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Phase transition in a self-gravitating planar gas

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Abstract

We consider a gas of Newtonian self-gravitating particles in two-dimensional space, finding a phase transition, with a high temperature homogeneous phase and a low temperature clumped one. We argue that the system is described in terms of a gas with fractal behaviour. © 1998 Elsevier Science B.V. All rights reserved.

1. Introduction

The statistical behaviour of systems interacting via classical (Newtonian) gravitational forces is rather peculiar as compared with other statistical systems, such as neutral gases and plasmas. The central distinguishing feature of the former is the fact that there is no shielding of the long range gravitational force, while Debye-screening rules the long distance behaviour of the electric Coulomb force. In lower dimensions such a characteristic of the gravitational force is dramatic, due to the rising of the two body potential in Newtonian gravity. This leads to conceptual problems, related to the extensive nature of energy, as required in statistical mechanics.

In general these statistical problems are not perceptible to exact treatment, and only numerical results are known. While classical gravitating systems in one space dimension are quite pathological, in two

space dimensions they become manageable, although the most realistic three dimensional case is still beyond reach. In particular, in the latter case the phase space volume diverges and one has to use a short distance cutoff.

In two-dimensional space, the thermodynamic functions are analytically computable. The thermodynamical properties of a gravitational gas has been studied in detail both from a theoretical [1] as well as from a numerical [2–5] point of view. In general, it has been shown [6,7] that a classical gas of gravitating particles, in the grand canonical formalism is related to a field theory described by a Liouville Lagrangian. In a two-dimensional space Liouville theory is well known [8] and several correlation functions can be exactly computed [9–11]. Indeed, in this particular case the theory is conformally invariant, and correlators can be computed in terms of known functions of mathematical physics, once properly regularized [12].

At low temperatures the gravitational force is strong, that is, the potential energy is big as com-

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pared to the kinetic energy, giving rise to a collapsing phase, identified by the presence of a single cluster of particles floating in a diluted homogeneous background. At high energy a homogeneous phase is recovered and the cluster disappears. Using the microcanonical ensemble one can show that in the transition region the system is characterized by a negative specific heat (the corresponding instability is of extreme relevance for astrophysics, [13]).

Hertel and Thirring [1] have shown that the canonical and microcanonical ensembles are not equivalent in the proximity of the transition. This thermodynamic inconsistency has been solved in Ref. [1], and these results have been successfully confirmed by numerical investigations on self gravitating non-singular systems with short range interactions [2,3].

Recently, using numerical calculations, a long range attractive potential has been considered, as constructed taking the first few terms of the Fourier expansion of the logarithmic potential. As a result of the simulations, the system turns out to exhibit a transition from a collapsed phase at low temperature, to a homogeneous phase [5].

Here we consider a two-dimensional gas of classically interacting particles via the Newtonian logarithmic potential, computing its thermodynamical properties. Using results known in the framework of two-dimensional conformally invariant euclidian field theory, the complex integrals may be computed in closed form, and we find a sophisticated structure of poles and zeros in terms of the temperature. Furthermore, we introduce an order parameter, corresponding to the expectation value of the square of the two-body distance. We find that at least at one of the singularities of the partition function the average distance vanishes, signaling a clumped phase. We are thus able to investigate the collapse of the gravitating system. The phase transition point is given in terms of the mass of the particles (m) and the gravitational constant G by $T_c = \frac{1}{4k} NGm^2$. We find the exponents of the phase transition.

We furthermore discuss the formulation of gravity in terms of the grand-canonical partition function and the related Liouville theory as proposed by de Vega et al [6], outlining the consequent relations. We find, using the same type of complex integrals, a critical temperature in this formulation.

2. Canonical partition function

We consider a gas of nonrelativistic particles with mass m interacting through Newtonian gravity at temperature T . Here, the number of particles is fixed and we shall work with the canonical ensemble. The canonical partition function of the system can be written as

$$Z = \int [d^2 r_i] [d^2 p_i] e^{-\beta H_N} \quad (1)$$

where $[d^2 r_i] = \prod_{i=1}^N d^2 r_i$, and the N -particle Hamiltonian H_N is obtained adding the Newtonian potential to the usual kinetic term, that is,

$$H_N = \sum_{i=1}^N \frac{p_i^2}{2m} + \frac{1}{2} \sum_{i \neq j} Gm^2 \log |r_i - r_j| \quad (2)$$

The potential term must be regularized, since in general there are divergences in either the infrared or in the ultraviolet domain. This is done implicitly defining the two-dimensional integrals in the complex plane, as done in Ref. [12] (see also Ref. [8]). We thus consider the two-dimensional variables as one single complex variable in the complex plane, denoting the procedure by the use of latin letters from the end of the alphabet, i.e., $d^2 r_i \rightarrow d^2 w_i$. As it turns out, there is an underlining conformal invariance in the present problem, easily seen using the complex variables, where it is characterized by a $SL(2, \mathbb{C})$ invariance of the Hamiltonian. Such an invariance is valid in the infinite volume limit. For finite volume one should take into account boundary effects, and in that case it would not be possible to treat the problem exactly. Fortunately, due to the fact that gravity on a plane constitutes a long range attractive force, the system is naturally bound to a region defined in terms of the typical bound state length. We thus expect the procedure to be exact for not too large values of the temperature. Therefore, well above the critical point departure might be expected from our results. Nevertheless, the critical temperature is expected to be correct on the above grounds.

Using that invariance, we set $w_{N+1} = 0, w_{N+2} = 1$ and $w_{N+3} = \infty$ (whose integration would lead to an infinite overall factor in the partition function which

can be discarded in the computation of physical quantities) and find

$$Z = (2\pi m)^{N+3} \times \int d^2 w_i |w_i|^{2\alpha} |1 - w_i|^{2\beta} \prod_{i < j} |w_i - w_j|^{4\rho} \quad (3)$$

where $\alpha = \beta = -\frac{Gm^2}{2kT}$ and $2\rho = \alpha$. We keep the use of the α , β and γ parameters in order to facilitate comparison with the corresponding integrals in the literature [8,12]. Using such well known formulae, the partition function Z can be written in terms of ρ as

$$Z \simeq \Gamma(N+1) [\Delta(1-\rho)]^N \prod_1^N \Delta(i\rho) \times \prod_0^{N-1} [\Delta(1+(i+2)\rho)]^2 \times \Delta(-1-(N+3+i)\rho) \quad (4)$$

where $\Delta(x) = \frac{\Gamma(x)}{\Gamma(1-x)}$. This function has poles at the points $x = -n = 0, -1, -2, \dots$ and zeros for $x = n = 1, 2, \dots$. This is an exact result, as shown in e.g. [12,8]. However, in order to obtain the singularity structure (poles and zeros) in further related integrals, it is worthwhile to classify them. They arise from the following kind of rather naive argumentation.

If one of w_i (say w) $\rightarrow 0$ (or $w \rightarrow 1$), it can be shown that the behaviour of Z is

$$Z \simeq \pi \Delta(1+\alpha) \Delta(1+\beta) \Delta(-1-\alpha-\beta) \quad (5)$$

namely, for α (or β) non negative integer a zero appears, while for α, β negative integers, we have poles. The poles/zeros in α and β are connected with behaviour at $w \rightarrow \infty$. We thus obtain some of the functions Δ given in (4). If we consider the neighbourhood of some of the w_i 's around 0 or 1, $w_i \sim 0, 1$, we find that Z behaves as

$$Z \sim \prod_i \frac{\Gamma(1+\alpha+i\rho)}{\Gamma(-\alpha-i\rho)} \prod_i \frac{\Gamma(1+\beta+i\rho)}{\Gamma(-\beta-i\rho)}, \quad (6)$$

where $i = 1, 2, \dots, N-1$.

Using the invariance of Z under $w \rightarrow 1/w$, one can also show that Z has the symmetry

$$Z(\alpha, \beta, \rho) = Z(-\alpha-\beta-2-\rho(N-1), \beta, \rho) \quad (7)$$

As is evident in the canonical partition function (4) the zeros and poles given in terms of ρ depend on the temperature. Those zeros and poles are, in general, related to phase transitions. It should be noted that at such points the free-energy becomes singular. The simplest possible phase transition in this system is the collapsing of particles into clumps. In order to investigate the collapsed phase, it is natural to introduce an order parameter describing the average distance of the particles with respect to one another. We therefore consider the parameter $\Sigma_{i,j}^N \langle r_{ij}^2 \rangle$, which allows investigation of the collapse of the system. As it turns out, $\langle r^2 \rangle$ has several zeros, one of them coinciding with a singularity of the partition function. We also show that a class of singularities of Z may be related to the zeroes of higher moments of $\langle r^2 \rangle$ i.e $\langle r^{2q} \rangle$. Using the invariance of the Hamiltonian under translation, we set the coordinate of one of the particles to zero and we can write the mean square distance as

$$\langle r^2 \rangle = Z^{-1} \langle r^2 \rangle_0 \quad (8)$$

where

$$\langle r^2 \rangle_0 = \int d^2 z \prod_{i=1}^{N-1} d^2 w_i |w_i|^{2\alpha} |1 - w_i|^{2\beta} \times \prod_{i \neq j}^{N-1} |w_i - w_j|^{4\rho} |z|^{2+2\alpha} |1 - z|^{2\beta} |z - w_i|^{4\rho} \quad (9)$$

Similarly to the case of the partition function, the integral displayed in (9) has a complex structure of poles and zeros. Let us classify such a singularity structure, using now the hints we have gotten based on the computation of the partition function, (4).

Supposing $w \rightarrow 0$ (or $w \rightarrow 1, \infty$) we find

$$\langle r^2 \rangle_0 \sim \prod_{i=1}^{N-1} \Delta(1+\alpha+i\rho) \Delta(1+\beta+i\rho) \times \Delta(-1-\alpha-\beta-(N-2+i)\rho) \quad (10)$$

At $w_N = z \rightarrow 0$, $\langle r^2 \rangle_0$ behaves as

$$\frac{\Gamma(\alpha+2)}{\Gamma(-1-\alpha)}.$$

For $z, w \rightarrow 0$, it behaves as

$$\frac{\Gamma(1 + \alpha + 1/2 + \rho)}{\Gamma(1 - (\alpha + 1/2 + \rho))}.$$

It is easy to show that analysing the case where several points tend to zero we see that $\langle r^2 \rangle_0$ must behave as

$$\frac{\Gamma\left(1 + \alpha + \frac{1}{i+1} + i\rho\right)}{\Gamma\left(1 - \left(\alpha + \frac{1}{i+1} + i\rho\right)\right)}$$

where $i = 0, 1, \dots, N-1$. The behaviour of $\langle r^2 \rangle_0$ at 1 similarly obtained.

Therefore, for the singularity structure of $\langle r^2 \rangle$ we arrive at the result

$$\begin{aligned} \langle r^2 \rangle_0 &\approx \prod_{i=0}^{N-2} \Delta(1 + \alpha + i\rho) \Delta(1 + \beta + i\rho) \\ &\times \Delta(-1 - \alpha - \beta - (N + i - 1)\rho) \\ &\times \prod_{i=0}^{N-1} \Delta\left(1 + \alpha + i\rho + \frac{1}{i+1}\right) \\ &\times \Delta\left(-1 - \alpha - \beta - (N - 1 + i)\rho - \frac{1}{N - 1 - i}\right) \end{aligned} \quad (11)$$

while the normalized $\langle r^2 \rangle$ has the form

$$\begin{aligned} \langle r^2 \rangle &\sim \prod_{i=0}^{N-1} \Delta\left(1 + (i+2)\rho + \frac{1}{i+1}\right) \\ &\times \frac{\Delta\left(-1 - (N+3+i)\rho - \frac{1}{N-i-1}\right)}{\Delta(1 + (N+1)\rho)^2 \Delta(-1 - 2(N+1)\rho)} \end{aligned} \quad (12)$$

for $i=0$ it has a zero at $1 + \alpha + 1 =$ positive integer $= 1, 2, \dots$ therefore the possible value of ρ is given by $2\rho = \alpha = -1, 0, 1, \dots$

The expression for r^2 has several zeros. They can certainly not be all physically relevant, since most are just consequences of the regularization of the integrals (9). After encountering a zero, we loose the physical relevance of the integral. Below the critical point, the theory is in a clumped phase, while above

the particles are far apart. Therefore, there should be no further critical point. This fixes our critical point to be, at large N

$$kT_c = \frac{1}{4} N G m^2. \quad (13)$$

For large N this coincides with an old result of Salzberg [14], from which a possible phase transition point can be obtained.

Note that $\langle r^2 \rangle$ behaves as $|T - T_c|$ in terms of the temperature.

The higher moments of $\langle r^2 \rangle$ have the singularity structure

$$\begin{aligned} \langle r^{2q} \rangle &\sim \prod_{i=0}^{N-1} \frac{\Delta\left(1 + (i+2)\rho + \frac{q}{i+1}\right)}{\Delta(1 + (N+1)\rho)} \\ &\times \frac{\Delta\left(-1 + (N+3+i)\rho - \frac{q}{N-i-1}\right)}{\Delta(-1 - 2(N+1)\rho)} \end{aligned} \quad (14)$$

There are zeros at the values $\rho = -\frac{1}{2}q, -\frac{1}{2}q + 1, \dots$. If q is a multiple of $(i+1)$, for $i = 1, \dots, N-1$ there are further singularities. Depending on the exact value of q the behaviour in terms of the departure from the critical temperature changes, which shows that the system has multi-fractal nature. The zeros of $\langle r^{2q} \rangle$ are coincident with a class of singularities of Z . For $q = 2(N-1)$ the behaviour of $\langle r^{2q} \rangle$ is not linear in $|T - T_c|$.

3. The grand partition function

The grand partition function of the system can be written as

$$Z_G = \sum_{N=0}^{\infty} \frac{z^N}{N!} \int [d^2 r_i] [d^2 p_i] e^{-\beta H_N} \quad (15)$$

where H_N is given by (2) and $z = e^\mu$ is the fugacity.

According to [6,7] this many body problem can be transformed into a field theoretic one. Using the definition of density $\rho(\mathbf{r}) = \sum_{i=1}^N \delta(\mathbf{r} - \mathbf{q}_i)$, it has been shown that Z_G can be written in terms of Liouville field theory,

$$Z_G = \int D\phi' \exp\left(-\frac{1}{T_{\text{eff}}} \int \left[\frac{1}{2}(\nabla\phi')^2 - \mu'^2 e^{\phi'}\right]\right) \quad (16)$$

where $T_{\text{eff}} = 2\pi \frac{Gm^2}{kT}$ and $\mu'^2 = zGm^3$. we rescale ϕ' as $\phi' = \sqrt{\frac{T_{\text{eff}}}{4\pi}} \phi$. Therefore, the action can be written as

$$S_L = \frac{1}{8\pi} \int d^2x (|\nabla\phi|^2 + \mu e^{b\phi}) \tag{17}$$

where $b = \sqrt{\frac{T_{\text{eff}}}{4\pi}}$ and $\mu = \frac{-8\pi\mu'^2}{T_{\text{eff}}}$. In Ref. [6], it has also been shown that the correlations of the density can be written in terms of vertex operators of the Liouville theory.

It is conventional to add term $\frac{Q}{4\pi} R\sqrt{g} \varphi$ to the Liouville Lagrangian density, where R is the scalar curvature of background metric $g_{\mu\nu}$, and the parameter Q adjusted to ensure that all physical quantities be independent of a particular choice of the background. However it is possible to choose a specific background which is flat everywhere except for few selected points [15].

The Liouville field $\phi(z, \bar{z})$ is a logarithmic operator [16] and varies under holomorphic coordinate transformation $z \rightarrow w(z)$ as $\phi(w, \bar{w}) = \phi(z, \bar{z}) - \frac{Q}{2} \log|\frac{dw}{dz}|^2$ where $Q = b + \frac{1}{b}$. Q parametrizes the central charge of the theory by the well known relation $c = 1 + 6Q^2$.

The introduction of Q in the Liouville action is done for taking into account the fact that the theory corresponds to a Coulomb gas where there is no zero charge sector, except for the vacuum. It corresponds to different boundary conditions, and takes account of the zero modes of the theory [8]. In the description of Ref. [6] there is no zero mode in the inverse propagator, fixed as being the classical gravity potential, which naturally matches the fact that the background curvature has to vanish in Minkovski space.

We suppose that the theory can be described as a compactified Euclidian space, in which case the Q -term can be seen as a requirement of renormalization. The question to be answered is whether the large compactification radius limit is smooth or not. Since our argumentation are based on local Green functions, as in (23) below, we do not expect any major difference to occur. On the other hand, the argumentation based on the conformal dimensions (as in (24)) below and following equations) may depend on the above limit. However, it is rewarding to see that the results are in accordance with general

expectations, leading to the conjecture that the large compactification radius limit is smooth.

It is nevertheless necessary to stress that the introduction of the Q term is non trivial, and might lead to a change of the problem. The aim here is to describe the results obtained comparing them with known results. We know in fact from [6] that Liouville theory describes gravity, and any effort in the direction of understanding the model is worth undertaking.

It is well known that the exponential Liouville operators $V_\alpha(x) = e^{2\alpha\varphi(x)}$ are the spinless primary conformal fields of dimension $\Delta_\alpha = \alpha(Q - \alpha)$. The two, three and four-point correlation functions of Liouville field theory for given α have been calculated in Refs. [9,15].

In addition, we find that the dependence on the scale μ of any correlation function in Liouville theory [17,11] is

$$\langle \prod_{i=1}^N e^{2\alpha_i\varphi(x_i)} \rangle_Q \sim (\pi\mu)^{(Q - \sum_i \alpha_i)/b}. \tag{18}$$

We note that the Liouville theory is self-dual under $b \rightarrow 1/b$. Indeed, we consider the partition function of the Liouville theory. According to [9] the partition function has the form

$$Z = - \frac{\mu}{\sqrt{2} \pi^2 (b + 1/b)} \left(\frac{\pi\mu\Gamma(b^2)}{\Gamma(1 - b^2)} \right)^{1/b^2} \times \frac{\Gamma(-1/b^2)}{\Gamma(1/b^2 - 1)} \tag{19}$$

Under transformation of b and μ as given by

$$b \rightarrow \frac{1}{b}, \quad \mu \rightarrow \bar{\mu} = \frac{1}{\pi\Delta(1/b^2)} (\pi\mu\gamma(b^2))^{1/b^2} \tag{20}$$

it is easy to show that the partition function transforms as

$$Z(b, \mu) \Rightarrow Z\left(\frac{1}{b}, \bar{\mu}\right) = -\frac{1}{b^2} Z(b, \mu) \tag{21}$$

This duality transformation was first observed by Zamolodchikov [15]. We further observe that there

exists a sequence of critical values for b , so that the partition function becomes singular, that is

$$b_N^2 = \frac{b_c^2}{N} \quad (22)$$

where $b_c = 1$. As in the case of the canonical partition function, the grand-canonical Z_G has zeros.

In order to understand the nature of phase transition in the grand-canonical ensemble we define the variance of the number of particles as a parameter, where in the clumping point $(\Delta N)^2 = (\langle N^2 \rangle - \langle N \rangle^2)$. This variance has a peculiar behaviour. It is easy to show that the $(\Delta N)^2$ can be written in terms of two-point density correlation functions as:

$$(\Delta N)^2 \sim \iint d^2 x_1 d^2 x_2 \langle \rho(x_1) \rho(x_2) \rangle \quad (23)$$

Now using the results of [9] for two point correlation functions of Liouville vertex operators, one can show that at the transition point $\langle \rho(x_1) \rho(x_2) \rangle \rightarrow 0$, which shows that particles collapse to local clusters.

Now it is possible to investigate the multi-fractal nature of 2D-Gravitating gas considering moments of the density in a given scale R . Using the conformal dimensions of Liouville operators we can show that

$$\int_R d^2 x \langle \rho^q \rangle \sim R^{-\tau(q)} \quad (24)$$

where $\tau(q) = 2(1 - b^2 q)(q - 1)$ which is valid only for $q \leq \frac{b^{-2} + 1}{2}$.

Eq. (24) allows us to determine the distribution functions of density $p(\rho)$, such that

$$\langle \rho^n \rangle = \int_0^\infty \rho^n p(\rho) d\rho \quad (25)$$

and

$$p(\rho) = f(\rho) \exp\left[-\frac{1}{\log R^a} \log^2(\rho R^d)\right] \quad (26)$$

where $f(\rho)$ is a smooth function of $\log(\rho)$. Moreover $a = 16b^6$, $d = 2b(1 + 4b^2)$ (see also [18]). This is the famous log-normal distribution, considered as a characteristic feature of a disordered system. It turns out that for a dilute gas $\rho \approx \frac{1}{R^{2b(1+4b^2)}}$, and we

find the Gaussian distribution functions, whose variance is controlled by R .

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