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The exact N-point generating function in Polyakov–Burgers turbulence

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Abstract

We find the exact N-point generating function in the Polyakov approach to Burgers turbulence.

1. Introduction

The theoretical understanding of turbulence has eluded physicists for a long time. An interesting approach is to model turbulence using stochastic partial differential equations [1,2]. In this direction, Polyakov [3] has recently offered a field theoretic method for deriving the probability distribution or density of states in (1 + 1)-dimensional turbulent systems. He formulates a new method for analyzing the inertial range correlation functions based on the two important ingredients in field theory and statistical physics, namely the operator product expansion (OPE) and anomalies. Despite the existence of many field theoretic approaches to turbulence [4–6], it appears that this new approach is more promising. Polyakov argues that in the limit of a high Reynold number, because of the existence of singularities at the coinciding point, dissipation remains finite and all subleading terms give vanishing contributions in the inertial range. By using the OPE one finds the leading singularities and can show that this approach is self-consistent. Here we consider Polyakov's approach [3] to Burgers turbulence when the pressure gradient is negligible and solve the N-point master equation, calculating the N-point generating function. Our result also applies to the Kardar–Parisi–Zhang (KPZ) equation in 1 + 1 dimensions to investigate crystal growth [7], nonlinear dynamics of a moving line [8], galaxy formation [9], dissipative transport [11], dynamics of a sine-Gordon chain [12], behavior of a magnetic flux line in superconductors [13], and spin glasses [14].

2. N-point generating functions

The Burgers equation has the following form,

$$u_t + u u_x = \nu u_{xx} + f(x, t), \tag{1}$$

where u is the velocity field, and ν is the viscosity and $f(x, t)$ is the Gaussian random force with the following correlation,

$$\langle f(x, t) f(x', t') \rangle = k(x - x') \delta(t - t'). \tag{2}$$

The transformation, $u(x, t) = -\lambda \partial_x h(x, t)$ maps Eq. (1) to the well known KPZ equation [7],

$$\partial_t h = \nu \partial_{xx} h + \tfrac{1}{2}\lambda [\partial_x h]^2 + f_2(x, t). \tag{3}$$

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It is noted that the parameter λ that appears in the above transformation is not renormalized under any renormalization procedure [10]. Following Polyakov [3] we consider the following generating functional,

$$Z_N(\lambda_1, \lambda_2, \dots \lambda_N, x_1, \dots x_N) = \left\langle \exp\left(\sum_{j=1}^{N} \lambda_j u(x_j, t) \right) \right\rangle. \tag{4}$$

Noting that the random force $f(x, t)$ has a Gaussian distribution, Z_N satisfies a closed differential equation provided that the viscosity ν tends to zero,

$$\dot{Z}_N + \sum \lambda_j \frac{\partial}{\partial \lambda_j}\left(\frac{1}{\lambda_j} \frac{\partial Z_N}{\partial x_j} \right) = \sum k(x_i - x_j) \lambda_i \lambda_j Z_N + D_N, \tag{5}$$

where D_N is

$$D_N = \nu \sum \lambda_j \left\langle u''(x_j, t) \exp\left(\sum \lambda_k u(x_k, t) \right) \right\rangle. \tag{6}$$

To remain in the inertial range we must, however, keep ν infinitesimal but nonzero. Polyakov argues that the anomaly mechanism implies that infinitesimal viscosity produces a finite effect. To compute this effect Polyakov makes the F-conjecture, which is the existence of an operator product expansion or fusion rules. A fusion rule is a statement concerning the behaviour of correlation functions, when some subset of points are put close together.

Let us use the following notation,

$$Z(\lambda_1, \lambda_2, \dots, x_1, \dots x_N) = \langle e_{\lambda_1}(x_1) \dots e_{\lambda_N}(x_N) \rangle. \tag{7}$$

Then Polyakov's F-conjecture is that in this case the OPE has the following form,

$$e_{\lambda_1}(x + \tfrac{1}{2}y) e_{\lambda_2}(x - \tfrac{1}{2}y) = A(\lambda_1, \lambda_2, y) e_{\lambda_1 + \lambda_2}(x) + B(\lambda_1, \lambda_2, y) \frac{\partial}{\partial x} e_{\lambda_1 + \lambda_2} + o(y^2). \tag{8}$$

This implies that Z_N fuses into functions Z_{N-1} as we fuse a couple of points together. The F-conjecture allows us to evaluate the following anomaly operator (i.e. the D_N-term in Eq. (5)),

$$a_\lambda(x) = \lim_{\nu \to 0} \nu(\lambda) u''(x) \exp[\lambda u(x)], \tag{9}$$

which can be written as

$$a_\lambda(x) = \lim_{\xi, y, \nu \to 0} \lambda \nu \frac{\partial^3}{\partial \xi \partial y^2} e_\xi(x + y) e_\lambda(x). \tag{10}$$

As discussed in Ref. [3] the only possible Galilean invariant expression is

$$a_\lambda(x) = a(\lambda) e_\lambda(x) + \tilde{\beta}(\lambda) \frac{\partial}{\partial x} e_\lambda(x). \tag{11}$$

Therefore in the steady state the master equation takes the following form,

$$\sum \left(\frac{\partial}{\partial \lambda_j} - \beta(\lambda_j) \right) \frac{\partial}{\partial x_j} Z_N - \sum k(x_i - x_j) \lambda_i \lambda_j Z_N = \sum a(\lambda_j) Z_N,$$

$$\beta(\lambda) = \tilde{\beta}(\lambda) + \frac{1}{\lambda}. \tag{12}$$

Polyakov has found the following explicit form of Z_2 in the case that $k(x_i - x_j) = K(0)[1 - (x_i - x_j)^2/l^2]$,

$$Z_2(\mu y) = e^{2(\mu y)^{3/2}/3}, \tag{13}$$

and the following expression for the density of states as the Laplace transform of Z_2,

$$W(u, y) = \int_{c - i\infty}^{c + i\infty} \frac{d\mu}{2\pi i} e^{-\mu u} Z_2(\mu y), \tag{14}$$

where

$$\mu = 2(\lambda_1 - \lambda_2), \quad y = x_1 - x_2.$$

It can be easily shown that with the following definition of variables the Polyakov master equation with the scaling conjecture [3] is

$$\left(\frac{\partial^2}{\partial \mu_2 \partial y_2} + \frac{\partial^2}{\partial \mu_3 \partial y_3} - (y_2 \mu_2 + y_3 \mu_3)^2 \right) f_3 = 0, \tag{15}$$

where

$$f_3 = (\lambda_1 \lambda_2 \lambda_3)^{-b} Z_3,$$
$$y_1 = \tfrac{1}{3}(x_1 + x_2 + x_3), \quad y_2 = x_1 - \tfrac{1}{2}(x_2 + x_3),$$
$$y_3 = x_2 - x_3, \quad \mu_1 = \tfrac{1}{3}(\lambda_1 + \lambda_2 + \lambda_3),$$
$$\mu_2 = \tfrac{3}{2}[\lambda_1 - \tfrac{1}{2}(\lambda_2 + \lambda_3)], \quad \mu_3 = 2(\lambda_2 - \lambda_3). \tag{16}$$

Now we set f_3 as

$$f_3 = \mu_2^{S_2}\mu_3^{S_3} g_3(\mu_2 y_2, \mu_3 y_3). \tag{17}$$

Inserting this in Eq. (15) results in

$$g_3(\mu_2 y_2, \mu_3 y_3) = \mathrm{e}^{2(\mu_2 y_2 + \mu_3 y_3)^{3/2}/3} \tag{18}$$

and

$$S_2 = S_3 = -\tfrac{5}{4}.$$

If we use the following transformation,

$$y_1 = \frac{x_1 + x_2 + x_3 + \dots x_N}{N},$$
$$y_2 = x_1 - \frac{x_2 + x_3 + \dots x_N}{N-1},$$
$$y_3 = x_2 - \frac{x_3 + x_4 + \dots x_N}{N-2}, \quad \dots,$$
$$y_N = x_{N-1} - x_N, \tag{19}$$

and

$$\mu_1 = \frac{\lambda_1 + \lambda_2 + \dots + \lambda_N}{N},$$
$$\mu_2 = \frac{N}{N-1}\left(\lambda_1 - \frac{\lambda_2 + \lambda_3 + \dots + \lambda_N}{N-1}\right),$$
$$\mu_3 = \frac{N-1}{N-2}\left(\lambda_2 - \frac{\lambda_3 + \lambda_4 + \dots + \lambda_N}{N-2}\right), \quad \dots,$$
$$\mu_N = 2(\lambda_{N-1} - \lambda_N), \tag{20}$$

we obtain the following partial differential equation for f_N,

$$\left(\frac{\partial^2}{\partial y_2 \partial \mu_2} + \dots + \frac{\partial^2}{\partial y_N \partial \mu_N}\right) f_N - (y_2 \mu_2 + \dots + y_N \mu_N)^2 f_N = 0, \tag{21}$$

which is solved by

$$f_N = (\mu_2 \mu_3 \dots \mu_N)^{-(2N-1)/2(N-1)} \times \mathrm{e}^{2(\mu_2 y_2 + \dots + \mu_N y_N)^{3/2}/3}. \tag{22}$$

In principle the parameter b in Eq. (16) can be evaluated by means of the exponents of the $(\mu_2 \dots \mu_N)$ term in Eq. (22), which for Z_2 turns out to be $b = \frac{3}{4}$.

It must be noted that although the particular choice of $k(x-x')$ used above is solvable it may not be appropriate to turbulence, due to its short-range correlation. A correlation function for a noise better suited to turbulence is given in Ref. [15]. Another relevant random force is the conservative one, which is important when studying the KPZ equation. Work in both these directions is on the way.

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