln \lambda + O(n^2) + \cdots, \tag{20}$$

where, $u = x_E'(0) - x_{\widetilde{E}}'(0)$. Now, it is straightforward to check that:

$$\lim_{n \to 0} : \phi_a \phi_b(\lambda r) := \lambda^{x_E(0)}[: \phi_a \phi_b(r) : - \ln \lambda(u\delta_{ab}E(r))]. \tag{21}$$

Therefore, $u\delta_{ab}E(r)$ and $: \phi_a \phi_b(r) :$ are degenerate fields in the $n \to 0$ limit which form a Jordan cell:

$$\begin{pmatrix} u\delta_{ab}E(r) \\ : \phi_a \phi_b(r) : \end{pmatrix}. \tag{22}$$

IV. TWO, THREE AND FOUR POINT CORRELATION FUNCTIONS

In this section we first derive the two-point correlation functions of the fields $\widetilde{E}_{ab}(r)$ and $E(r)$, for arbitrary n. As shown in Section 2, these fields have the scaling dimensions $x_{\widetilde{E}}(n)$ and $x_E(n)$, respectively. However, to fix the amplitude of the two-point correlation function and their tensorial structure we note that:

$$\langle E(r_1)E(r_2)\rangle = \langle (\sum_i : \phi_i^2(r_1) :)(\sum_j : \phi_j^2(r_2) :)\rangle, \tag{23}$$

where normal ordered is defined as $:: = 1- <>$. The two-point correlation function of the field $\widetilde{E}_{ab}(r)$ is given by:

$$\langle \widetilde{E}_{ab}(r_1)\widetilde{E}_{cd}(r_2)\rangle = \langle (: \phi_a\phi_b(r_1) : -\frac{\delta_{ab}}{n}\sum_i : \phi_i^2(r_1) :)(: \phi_c\phi_d(r_2) : -\frac{\delta_{cd}}{n}\sum_j : \phi_j^2(r_2) :)\rangle, \tag{24}$$

and similarly for $\langle \widetilde{E}_{ab}(r_1)E(r_2)\rangle$. The right-hand side of the above equations can be evaluated using Wick's theorem and noting that $\langle \phi_a\phi_b\rangle \sim \delta_{ab}$. Therefore:

$$\langle : \phi_a\phi_b :: \phi_c\phi_d :\rangle \sim D_{ab,cd},$$
$$\sum_i \langle : \phi_a\phi_b :: \phi_i^2 :\rangle \sim 2\delta_{ab},$$
$$\sum_{ij} \langle : \phi_i^2 :: \phi_j^2 :\rangle \sim 2n, \tag{25}$$

where $D_{ab,cd} = \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}$. However, we have fixed their tensorial structures by calculation of the two-point correlation function using the free Hamiltonian. Indeed, the interaction will change the amplitude and the scaling exponent, but dose not affect the tensorial structures.
Using the Eqs.(25), we find that:

$$\langle \widetilde{E}_{ab}(r_1)E(r_2)\rangle = 0,$$
$$\langle E(r_1)E(r_2)\rangle = 2nA(n)r_{12}^{-2x_E(n)},$$
$$\langle \widetilde{E}_{ab}(r_1)\widetilde{E}_{cd}(r_2)\rangle = [D_{ab,cd} - \frac{2}{n}\delta_{cd}\delta_{ab}]\widetilde{A}(n)r_{12}^{-2x_{\widetilde{E}}(n)}, \tag{26}$$

where $A(n)$ and $\widetilde{A}(n)$ are two functions have Taylor expansion at $n = 0$.
We can now derive two-point correlation functions of fields E (scaling operator) and $: \phi_a\phi_b :$ (logarithmic operator) in the $n \to 0$ limit. Singularity of the two point functions at $n \to 0$ limit can be removed by choosing $A(0) = \widetilde{A}(0)$, as has recently been shown by Cardy [7]:

$$\lim_{n\to 0}\langle E(r_1)E(r_2)\rangle = 0,$$
$$\lim_{n\to 0}\langle : \phi_a\phi_b(r_1) : E(r_2)\rangle = 2A(0)\delta_{ab}r_{12}^{-2x_E(0)},$$
$$\lim_{n\to 0}\langle : \phi_a\phi_b(r_1) :: \phi_c\phi_d(r_2) :\rangle = \left\{ A(0)\left[D_{ab,cd} - 4u\delta_{ab}\delta_{cd}\ln r_{12}\right] + 2\delta_{ab}\delta_{cd}[A'(0) - \widetilde{A}'(0)]\right\} r_{12}^{-2x_E(0)}. \tag{27}$$

This is a general property of correlation functions of the Jordan cell components in a LCFT [9].
In what follows we derive various three- and four-point correlation functions of the Jordan cell components. In the CFT, due to the conformal symmetry, the three-point correlation function of the scaling fields has the following form:

$$\langle \varphi_1(r_1)\varphi_2(r_2)\varphi_3(r_3)\rangle = \frac{C_{123}}{r_{12}^{\Delta_1+\Delta_2-\Delta_3}r_{13}^{\Delta_1+\Delta_3-\Delta_2}r_{23}^{\Delta_2+\Delta_3-\Delta_1}}, \tag{28}$$

where Δ_1, Δ_2 and Δ_3 are scaling dimensions of φ_1, φ_2 and φ_3, respectively, and C_{123} is a parameter that depends on the model.
We are interested in various three-point correlation functions of scaling field E and $: \phi_a\phi_b :$ which is its logarithmic partner at $n \to 0$ limit. Using Eq.(18) and substitute of $: \phi_i\phi_j :$ in terms of E and $\widetilde{E}$, three-point correlation function of logarithmic field $\langle : \phi_a\phi_b(r_1) :: \phi_c\phi_d(r_2) :: \phi_e\phi_f(r_3) :\rangle$ can be written as follows:

$$\langle : \phi_a\phi_b(r_1) :: \phi_c\phi_d(r_2) :: \phi_e\phi_f(r_3) : \rangle = A(n)(r_{12}r_{13}r_{23})^{-x_{\tilde{E}}(n)}$$
$$\times \left[D_{ab,cd,ef} - \tfrac{4}{n}(\delta_{ab}D_{cd,ef} + \delta_{cd}D_{ab,ef} + \delta_{ef}D_{ab,cd}) + \tfrac{16}{n^2}\delta_{ab}\delta_{cd}\delta_{ef} \right]$$
$$+ \tfrac{\delta_{ab}}{n}\left(4D_{cd,ef} - \tfrac{8}{n}\delta_{cd}\delta_{ef} \right) B(n)(r_{12}r_{13})^{-x_E(n)}r_{23}^{x_E(n)-2x_{\tilde{E}}(n)}$$
$$+ \tfrac{\delta_{cd}}{n}\left(4D_{ab,ef} - \tfrac{8}{n}\delta_{ab}\delta_{ef} \right) B(n)(r_{12}r_{23})^{-x_E(n)}r_{13}^{x_E(n)-2x_{\tilde{E}}(n)}$$
$$+ \tfrac{\delta_{ef}}{n}\left(4D_{ab,cd} - \tfrac{8}{n}\delta_{ab}\delta_{cd} \right) B(n)(r_{13}r_{23})^{-x_E(n)}r_{12}^{x_E(n)-2x_{\tilde{E}}(n)}$$
$$+ \tfrac{1}{n^3}8n\delta_{ab}\delta_{cd}\delta_{ef}C(n)(r_{12}r_{13}r_{23})^{-x_E(n)}, \tag{29}$$

where $A(n)$, $B(n)$, $C(n)$ are functions that have Taylor expansion near $n=0$ (see appendix B for details). To obtain the $n \to 0$ limit, we expand above expression about $n=0$ by writing down the Taylor expansion of $A(n)$, $B(n)$ and $C(n)$, and $r_{ij}^{x_E(n)-x_{\tilde{E}}(n)}$. It is not difficult to see that resulting relation will be divergent in the $n \to 0$ limit for arbitrary values of $A(0)$, $B(0)$, $C(0)$, $A'(0)$, $B'(0)$ and $C'(0)$. However, the divergent terms will cancel each other if we choose a special case in which $A(0) = B(0) = C(0)$, and $A'(0) = B'(0) = C'(0)$, so that:

$$\lim_{n\to 0}\langle : \phi_a\phi_b(r_1) :: \phi_c\phi_d(r_2) :: \phi_e\phi_f(r_3) : \rangle = (r_{12}r_{13}r_{23})^{-x_E(0)}$$
$$\times \left\{ A(0)\left[D_{ab,cd,ef} - 4u\left(\delta_{ab}D_{cd,ef}\ln\tfrac{r_{12}r_{13}}{r_{23}} + \delta_{cd}D_{ab,ef}\ln\tfrac{r_{12}r_{23}}{r_{13}} + \delta_{ef}D_{ab,cd}\ln\tfrac{r_{13}r_{23}}{r_{12}} \right) \right. \right.$$
$$+ 8u^2\delta_{ab}\delta_{cd}\delta_{ef}[4(\ln r_{12}\ln r_{13} + \ln r_{12}\ln r_{23} + \ln r_{13}\ln r_{23}) - \ln^2(r_{12}r_{13}r_{23})]]$$
$$\left. + \delta_{ab}\delta_{cd}\delta_{ef}\left[8A''(0) - 12B''(0) + 4C''(0) \right] \right\}. \tag{30}$$

In the same manner other three-point correlation functions obtain at $n \to 0$ limit as follows:

$$\lim_{n\to 0}\langle E(r_1)E(r_2)E(r_3) \rangle = 0,$$
$$\lim_{n\to 0}\langle : \phi_a\phi_b(r_1) : E(r_2)E(r_3) \rangle = 8A(0)\delta_{ab}(r_{12}r_{13}r_{23})^{-x_E(0)}, \tag{31}$$
$$\lim_{n\to 0}\langle : \phi_a\phi_b(r_1) :: \phi_c\phi_d(r_2) : E(r_3) \rangle = A(0)\left[4D_{ab,cd} - 16u\delta_{ab}\delta_{cd}\ln r_{12} \right](r_{12}r_{13}r_{23})^{-x_E(0)}.$$

Finally, the four-point correlation functions of E and $: \phi_a\phi_b :$ can be calculated just as the same as three-point correlation functions. Some key-functions which are useful in this calculation given in appendix B.

In the $n \to 0$ limit we encounter divergent terms in four-point correlation functions, but the divergent terms will cancel each other if we choose a special case in which $A(0) = B(0) = C(0) = D(0)$, $A'(0) = B'(0) = C'(0) = D'(0)$, $A''(0) = B''(0) = C''(0) = D''(0)$ and $f_1(\eta) = f_2(\eta) = f_3(\eta) = f_4(\eta)$, Therefore:

$$\lim_{n\to 0}\langle E(r_1)E(r_2)E(r_3)E(r_4) \rangle = 0,$$
$$\lim_{n\to 0}\langle : \phi_a\phi_b(r_1) : E(r_2)E(r_3)E(r_4) \rangle = 48A(0)f(\eta)\delta_{ab}(r_{12}r_{13}r_{14}r_{23}r_{24}r_{34})^{-2x_E(0)/3},$$
$$\lim_{n\to 0}\langle : \phi_a\phi_b(r_1) :: \phi_c\phi_d(r_2) : E(r_3)E(r_4) \rangle = A(0)f(\eta)(24D_{ab,cd} + 8 - 16u\ln\tfrac{r_{13}r_{23}r_{14}r_{24}r_{12}^4}{r_{34}^2}\delta_{ab}\delta_{cd})$$
$$\times (r_{12}r_{13}r_{14}r_{23}r_{24}r_{34})^{-2x_E(0)/3}, \tag{32}$$

$$\lim_{n\to 0}\langle : \phi_a\phi_b(r_1) :: \phi_c\phi_d(r_2) :: \phi_e\phi_f(r_3) : E(r_4) \rangle =$$
$$A(0)f(\eta)\left\{ \left[6D_{ab,cd,ef} + 2\delta_{ab}D_{cd,ef} + 2\delta_{cd}D_{ab,ef} + 2\delta_{ef}D_{ab,cd} \right] \right.$$
$$+ 8u[\delta_{ab}D_{cd,ef}\ln\tfrac{r_{23}r_{24}r_{34}}{(r_{12}r_{13}r_{14})^2} + \delta_{cd}D_{ab,ef}\ln\tfrac{r_{13}r_{14}r_{34}}{(r_{12}r_{23}r_{24})^2} + \delta_{ef}D_{ab,cd}\ln\tfrac{r_{12}r_{24}r_{14}}{(r_{23}r_{13}r_{34})^2}$$
$$- \delta_{ab}\delta_{cd}\delta_{ef}\ln r_{12}r_{13}r_{23}] - 16u^2\delta_{ab}\delta_{cd}\delta_{ef}[-2\ln r_{12}\ln r_{34} - 3\ln r_{12}\ln r_{23}$$
$$- 2\ln r_{13}\ln r_{24} - 3\ln r_{12}\ln r_{13} - 2\ln r_{23}\ln r_{14} - 3\ln r_{13}\ln r_{23}$$
$$+ \ln r_{12}\ln r_{14}r_{24} - \ln r_{14}\ln r_{24} + \ln r_{13}\ln r_{14}r_{34} - \ln r_{14}\ln r_{34}$$
$$\left. + \ln r_{23}\ln r_{24}r_{34} - \ln r_{24}\ln r_{34} + \ln^2 r_{14} + \ln^2 r_{24} + \ln^2 r_{34}] \right\}$$
$$\times (r_{12}r_{13}r_{14}r_{23}r_{24}r_{34})^{-2x_E(0)/3}, \tag{33}$$

and

$$\lim_{n\to 0}\langle: \phi_a\phi_b(r_1) :: \phi_c\phi_d(r_2) :: \phi_e\phi_f(r_3) :: \phi_g\phi_h(r_4) :\rangle = A(0)f(\eta)$$

$$\times \left\{ D_{ab,cd,ef,gh} + \frac{2}{3}u\left[\left(3\delta_{ab}D_{cd,ef,gh}\ln\frac{r_{23}r_{24}r_{34}}{(r_{12}r_{13}r_{14})^2} + three\ terms\right) - \left(\delta_{ab}\delta_{cd}D_{ef,gh}\ln\frac{r_{12}^4 r_{13}r_{14}r_{23}r_{24}}{r_{34}^2} + five\ terms\right)\right]\right.$$

$$+ \frac{8}{3}u^2\left\{-\delta_{ab}\delta_{cd}D_{ef,gh}\left(2(\ln r_{13} + \ln r_{14})^2 + 2(\ln r_{23} + \ln r_{24})^2 - (2\ln r_{12} - \ln r_{34})^2\right.\right.$$

$$+ \ln(r_{13}r_{14}r_{23}r_{24})\ln\frac{r_{34}}{r_{12}^2} - 5(\ln r_{13}\ln r_{24} + \ln r_{14}\ln r_{24} + \ln r_{13}\ln r_{23} + \ln r_{14}\ln r_{23})\bigg) + five\ terms]$$

$$+ \delta_{ab}\delta_{cd}\delta_{ef}\delta_{gh}\left[-\left(\ln^2 r_{12} + five\ terms\right) + (\ln r_{12}\ln r_{13} + \ln r_{12}\ln r_{14} + \ln r_{13}\ln r_{14}\right.$$

$$+ three\ terms) + (4\ln r_{12}\ln r_{34} + two\ terms)]\}$$

$$+ \frac{16}{9}u^3\delta_{ab}\delta_{cd}\delta_{ef}\delta_{gh}\left\{(3\ln^2 r_{12}\ln(r_{13}r_{14}r_{23}r_{24}) + 4\ln^3 r_{12} - 6\ln^2 r_{12}\ln r_{34} + five\ terms)\right.$$

$$- (12\ln(r_{12}r_{13})\ln r_{24}\ln r_{34} + five\ terms) + (24\ln r_{12}\ln r_{13}\ln r_{23} - 30\ln r_{12}\ln r_{23}\ln r_{24} + three\ terms)\}$$

$$+ \delta_{ab}\delta_{cd}\delta_{ef}\delta_{gh}[-24A'''(0) + 64B'''(0) - 48C'''(0) + 8D'''(0)]\}\,(r_{12}r_{13}r_{14}r_{23}r_{24}r_{34})^{-2x_E(0)/3}. \tag{34}$$

Because of avoiding lengthy expression we didn't write explicit form of all terms in the above equation, but it is easy to write them by symmetry considerations.

V. SUMMARY

We have studied the correlation functions of self-avoiding walks and derived their two-, three-, and four-point correlation functions using the $O(n)$ model in the limit $n \to 0$. One can directly check that the three- and four-point correlation functions have the general properties of a logarithmic conformal field theory, and that the logarithmic partner can be regarded as the formal derivative of the ordinary fields (top field) with respect to their conformal weight [9]. In this case, one can consider the field $: \phi_a\phi_b :$ as the derivatives of field E with respect to n. We emphasize that the derivative with respect to the scaling weight can be written in terms of the derivative with respect to n. These properties enable us to calculate any N-point correlation function that contains the logarithmic field $: \phi_a\phi_b :$, in terms of the correlation functions of the top fields. The general expression of the correlation functions of the LCFT are given in Ref. [9]. Here, we have determined the unknown constants in the logarithmic correlation functions in terms of the details of the SAWs. It is noted that the formal derivations with respect to the scaling dimensions cannot predict the unknown constants in the quenched averaged correlation functions of the local energy density operators. The constants depend on the detail of the statistical model.

Our analytical results can also be checked numerically. Our analysis is valid in all dimensions below the upper critical dimension. These results can be generalized to other problems, such as percolation, random phase sine-Gordon model, etc.

This paper is dedicated to Professor Ian Kogan.

VI. APPENDIX A

Here, we present the details of the calculations for the operator product expansion of $\Phi\cdot\Phi$, $E\cdot\Phi$ and $\widetilde{E}_{ab}\cdot\Phi$. Using the definition Φ, one finds that:

$$\Phi.\Phi = (\sum_{ab} : \phi_a^2\phi_b^2 :).(\sum_{cd} : \phi_c^2\phi_d^2 :)$$

$$= 4\sum_{abcd}(\delta_{ac} + \delta_{ac}\phi_a)\phi_a\phi_c\phi_b^2\phi_d^2 + 4\sum_{abcd}(\delta_{ad} + \delta_{ad}\phi_a)\phi_a\phi_d\phi_b^2\phi_c^2 + 4\sum_{abcd}(\delta_{bc} + \delta_{bc}\phi_b)\phi_b\phi_c\phi_a^2\phi_d^2$$

$$+ 4\sum_{abcd}(\delta_{bd} + \delta_{bd}\phi_b)\phi_b\phi_d\phi_a^2\phi_c^2 + 2\sum_{abcd}(\delta_{ac} + \delta_{ac}\phi_a)(\delta_{ac} + \delta_{ac}\phi_a)\phi_b^2\phi_d^2$$

$$+ 2\sum_{abcd}(\delta_{ad} + \delta_{ad}\phi_a)(\delta_{ad} + \delta_{ad}\phi_a)\phi_b^2\phi_c^2 + 2\sum_{abcd}(\delta_{bc} + \delta_{bc}\phi_b)(\delta_{bc} + \delta_{bc}\phi_b)\phi_a^2\phi_d^2$$

$$+ 2\sum_{abcd}(\delta_{bd} + \delta_{bd}\phi_b)(\delta_{bd} + \delta_{bd}\phi_b)\phi_a^2\phi_c^2 + 8\sum_{abcd}(\delta_{ac} + \delta_{ac}\phi_a)(\delta_{ad} + \delta_{ad}\phi_a)\phi_b^2\phi_c\phi_d$$

$$+ 8\sum_{abcd}(\delta_{bc} + \delta_{bc}\phi_b)(\delta_{bd} + \delta_{bd}\phi_b)\phi_a^2\phi_c\phi_d + 16\sum_{abcd}(\delta_{ac} + \delta_{ac}\phi_a)(\delta_{bd} + \delta_{bd}\phi_b)\phi_a\phi_b\phi_c\phi_d$$

$$+ 16\sum_{abcd}(\delta_{ad} + \delta_{ad}\phi_a)(\delta_{bc} + \delta_{bc}\phi_b)\phi_a\phi_b\phi_c\phi_d + 8\sum_{abcd}(\delta_{ac} + \delta_{ac}\phi_a)(\delta_{bc} + \delta_{bc}\phi_b)\phi_a\phi_b\phi_d^2$$

$$+ 8\sum_{abcd}(\delta_{ad} + \delta_{ad}\phi_a)(\delta_{bd} + \delta_{bd}\phi_b)\phi_a\phi_b\phi_c^2 + ...,$$

$$= 24n^2 + 96nE + (8n + 64)\Phi + ... \tag{35}$$

$$E.\Phi = (\sum_{a} : \phi_a^2 :).(\sum_{bc} : \phi_b^2\phi_c^2 :)$$

$$= 4\sum_{abc}(\delta_{ab} + \delta_{ab}\phi_a)\phi_a\phi_b\phi_c^2 + 4\sum_{abc}(\delta_{ac} + \delta_{ac}\phi_a)\phi_a\phi_c\phi_b^2$$

$$+ 2\sum_{abc}(\delta_{ab} + \delta_{ab}\phi_a)(\delta_{ab} + \delta_{ab}\phi_a)\phi_c^2 + 2\sum_{abc}(\delta_{ac} + \delta_{ac}\phi_a)(\delta_{ac} + \delta_{ac}\phi_a)\phi_b^2$$

$$+ 8\sum_{abc}(\delta_{ab} + \delta_{ab}\phi_a)(\delta_{ac} + \delta_{ac}\phi_a)\phi_b\phi_c,$$

$$= 8\Phi + (4n + 8)E + ... \tag{36}$$

and,

$$\widetilde{E}_{ab}.\Phi = (: \phi_a\phi_b : -\frac{\delta_{ab}}{n}\sum_{i} : \phi_i^2 :).(\sum_{cd} : \phi_c^2\phi_d^2 :)$$

$$+ 2\sum_{cd}(\delta_{ac} + \delta_{ac}\phi_a)(\delta_{bc} + \delta_{bc}\phi_b)\phi_d^2 + 2\sum_{cd}(\delta_{ad} + \delta_{ad}\phi_a)(\delta_{bd} + \delta_{bd}\phi_b)\phi_c^2$$

$$+ 4\sum_{cd}(\delta_{ac} + \delta_{ac}\phi_a)(\delta_{bd} + \delta_{bd}\phi_b)\phi_c\phi_d + 4\sum_{cd}(\delta_{ad} + \delta_{ad}\phi_a)(\delta_{bc} + \delta_{bc}\phi_b)\phi_c\phi_d \tag{37}$$

$$- \frac{\delta_{ab}}{n}(4n + 8)\sum_{i} : \phi_i^2 : +...,$$

$$= -\frac{8}{n}\delta_{ab}\Phi + 8\widetilde{E}_{ab} + ...$$

VII.   APPENDIX B

The detail of the derivation of the three- and four-points correlation functions are presented in this Appendix. The three-point correlation function of the field $\widetilde{E}_{ab}(r)$ has the following explicit expression in terms of the fields : $\phi_a\phi_b$ ::

$$\langle\widetilde{E}_{ab}(r_1)\widetilde{E}_{cd}(r_2)\widetilde{E}_{ef}(r_3)\rangle = \Big\langle (: \phi_a\phi_b(r_1) : -\tfrac{\delta_{ab}}{n}\sum_i : \phi_i^2(r_1) :)(: \phi_c\phi_d(r_2) : -\tfrac{\delta_{cd}}{n}\sum_j : \phi_j^2(r_2) :)$$
$$\times(: \phi_e\phi_f(r_3) : -\tfrac{\delta_{ef}}{n}\sum_k : \phi_k^2(r_3) :)\Big\rangle. \tag{38}$$

Tensorial structure of the right-hand side of the above equation has the following terms:

$$\langle : \phi_a\phi_b :: \phi_c\phi_d :: \phi_e\phi_f :\rangle \sim D_{ab,cd,ef},$$
$$\sum_i \langle : \phi_a\phi_b :: \phi_c\phi_d :: \phi_i^2 :\rangle \sim 4D_{ab,cd},$$
$$\sum_{ij} \langle : \phi_a\phi_b :: \phi_i^2 :: \phi_j^2 :\rangle \sim 8\delta_{ab},$$
$$\sum_{ijk} \langle : \phi_i^2 :: \phi_j^2 :: \phi_k^2 :\rangle \sim 8n, \tag{39}$$

where $D_{ab,cd,ef} = \delta_{ac}D_{bd,ef} + \delta_{ad}D_{bc,ef} + \delta_{ae}D_{cd,bf} + \delta_{af}D_{be,cd}$. According to Eq.(28) and the above equations:

$$\langle\widetilde{E}_{ab}(r_1)\widetilde{E}_{cd}(r_2)\widetilde{E}_{ef}(r_3)\rangle = \left[D_{ab,cd,ef} - \frac{4}{n}(\delta_{ab}D_{cd,ef} + \delta_{cd}D_{ab,ef} + \delta_{ef}D_{ab,cd}) + \frac{16}{n^2}\delta_{ab}\delta_{cd}\delta_{ef}\right]$$
$$\times A(n)(r_{12}r_{13}r_{23})^{-x_{\tilde{E}}(n)}. \tag{40}$$

Other three-point correlation functions which are used in deriving Eqs.(30,31), obtain by the same method:

$$\langle\widetilde{E}_{ab}(r_1)E(r_2)E(r_3)\rangle = 0,$$
$$\langle E(r_1)E(r_2)E(r_3)\rangle = 8nC(n)(r_{12}r_{13}r_{23})^{-x_E(n)},$$
$$\langle\widetilde{E}_{ab}(r_1)\widetilde{E}_{cd}(r_2)E(r_3)\rangle = \left[4D_{ab,cd} - \frac{8}{n}\delta_{ab}\delta_{cd}\right]B(n)(r_{13}r_{23})^{-x_E(n)}r_{12}^{x_E(n)-2x_{\tilde{E}}(n)}. \tag{41}$$

Four-point correlation function of four scaling fields φ_1, φ_2, φ_3 and φ_4 with scaling dimensions Δ_1, Δ_2, Δ_3 and Δ_4, respectively, has the form:

$$\langle\varphi_1(r_1)\varphi_2(r_2)\varphi_3(r_3)\varphi_4(r_4)\rangle = \prod_{1\leq i<j\leq 4} r_{ij}^{\Delta_{ij}} f(\eta), \tag{42}$$

with $\Delta_{ij} = \frac{1}{3}\sum_{k=1}^4 \Delta_k - \Delta_i - \Delta_j$ and $f(\eta)$ is the unknown function of cross ratio $\eta = \frac{r_{12}r_{34}}{r_{13}r_{24}}$.

According to above equation some key functions that are used for deriving the four-point correlation functions are as follows:

$$\langle\widetilde{E}_{ab}(r_1)E(r_2)E(r_3)E(r_4)\rangle = 0,$$
$$\langle E(r_1)E(r_2)E(r_3)E(r_4)\rangle = A(n)f_1(\eta)(r_{12}r_{13}r_{14}r_{23}r_{24}r_{34})^{-2x_E(n)/3}(48n + 12n^2),$$
$$\langle\widetilde{E}_{ab}(r_1)\widetilde{E}_{cd}(r_2)\widetilde{E}_{ef}(r_3)E(r_4)\rangle = B(n)f_2(\eta)(r_{12}r_{13}r_{23})^{-x_{\tilde{E}}(n)+x_E(n)/3}(r_{14}r_{24}r_{34})^{-2x_E(n)/3}$$
$$\times \left[6D_{ab,cd,ef} - \frac{24}{n}(\delta_{ab}D_{cd,ef} + \delta_{cd}D_{ab,ef} + \delta_{ef}D_{ab,cd}) + \frac{96}{n^2}\delta_{ab}\delta_{cd}\delta_{ef}\right],$$
$$\langle\widetilde{E}_{ab}(r_1)\widetilde{E}_{cd}(r_2)E(r_3)E(r_4)\rangle = C(n)f_3(\eta)\left[(2n + 24)D_{ab,cd} - \left(4 + \tfrac{48}{n}\right)\delta_{ab}\delta_{cd}\right]$$
$$\times(r_{12}^{-4x_{\tilde{E}}(n)/3+2x_E(n)/3})(r_{34}^{2x_{\tilde{E}}(n)/3-4x_E(n)/3})(r_{13}r_{14}r_{23}r_{24})^{-x_{\tilde{E}}(n)/3-x_E(n)/3},$$
$$\langle\widetilde{E}_{ab}(r_1)\widetilde{E}_{cd}(r_2)\widetilde{E}_{ef}(r_3)\widetilde{E}_{gh}(r_4)\rangle = D(n)f_4(\eta)(r_{12}r_{13}r_{14}r_{23}r_{24}r_{34})^{-2x_{\tilde{E}}(n)/3}$$
$$\times \left[D_{ab,cd,ef,gh} - \frac{6}{n}\alpha_{abcdefgh} + \left(\tfrac{24}{n^2} - \tfrac{2}{n}\right)\beta_{abcdefgh} + \left(\tfrac{12}{n^2} - \tfrac{144}{n^3}\right)\delta_{ab}\delta_{cd}\delta_{ef}\delta_{gh}\right]. \tag{43}$$

where

$$D_{ab,cd,ef,gh} = \delta_{ac}(\delta_{bd}D_{ef,gh} + D_{bd,ef,gh}) + \delta_{ad}(\delta_{bc}D_{ef,gh} + D_{bc,ef,gh}) + \delta_{ae}(\delta_{bf}D_{cd,gh} + D_{bf,cd,gh})$$
$$+ \delta_{af}(\delta_{be}D_{cd,gh} + D_{be,cd,gh}) + \delta_{ag}(\delta_{bh}D_{cd,ef} + D_{bh,cd,ef}) + \delta_{ah}(\delta_{bg}D_{cd,ef} + D_{bg,cd,ef}),$$
$$\alpha_{abcdefgh} = \delta_{ab}D_{cd,ef,gh} + \delta_{cd}D_{ab,ef,gh} + \delta_{ef}D_{ab,cd,gh} + \delta_{gh}D_{ab,cd,ef},$$
$$\beta_{abcdefgh} = \delta_{ab}\delta_{cd}D_{ef,gh} + \delta_{ab}\delta_{ef}D_{cd,gh} + \delta_{ab}\delta_{gh}D_{cd,ef} + \delta_{cd}\delta_{ef}D_{ab,gh} + \delta_{cd}\delta_{gh}D_{ab,ef} + \delta_{ef}\delta_{gh}D_{ab,cd}. \tag{44}$$

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