

روش های

- (روفازی) leap - frog -
- (روفازی) IMPES -
- (سرفازی) IMPES -
- (سرفازی) Full Implicit -
- (سرفازی) Adaptive Implicit -

جلسه دوم
شبهه های مدل سازی

7

Multiphase Flow

*These notions everywhere in nature
Must surely be
The circulations of God.*

Thoreau

7.1 Introduction

The principles outlined in the preceding chapter can be readily extended to multiphase, multidimensional flow. We begin our discussion with two-phase incompressible flow and follow this with a treatment of three-phase compressible flow. We confine ourselves to heterogeneous, anisotropic media in 3-D. Actually, the three-phase, compressible model represents the most general case for such an environment in a black-oil reservoir. A simulator constructed to treat the latter can equally well handle the incompressible case by supplying appropriate input data. Furthermore, flow regimes in either 1, 2, or 3-D can be similarly treated.

We outline three methods of solution for the two-phase flow problem, a simultaneous solution technique, the "leap-frog" approach, and an implicit pressure, explicit saturation procedure (IMPES). For the three-phase compressible problem, we discuss the IMPES method, a fully implicit approach, and an adaptive implicit method.

7.2 Two-Phase Flow

For a two-phase flow system, we consider the water and oil equations given by Eqs. 4.44 and 4.45 and treat first a simultaneous solution procedure similar to that given by Coats et al.¹

روش حل معادلات اریتمیک صفر

$$\nabla \cdot \left\{ \frac{k k_{rw}}{\mu_w} \nabla \Phi_w \right\} + B_w Q_w = -S' \phi \frac{\partial \Phi_w}{\partial t} + S' \phi \frac{\partial \Phi_o}{\partial t} \quad (7.11)$$

$$\nabla \cdot \left\{ \frac{k k_{ro}}{\mu_o} \nabla \Phi_o \right\} + B_o Q_o = S' \phi \frac{\partial \Phi_w}{\partial t} - S' \phi \frac{\partial \Phi_o}{\partial t} \quad (7.12)$$

which have the finite difference approximations

$$\Delta T_w \Delta \Phi_w + B_w q_w = G \Delta_t \Phi_w - G \Delta_t \Phi_o \quad (7.13)$$

$$\Delta T_o \Delta \Phi_o + B_o q_o = -G \Delta_t \Phi_w + G \Delta_t \Phi_o \quad (7.14)$$

where

$$G \equiv - \frac{S' \phi \Delta x_l \Delta y_l \Delta z_k}{\Delta t 5.61416} \quad \text{، دبر و عبارات عاریه } D$$

$$q_w \equiv \frac{\Delta x_l \Delta y_l \Delta z_k Q_w}{5.61416}$$

$$q_o \equiv \frac{\Delta x_l \Delta y_l \Delta z_k Q_o}{5.61416}$$

$$\Delta_t \Phi_l \equiv \Phi_l^{n+1} - \Phi_l^n$$

$$\Delta T_l \Delta \Phi_l \equiv \Delta_r T_l \Delta_r \Phi_l + \Delta_v T_l \Delta_v \Phi_l + \Delta_z T_l \Delta_z \Phi_l, \quad l = o, w$$

$$(T_{\alpha\alpha})_{l=1/2} \equiv \left\{ \frac{\alpha k k_{ro}}{\Delta x_l \mu_o} \right\}_{l=1/2} \Delta y_l \Delta z_k$$

and $\alpha \equiv 0.001127$, for k in md., $\Delta x_l, \Delta y_l, \Delta z_k$ in ft. and μ_o in cp. T_l and $T_{l\alpha}$, $l = o, w$, have definitions similar to $T_{\alpha\alpha}$.

We assume the transmissibilities are evaluated at time level n ; i.e., they are computed explicitly. The transmissibilities contain saturation dependent terms ($k_{r,l}$). These are evaluated at the half-nodal points by a weighting technique, i.e.,

$$(k_{r,l})_{i+1/2} = \omega (k_{r,l})_{upstream} + (1 - \omega) (k_{r,l})_{downstream} \quad (7.15)$$

where $0 \leq \omega \leq 1$. If $\omega = 1/2$ then Eq. 7.15 produces the arithmetic average. In Fig. 7.1, the i^{th} block is the upstream block while the $(i + 1)^{th}$ block is downstream.

To obtain physically realistic results it is necessary to employ upstream

Water Equation: $\nabla \cdot \left\{ \frac{k k_{rw}}{\mu_w B_w} \nabla \Phi_w \right\} + Q_w = \phi \frac{\partial (S_w/B_w)}{\partial t}$ (7.1)

Oil Equation: $\nabla \cdot \left\{ \frac{k k_{ro}}{\mu_o B_o} \nabla \Phi_o \right\} + Q_o = \phi \frac{\partial (S_o/B_o)}{\partial t}$ (7.2)

For simplicity, we associate only plus signs with the source/sink terms since the choice is arbitrary. If we assume the fluids are incompressible then B_o and B_w are constants, and

$$\nabla \cdot \left\{ \frac{k k_{r,l}}{\mu_l} \nabla \Phi_l \right\} + B_l Q_l = \phi \frac{\partial S_l}{\partial t}, \quad l = o, w. \quad (7.3)$$

We also invoke the following relations:

$$S_o + S_w = 1 \quad (7.4)$$

(water-wet system) capillary pressure $P_{rcw} = p_o - p_w$ (7.5)

$$\Phi_o = p_o - \gamma_o d \quad (7.6)$$

$$\Phi_w = p_w - \gamma_w d \quad (7.7)$$

Then,

$$P_{cwo} = \Phi_o - \Phi_w - \delta \gamma d; \quad \delta \gamma \equiv \gamma_w - \gamma_o. \quad \text{(مقدار مایل)} \quad (7.8)$$

At the water-oil contact (WOC) $P_{cwo} = 0$; thus we have the initial conditions

$$\Phi_o = \Phi_w + (\delta \gamma) d_{iwc}. \quad (7.9)$$

7.2.1 Simultaneous Solution Method

Let $S_w = S$ such that $S_o = 1 - S$ from Eq. 7.4. The right-hand sides of Eq. 7.3 become $\phi \partial S / \partial t$ and $-\phi \partial S / \partial t$, respectively. Now

$$\frac{\partial S}{\partial t} = \frac{\partial S}{\partial P_{cwo}} \frac{\partial P_{cwo}}{\partial t} = S' \frac{\partial P_{cwo}}{\partial t} = S' \left\{ \frac{\partial \Phi_o}{\partial t} - \frac{\partial \Phi_w}{\partial t} \right\} = S' \left\{ \frac{\partial \Phi_o}{\partial t} - \frac{\partial \Phi_w}{\partial t} \right\}$$

حرکت است. این 6 فرکانس است

using Eq. 7.8. If Eq. 7.10 is substituted in Eq. 7.3 then,

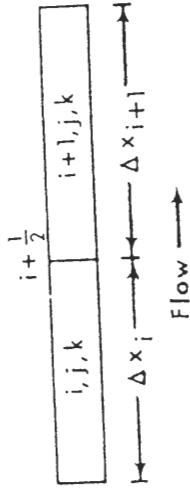


Fig. 7.1 Upstream and Downstream Blocks.

weighting on k_{r1} , i.e., $w = 1.0$ for the situation in Fig. 7.1. Moreover, for 1-D problems in the absence of capillary pressure, upstream weighting insures stability for Δt sufficiently small.² Thus, if $w = 1$, *single-point* upstream weighting is applied to the relative permeability which is first order correct. A *two-point* weighting scheme which is second order correct has been presented by Todd, et al.³ The absolute permeability k is computed as a harmonic average,

$$(\bar{\alpha}) = \frac{1}{\bar{\alpha}} = \frac{1}{\alpha_1} + \frac{1}{\alpha_2} \quad ; \quad \left(\frac{k}{\Delta x}\right)_{i+1/2} = \frac{2 k_i k_{i+1}}{k_i \Delta x_{i+1} + k_{i+1} \Delta x_i} \quad (7.16)$$

In incompressible flow, the viscosities μ_o and μ_w are constants. In the compressible flow case they are pressure dependent functions. They can be satisfactorily computed at the block interfaces by averaging between blocks.

If we expand Eqs. 7.13 and 7.14 we obtain

$$\Delta T_w \Delta \phi_w^{n+1} - C \phi_w^{n+1} + C \phi_o^{n+1} = -B_w q_w^n - C \phi_w^n + C \phi_o^n \quad (7.17)$$

$$\Delta T_o \Delta \phi_o^{n+1} + C \phi_w^{n+1} - C \phi_o^{n+1} = -B_o q_o^n + C \phi_w^n - C \phi_o^n \quad (7.18)$$

We formulate the solution of Eqs. 7.17-7.18 in terms of the Douglas-Rachford iterative procedure presented in chapter 6. For the water equation (Eq. 7.17) we have

$$\begin{aligned} \Delta_x T_{wx} \Delta_x \phi_w^{k+1/3} - C \phi_w^{k+1/3} + C \phi_o^{k+1/3} + \Delta_y T_{wy} \Delta_y \phi_w^k + \Delta_z T_{wz} \Delta_z \phi_w^k \\ = -B_w q_w^n - C \phi_w^n + C \phi_o^n + a \end{aligned} \quad (7.19)$$

x-sweep:

y-sweep:

$$\begin{aligned} \Delta_x T_{wx} \Delta_x \phi_w^{k+1/3} - C \phi_w^{k+2/3} + C \phi_o^{k+2/3} + \Delta_y T_{wy} \Delta_y \phi_w^{k+2/3} + \Delta_z T_{wz} \Delta_z \phi_w^k \\ = -B_w q_w^n - C \phi_w^n + C \phi_o^n + b \end{aligned} \quad (7.20)$$

z-sweep:

$$\begin{aligned} \Delta_x T_{wx} \Delta_x \phi_w^{k+1/3} - C \phi_w^{k+1} + C \phi_o^{k+1} + \Delta_y T_{wy} \Delta_y \phi_w^{k+2/3} + \Delta_z T_{wz} \Delta_z \phi_w^{k+1} \\ = -B_w q_w^n - C \phi_w^n + C \phi_o^n + c \end{aligned} \quad (7.21)$$

where

$$\begin{aligned} a &= \sigma_k(\Sigma T_w)(\phi_w^{k+1/3} - \phi_w^k) \\ b &= \sigma_k(\Sigma T_w)(\phi_w^{k+2/3} - \phi_w^k) \\ c &= \sigma_k(\Sigma T_w)(\phi_w^{k+1} - \phi_w^k) \end{aligned}$$

As in chapter 6, ΣT_w means the sums of the water transmissibilities over the faces of a grid block. Similar expressions to Eq. 7.21 are written for Eq. 7.18. We employ the following definitions to minimize round-off and cancellation errors:

$$PX = \phi_w^{k+1/3} - \phi_w^k \quad (7.22a)$$

$$PY = \phi_w^{k+2/3} - \phi_w^k \quad (7.22b)$$

$$PZ = \phi_w^{k+1} - \phi_w^k \quad (7.22c)$$

$$RX = \phi_o^{k+1/3} - \phi_o^k \quad (7.22d)$$

$$RY = \phi_o^{k+2/3} - \phi_o^k \quad (7.22e)$$

$$RZ = \phi_o^{k+1} - \phi_o^k \quad (7.22f)$$

If we use Eqs. 7.22a-7.22c in Eqs. 7.19-7.21 then,

x-sweep:

$$\begin{aligned} \Delta_x T_{wx} \Delta_x PX + \Delta T_w \Delta \phi_w^k - C PX + C RX \\ = -B_w q_w^n + C(\phi_w^k - \phi_w^n) - C(\phi_o^k - \phi_o^n) + \sigma_k(\Sigma T_w) PX \end{aligned} \quad (7.23)$$

y-sweep:

$$\begin{aligned} \Delta_x T_{wx} \Delta_x PX + \Delta T_w \Delta \phi_w^k - C PY + C RY + \Delta_y T_{wy} \Delta_y PY \\ = -B_w q_w^n + C(\phi_w^k - \phi_w^n) - C(\phi_o^k - \phi_o^n) + \sigma_k(\Sigma T_w) PY \end{aligned} \quad (7.24)$$

z-sweep:

$$\begin{aligned} \Delta_x T_{wx} \Delta_x PX + \Delta_y T_{wy} \Delta_y PY + \Delta_z T_{wz} \Delta_z PZ + \Delta T_w \Delta \phi_w^k - C PZ + C RZ \\ = -B_w q_w^n + C(\phi_w^k - \phi_w^n) - C(\phi_o^k - \phi_o^n) + \sigma_k(\Sigma T_w) PZ \end{aligned} \quad (7.25)$$

Again, Eqs. 7.22d-7.22f result in a similar expression for the oil equation. Finally, we employ the following definitions:

$$RWX \equiv \Delta T_w \Delta \Phi_w^k + B_w q_w^n - G(\Phi_w^k - \Phi_w^n) + G(\Phi_o^k - \Phi_o^n)$$

$$ROX \equiv \Delta T_o \Delta \Phi_o^k + B_o q_o^n + G(\Phi_w^k - \Phi_w^n) - G(\Phi_o^k - \Phi_o^n)$$

$$RWY \equiv (G + \sigma_k \Sigma T_w) PX - GRX$$

$$ROY \equiv (G + \sigma_k \Sigma T_o) RX - GPX$$

$$RWZ \equiv (G + \sigma_k \Sigma T_w) PY - GRY$$

$$ROZ \equiv (G + \sigma_k \Sigma T_o) RY - GPY$$

and ultimately obtain

$$x\text{-sweep:} \quad \Delta_x T_{wx} \Delta_x PX - (G + \sigma_k \Sigma T_w) PX + GRX = -RWX \quad (7.26a)$$

$$\Delta_x T_{ox} \Delta_x RX - (G + \sigma_k \Sigma T_o) RX + GPX = -ROX \quad (7.26b)$$

y-sweep:

$$\Delta_y T_{wy} \Delta_y PY - (G + \sigma_k \Sigma T_w) PY + GRY = -RWY \quad (7.27a)$$

$$\Delta_y T_{oy} \Delta_y RY - (G + \sigma_k \Sigma T_o) RY + GPY = -ROY \quad (7.27b)$$

z-sweep:

$$\Delta_z T_{wz} \Delta_z PZ - (G + \sigma_k \Sigma T_w) PZ + GRZ = -RWZ \quad (7.28a)$$

$$\Delta_z T_{oz} \Delta_z RZ - (G + \sigma_k \Sigma T_o) RZ + GPZ = -ROZ \quad (7.28b)$$

for both the water and oil equations. The right-hand sides of Eqs. 7.26-7.28 are residuals. Compare the definitions of RWX and ROX to Eqs. 7.17 and 7.18, and note that these should approach zero when convergence is achieved. Thus, satisfactory normalized closure criteria are

$$\left| \frac{\Sigma B_w q_w - \Sigma RWX}{\Sigma B_w q_w} \right| \leq \epsilon_w \quad (7.29)$$

$$\left| \frac{\Sigma B_o q_o - \Sigma ROX}{\Sigma B_o q_o} \right| \leq \epsilon_o \quad (7.30)$$

where the sums are over all active grid blocks.

Eqs. 7.26-7.28 constitute a bi-tridiagonal system that can be solved by an algorithm given by Douglas, Peaceman, and Rachford.⁴ In applying the procedure, the iteration parameters are varied cyclically until convergence is achieved. To compute the values of G , we must have S' which is updated after every iteration using

$$(S')^{k+1} = \frac{k}{N_i} \left(\frac{S^k - S^n}{P_{cwo}^k - P_{cwo}^n} \right) + (1 - k/N_i)(S')^k \quad (7.31)$$

where k is the iteration level and N_i is the total number of iterations per cycle.

7.2.2 Leap-Frog Technique

Douglas, Peaceman, and Rachford⁴ propose an alternative procedure to that given in Eqs. 7.26-7.28 known as the "leap-frog" technique. Its principal advantage is that substantially less work is required to arrive at a solution. The idea is to eliminate one of the two unknowns in Eqs. 7.11 and 7.12 and thus reduce the computing time. Define two new independent variables: $P \equiv (\Phi_o + \Phi_w)/2$ and $R \equiv (\Phi_o - \Phi_w)/2$ and the functions,

$$M \equiv k \left\{ \frac{k_{ro}}{\mu_o} + \frac{k_{rw}}{\mu_w} \right\}, \quad N \equiv k \left\{ \frac{k_{ro}}{\mu_o} - \frac{k_{rw}}{\mu_w} \right\}.$$

Now add and subtract Eqs. 7.11 and 7.12 using these definitions to get

$$\nabla \cdot M \nabla P + \nabla \cdot N \nabla R + B_w Q_w + B_o Q_o = 0. \quad (7.32)$$

$$\nabla \cdot M \nabla R + \nabla \cdot N \nabla P + B_o Q_o - B_w Q_w = -4 \phi S' \frac{\partial R}{\partial t} \quad (7.33)$$

The finite difference approximations of these equations are

$$\Delta M^m \Delta P^{m+1} + \Delta N^m \Delta R^m + B_w q_w^m + B_o q_o^m = 0 \quad (7.34)$$

$$\Delta M^m \Delta R^{m+1} + \Delta N^m \Delta P^{m+1} + B_o q_o^m - B_w q_w^m = -\frac{4\phi S'}{\Delta t} [R_{ijk}^{m+1} - R_{ijk}^m] \quad (7.35)$$

Eq. 7.34 is used to determine P at the $(m+1)^{th}$ time level using known values of R , N , M from the previous level. Then P^{m+1} is used in Eq. 7.35

$$y: \Delta_y \tilde{T} \Delta_y P Y - \sigma_k(\Sigma \tilde{T}) P Y = R Y \tag{7.41}$$

$$z: \Delta_z \tilde{T} \Delta_z P Z - \sigma_k(\Sigma \tilde{T}) P Z = R Z \tag{7.42}$$

where PX, PY, PZ are defined in Eqs. 7.22a-7.22c and $RX = G - \Delta \tilde{T} \Delta \Phi^k$, $RY = -\sigma_k(\Sigma \tilde{T}) PX$, $RZ = -\sigma_k(\Sigma \tilde{T}) PY$. The steps involved in arriving at Eqs. 7.40-7.42 are identical to those used to get Eqs. 7.26-7.28. At the end of each iteration, we update the potential using $\phi^{k+1} = \phi^k + PZ$. Convergence is achieved when

$$\left| \frac{\Sigma RX}{\Sigma B_w q_w} \right| \leq \epsilon$$

since RX is identical to Eq. 7.39 when $\phi^k \rightarrow \phi^{n+1}$. Since ϕ is the water potential, we also get the oil potential at the new time level using the capillary-pressure relationship in Eq. 7.8. If $P_{cwo} = 0$, then they differ only by the gravity term. The water saturation is computed using the explicit formula

$$S_w^{n+1} = S_w^n + \frac{\Delta t}{V_p} [\Delta T_w \Delta \Phi_w^{n+1} + B_w q_w] \tag{7.43}$$

and the oil saturation is obtained by difference, i.e., $S_o^{n+1} = 1 - S_w^{n+1}$. Conversely, one can use an explicit formula for S_o^{n+1} , and compute S_w^{n+1} by difference. As an alternative to an ADI approach, Eq. 7.39 could be solved by any of the solution techniques discussed in chapter 6. The matrix generated by the left-hand side is banded with each entry a single element, identical in structure to that coming from a single-phase flow problem.

7.3 Three-Phase Flow

The equations for three-phase flow in a black-oil reservoir are given in Eqs. 4.44-4.46. Their finite difference analogs are

$$\Delta T_w (\Delta P_w - \gamma_w \Delta d) + q_w = \frac{V_b}{\Delta t} \Delta_t (b_w S_w \phi) \tag{7.44}$$

$$\Delta T_o (\Delta P_o - \gamma_o \Delta d) + q_o = \frac{V_b}{\Delta t} \Delta_t (b_o S_o \phi) \tag{7.45}$$

مصفوفة راجع المسألة.

to solve for the remaining unknown R^{n+1} . Douglas et al⁴ experienced an approximate four-fold decrease in computing time using this technique.

7.2.3 IMPES Formulation

The implicit pressure-explicit saturation (IMPES) technique and its variations are widely used in commercial reservoir simulators. This method involves eliminating the saturation terms from the flow equations to get an equation that involves only one dependent variable, usually potential or pressure. This is solved implicitly. The saturation is then computed explicitly by referring back to one of the flow equations. The technique was originally proposed by Sheldon et al⁵ and has been extended by Stone and Gardner.⁶

Except in reservoir situations having a high degree of stratification, capillary forces may not have a significant effect on the computed results. The methods discussed thus far always require a non-constant value of capillary pressure because $\Delta_t S$ terms are replaced by $S' \Delta_t P_c$, and for $P_c = \text{constant}$, $S' = \infty$ which, of course, is invalid. Thus, in those cases where the capillary forces have little effect on the results, we cannot set $P_c = 0$ in the models treated in the preceding sections. This means there will always be the additional overhead in computer time required to evaluate S' . One advantage of the IMPES method is that zero capillary pressures can be employed and such simulators generally run faster. We briefly sketch the technique below. More detail is provided when we treat three-phase flow.

Adding Eqs. 7.13 and 7.14 yields

$$\Delta T_w \Delta \Phi_w + \Delta T_o \Delta \Phi_o = -B_w q_w - B_o q_o \equiv -\bar{q} \tag{7.36}$$

We solve Eq. 7.8 for Φ_o and write

$$\Delta \Phi_o = \Delta \Phi_w + \Delta (P_{cwo} + \delta \gamma d) \equiv \Delta \Phi + \Delta R \tag{7.37}$$

which when substituted in Eq. 7.36 gives

$$\Delta \tilde{T} \Delta \Phi + \Delta T_o \Delta R = -\bar{q} \tag{7.38}$$

where $\tilde{T} = T_o + T_w$. The transmissibilities, T_o and T_w , R and \bar{q} are evaluated explicitly, i.e., at time level n . Thus we have the elliptic form

$$\Delta \tilde{T}^n \Delta \Phi^{n+1} = -\Delta T_o^n \Delta R^n - \bar{q}^n \equiv G \tag{7.39}$$

in the single unknown Φ . The Douglas-Rachford ADI formulation yields

$$\tau \cdot \Delta \tilde{T}_r \Delta_r P X - \sigma_k(\Sigma \tilde{T}) P X = R X \tag{7.40}$$

$$\Delta_t [(1 + c_r p_w) b_1 S_1] = (1 + c_r p_w)^{n+1} \Delta_t (b_1 S_1) + (b_1 S_1)^n \Delta_t (1 + c_r p_w)$$

which on further expansion becomes

$$(1 + c_r p_w)^{n+1} [b_1^{n+1} \Delta_t S_1 + S_1^n \Delta_t b_1] + (b_1 S_1)^n \Delta_t (c_r p_w).$$

Now since $b_1 = b_1(p)$ then $\Delta_t b_1 = \Delta_p b_1 \Delta_t p_1$ where

$$\Delta_p b_1 = (b_1^{n+1} - b_1^n) / (p_1^{n+1} - p_1^n) \equiv b_1.$$

Finally, for the right-hand sides of Eqs. 7.44 and 7.45,

$$\begin{aligned} \Delta_t [(1 + c_r p_w) b_1 S_1] &= (1 + c_r p_w)^{n+1} [b_1^{n+1} \Delta_t S_1 + S_1 b_1 \Delta_t p_1] \\ &\quad + (b_1 S_1)^n \Delta_t (c_r p_w), \quad \mathbf{1} = o, w. \quad (7.54) \end{aligned}$$

For the right-hand side of Eq. 7.46 we have

$$\begin{aligned} \Delta_t [(1 + c_r p_w) b_y S_y + (1 + c_r p_w) R_s b_o S_o] &= (1 + c_r p_w)^{n+1} [b_y^{n+1} \Delta_t S_y \\ &\quad + S_y b_y \Delta_t p_y] + (b_y S_y)^n \Delta_t (c_r p_w) + \Delta_t [(1 + c_r p_w) R_s b_o S_o] \\ &= E_1 + (1 + c_r p_w)^{n+1} \Delta_t (R_s b_o S_o) + (R_s b_o S_o)^n \Delta_t (c_r p_w) \end{aligned}$$

where

$$E_1 \equiv (1 + c_r p_w)^{n+1} [b_y^{n+1} \Delta_t S_y + S_y b_y \Delta_t p_y] + (b_y S_y)^n \Delta_t (c_r p_w).$$

Further expansion of the terms having triple factors yields finally

$$\begin{aligned} \Delta_t [(1 + c_r p_w) b_y S_y + (1 + c_r p_w) R_s b_o S_o] \\ = (1 + c_r p_w)^{n+1} [b_y^{n+1} \Delta_t S_y + S_y b_y \Delta_t p_y + R_s^{n+1} b_o^{n+1} \Delta_t (S_o) \\ + b_o^n R_s^{n+1} S_o^n \Delta_t p_o + (b_o S_o)^n R_s \Delta_t p_o] + (b_y S_y + R_s b_o S_o)^n \Delta_t (c_r p_w) \quad (7.55) \end{aligned}$$

where we have used the definitions

$$\Delta_t R_s = \Delta_p R_s \Delta_t p_o \equiv R_s' \Delta_t p_o$$

$$\Delta_t b_y = \Delta_p b_y \Delta_t p_y \equiv b_y' \Delta_t p_y.$$

The final form for the right-hand sides is then

$$\frac{V_p}{\Delta t} \Delta_t [(1 + c_r p_w) b_w S_w] = \alpha_{10} \Delta_t p + \sum_{m=1}^3 \alpha_{1m} \Delta_t S_m \quad (7.56)$$

$$\begin{aligned} \Delta R_s T_o (\Delta p_o - \gamma_o \Delta d) + \Delta T_o (\Delta p_y - \gamma_o \Delta d) + q_g + R_s q_o \\ = \frac{V_o}{\Delta t} \Delta_t (\phi b_y S_y + \phi R_s b_o S_o) \quad (7.46) \end{aligned}$$

where

$$\begin{aligned} b_1 &= 1/B_1 \\ q_1 &= \frac{\Delta x_i \Delta y_j \Delta z_k Q_1}{5.61416} \quad \mathbf{1} = o, w, g \\ V_o &= \frac{\Delta x_i \Delta y_j \Delta z_k}{5.61416}, \quad \text{the bulk volume} \end{aligned}$$

The auxiliary relations are

$$S_o + S_w + S_y = 1 \quad (7.47)$$

$$P_{cwo} = p_o - p_w \quad (7.48)$$

$$P_{cgo} = p_g - p_o \quad (7.49)$$

7.3.1 IMPES Formulation

The development in this section follows that of Coats.⁷ If we assume ϕ and B_1 are pressure dependent then

$$\phi = \phi_o (1 + c_r p_w) \quad (7.50)$$

where c_r is the rock compressibility and ϕ_o is the porosity at a datum, e.g., surface conditions (see chapter 3). The right-hand sides of Eqs. 7.44-7.46 then become

$$\frac{V_p}{\Delta t} \Delta_t [(1 + c_r p_w) b_1 S_1], \quad \mathbf{1} = o, w, \quad (7.51)$$

$$\frac{V_p}{\Delta t} \Delta_t [(1 + c_r p_w) (b_y S_y + R_s B_o S_o)] \quad (\text{for gas}) \quad (7.52)$$

V_p represents the pore volume of a block at the datum condition. As Coats⁷ demonstrates a consistent expansion of Eqs. 7.51-7.52 requires that

$$\Delta_t (a b) = a^{n+1} \Delta_t b + b^n \Delta_t a. \quad (7.53)$$

Applying this definition to the right-hand side expressions we have (omitting $V_p/\Delta t$ for the time being)

Water Equation:
(4.44)

$$\nabla \cdot \left\{ \frac{[k] k_{rw}}{\mu_w B_w} (\nabla P_w - \gamma_w \nabla d) \right\} \pm Q_w = \phi \frac{\partial}{\partial t} \left(\frac{v_w}{B_w} \right)$$

$$\nabla \cdot \left\{ \frac{[k] k_{ro}}{\mu_o B_o} (\nabla P_o - \gamma_o \nabla d) \right\} \pm Q_o = \phi \frac{\partial}{\partial t} \left(\frac{v_o}{B_o} \right)$$

Oil Equation:
(4.45)

$$\nabla \cdot \left\{ \frac{[k] k_{rg} R_s}{\mu_o B_o} (\nabla P_o - \gamma_o \nabla d) \right\} + \nabla \cdot \left\{ \frac{[k] k_{rg}}{\mu_g B_g} (\nabla P_g - \gamma_g \nabla d) \right\} \pm (R_s Q_o + Q_g) = \phi \frac{\partial}{\partial t} \left(\frac{S_g}{B_g} + \frac{S_o R_s}{B_o} \right)$$

Gas Equation:
(4.46)

$$\begin{aligned} & \Delta [T_g(\Delta p_g - \gamma_g \Delta d)] + \Delta [R_i T_o(\Delta p_o - \gamma_o \Delta d)] + q_g + R_i q_o \\ & = \alpha_{30} \Delta_i p + \sum_{m=1}^3 \alpha_{3m} \Delta_i S_m \end{aligned} \quad (7.61)$$

Multiply Eq. 7.59 by β_1 , Eq. 7.60 by β_2 , Eq. 7.61 by β_3 and add to get

$$\begin{aligned} & \beta_1 \Delta [T_w(\Delta p_w - \gamma_w \Delta d)] + \beta_2 \Delta [T_o(\Delta p_o - \gamma_o \Delta d)] + \beta_3 \Delta [T_g(\Delta p_g - \gamma_g \Delta d)] \\ & + \beta_3 \Delta [R_i T_o(\Delta p_o - \gamma_o \Delta d)] + \beta_1 q_w + \beta_2 q_o + \beta_3 (q_g + R_i q_o) \\ & = \beta \Delta_i p + \beta_1 \sum \alpha_{1,m} \Delta_i S_m + \beta_2 \sum \alpha_{2,m} \Delta_i S_m + \beta_3 \sum \alpha_{3,m} \Delta_i S_m \end{aligned} \quad (7.62)$$

where

$$\beta = \beta_1 \alpha_{10} + \beta_2 \alpha_{20} + \beta_3 \alpha_{30}.$$

If we expand the right-hand side of Eq. 7.62 and use $\Delta_i S_g = -\Delta_i S_o - \Delta_i S_w$ (which follows from Eq. 7.42) then we get

$$RHHS = \beta \Delta_i p + (\beta_1 \alpha_{11} - \beta_3 \alpha_{33}) \Delta_i S_w + (\beta_2 \alpha_{22} + \beta_3 \alpha_{32} - \beta_3 \alpha_{33}) \Delta_i S_o. \quad (7.63)$$

The coefficients of $\Delta_i S_w$ and $\Delta_i S_o$ are set to zero. This yields two expressions in the three unknowns β_i , $i = 1, 2, 3$. We select $\beta_1 = 1$ arbitrarily which implies that $\beta_2 = (\alpha_{11}/\alpha_{22})(1 - \alpha_{32}/\alpha_{33})$ and $\beta_3 = \alpha_{11}/\alpha_{33}$. From Eqs. 7.48 and 7.49 we have $\Delta p_o = \Delta p_w + \Delta P_{cwo} = \Delta p_g - \Delta P_{cgo} \equiv \Delta p$ which is used in the left-hand side of Eq. 7.62. Finally,

$$\Delta \tilde{T} \Delta p^{n+1} = \beta \Delta_i p + G \quad (7.64)$$

where $\tilde{T} \equiv T_w + \beta_2 T_o + \beta_3 (T_g + R_i T_o)$, $G \equiv \Delta (T_w \Delta P_{cwo}) - \beta_3 \Delta (T_g \Delta P_{cgo}) - \Delta [\gamma_w T_w + \beta_3 \gamma_o T_o + \gamma_o R_i T_o] \Delta d - \tilde{q}$, $\tilde{q} \equiv q_w + \beta_2 q_o + \beta_3 (q_g + R_i q_o)$ and all terms in G , \tilde{q} and \tilde{T} , are evaluated at time level n . Eq. 7.64 involves a single unknown, namely the pressure in the oil phase, p , which is computed implicitly. We remark that β_2 and β_3 in \tilde{T} are regarded as constants relative to the difference operator Δ when evaluating the left-hand side; i.e., they are taken outside of the operator.

As in the two-phase incompressible case, Eq. 7.64 leads to a banded matrix problem (tridiagonal for 1-D, pentadiagonal for 2-D, and heptadiagonal for 3-D). In fact, this structure is typical of reservoir simulators, even for other formulations. An IMPES formulation is characterized by single elements in each entry of the matrix whereas the simultaneous formulation has entries which are in fact $r \times r$ submatrices (for r unknowns) as will be subsequently seen. Consequently, solution of the matrix problem for an IMPES model entails slightly more work (in coefficient generation) than

$$\frac{V_p}{\Delta t} \Delta_i [(1 + c_r p_w) b_o S_o] = \alpha_{20} \Delta_i p + \sum_{m=1}^3 \alpha_{2m} \Delta_i S_m \quad (7.57)$$

$$\frac{V_p}{\Delta t} \Delta_i [(1 + c_r p_w)(b_g S_g + R_s b_o S_o)] = \alpha_{30} \Delta_i p + \sum_{m=1}^3 \alpha_{3m} \Delta_i S_m \quad (7.58)$$

where $\Delta_i p_g = \Delta_i p_w = \Delta_i p_o \equiv \Delta_i p$ i.e., we've neglected the change in P_c in a time step. In Eqs. 7.56-7.58 the index m takes on values 1, 2, 3 which corresponds to water (w), oil (o), and gas (g), respectively. The values of alpha are defined as follows:

$$\alpha_{10} = \frac{V_p}{\Delta t} [(1 + c_r p_w^{n+1}) S_w^n b_w' + (b_w S_w)^n c_r]$$

$$\alpha_{11} = \frac{V_p}{\Delta t} (1 + c_r p_w^{n+1}) b_w^{n+1}; \alpha_{12} = \alpha_{13} = 0$$

$$\alpha_{20} = \frac{V_p}{\Delta t} [(1 + c_r p_w^{n+1}) S_o^n b_o' + (b_o S_o)^n c_r]$$

$$\alpha_{22} = \frac{V_p}{\Delta t} (1 + c_r p_w^{n+1}) b_o^{n+1}; \alpha_{21} = \alpha_{23} = 0$$

$$\alpha_{30} = \frac{V_p}{\Delta t} [(1 + c_r p_w^{n+1}) (S_g^n b_g' + S_o^n (b_o R_s)') + (b_g S_g + R_s b_o S_o)^n c_r]$$

$$\alpha_{31} = 0$$

$$\alpha_{32} = \frac{V_p}{\Delta t} [(1 + c_r p_w^{n+1}) (R_s b_o)^{n+1}]$$

$$\alpha_{33} = \frac{V_p}{\Delta t} [(1 + c_r p_w^{n+1}) b_g^{n+1}]$$

Thus we have finally for Eqs. 7.44-7.46,

$$\Delta [T_w(\Delta p_w - \gamma_w \Delta d)] + q_w = \alpha_{10} \Delta_i p + \sum_{m=1}^3 \alpha_{1m} \Delta_i S_m \quad (7.59)$$

$$\Delta [T_o(\Delta p_o - \gamma_o \Delta d)] + q_o = \alpha_{20} \Delta_i p + \sum_{m=1}^3 \alpha_{2m} \Delta_i S_m \quad (7.60)$$

After solving for p^{n+1} , the water and oil saturations are computed directly using Eqs. 7.59 and 7.60 in a manner analogous to that outlined for the two-phase flow problem. The gas saturation S_g^{n+1} is computed by difference. S_w^{n+1} and S_o^{n+1} are also used to compute new capillary pressures at time level $n + 1$. This completes the calculations over a time step.

7.3.2 Stability of an IMPES Formulation

For problems involving rapid changes in pressure and saturation (such as well coning) the IMPES formulation becomes unstable for practical time-step sizes. To convey the gist of the time-step limitations, we linearize Eqs. 7.44-7.46 by invoking some simplifying assumptions. A nonlinear stability analysis involves considerable mathematical development⁶ and is not required for our purposes. The assumptions are: (1) flow is incompressible; (2) no phase transfers occur; (3) the transmissibilities are constant and uniform; (4) the gravity effects are negligible. Again, we use a development due to Coats.⁷ Eqs. 7.44-7.46 then become

$$T_w \Delta^2 p_w + B_w q_w = \frac{V_p}{\Delta t} \Delta_t S_w \tag{7.71}$$

$$T_o \Delta^2 p_o + B_o q_o = \frac{V_p}{\Delta t} \Delta_t S_o \tag{7.72}$$

$$T_g \Delta^2 p_g + B_g q_g = \frac{V_p}{\Delta t} \Delta_t S_g \tag{7.73}$$

Using the capillary-pressure relationships given in Eqs. 7.48 and 7.49 we have

$$\Delta p_w = \Delta p_o - P'_{cwo} \Delta S_w$$

$$\Delta p_g = \Delta p_o + P'_{cgo} \Delta S_g$$

where

$$P'_{cwo} = \frac{dP_{cwo}}{dS_w}$$

$$P'_{cgo} = \frac{dP_{cgo}}{dS_g}$$

Assuming that P'_{cwo} and P'_{cgo} are linear for every S_w and S_g we have

that of a single-phase flow simulator. Of course additional solution time is encountered in performing the subsequent saturation computations. Nevertheless, IMPES models are the cheapest to run and provide adequate results in many instances.

If we use the Douglas-Rachford procedure, to solve Eq. 7.64 then the expansions are

$$x\text{-sweep:} \quad \Delta_x \tilde{T}_x \Delta_x p^{k+1/3} + \Delta_y \tilde{T}_y \Delta_y p^k + \Delta_z \tilde{T}_z \Delta_z p^k = \beta(p^{k+1/3} - p^n) + (\Sigma \tilde{T}) \sigma_k (p^{k+1/3} - p^k) + G \tag{7.65}$$

$$y\text{-sweep:} \quad \Delta_x \tilde{T}_x \Delta_x p^{k+1/3} + \Delta_y \tilde{T}_y \Delta_y p^{k+2/3} + \Delta_z \tilde{T}_z \Delta_z p^k = \beta(p^{k+2/3} - p^n) + (\Sigma \tilde{T}) \sigma_k (p^{k+2/3} - p^k) + G \tag{7.66}$$

$$z\text{-sweep:} \quad \Delta_x \tilde{T}_x \Delta_x p^{k+1/3} + \Delta_y \tilde{T}_y \Delta_y p^{k+2/3} + \Delta_z \tilde{T}_z \Delta_z p^{k+1} = \beta(p^{k+1} - p^n) + (\Sigma \tilde{T}) \sigma_k (p^{k+1} - p^k) + G \tag{7.67}$$

where

$$\Sigma \tilde{T} = \sum_{i=1}^3 (T_{i+1/2jk} + T_{i-1/2jk} + T_{i0+1/2k} + T_{i0-1/2k} + T_{i0k+1/2} + T_{i0k-1/2})$$

Using steps similar to those employed before, Eqs. 7.65-7.67 can be put in the form

$$x: \Delta_x \tilde{T}_x \Delta_x PX - (\beta + \sigma_k \Sigma \tilde{T}) PX = -G \tag{7.68}$$

$$y: \Delta_y \tilde{T}_y \Delta_y PY - (\beta + \sigma_k \Sigma \tilde{T}) PY = -(\beta + \sigma_k \Sigma \tilde{T}) PX \tag{7.69}$$

$$z: \Delta_z \tilde{T}_z \Delta_z PZ - (\beta + \sigma_k \Sigma \tilde{T}) PZ = -(\beta + \sigma_k \Sigma \tilde{T}) PY \tag{7.70}$$

where PX , PY and PZ are given in Eqs. 7.22a-7.22c with p substituting for ϕ_w and $\tilde{C} = \Delta \tilde{T} \Delta p - \beta(p^k - p^n) - G$. At the end of each iteration the pressures are updated using $p^{k+1} = p^k + PZ$. Convergence is achieved when

$$\frac{\Sigma |G|}{\Sigma |q_{wi}|} \leq \epsilon$$

$$\gamma_x = 4 \sin^2 \left(\frac{\alpha_x}{2} \right)$$

$$\gamma_y = 4 \sin^2 \left(\frac{\alpha_y}{2} \right)$$

$$\gamma_z = 4 \sin^2 \left(\frac{\alpha_z}{2} \right)$$

Thus,

$$T_1 \Delta^2 \epsilon_m = T_{1x} \Delta^2 \epsilon_m + T_{1y} \Delta^2 \epsilon_m + T_{1z} \Delta^2 \epsilon_m = -\lambda_l \gamma_m \epsilon_m, l = w, o, g; m = 1, 2, 3$$

where $\lambda_l = k_{rl} / \mu_l$ and

$$\gamma_m = \frac{4 k_x \Delta y \Delta z}{\Delta x} \sin^2 \left(\frac{\alpha_{mx}}{2} \right) + \frac{4 k_y \Delta x \Delta z}{\Delta y} \sin^2 \left(\frac{\alpha_{my}}{2} \right) + \frac{4 k_z \Delta x \Delta y}{\Delta z} \sin^2 \left(\frac{\alpha_{mz}}{2} \right).$$

Consequently, the error equations become

$$-\lambda_w \gamma_3 \epsilon_3^{n+1} - \bar{\lambda}_w \gamma_1 \epsilon_1^n = \beta (\epsilon_1^{n+1} - \epsilon_1^n) \quad (7.86)$$

$$-\lambda_o \gamma_3 \epsilon_3^{n+1} = -\beta (\epsilon_1^{n+1} - \epsilon_1^n) - \beta (\epsilon_2^{n+1} - \epsilon_2^n) \quad (7.87)$$

$$-\lambda_g \gamma_3 \epsilon_3^{n+1} - \bar{\lambda}_g \gamma_2 \epsilon_2^n = \beta (\epsilon_2^{n+1} - \epsilon_2^n) \quad (7.88)$$

where $\bar{\lambda}_w = -P'_{cwo} \lambda_w$, $\bar{\lambda}_g = -P'_{cgo} \lambda_g$, and $\beta = V_p / \Delta t$.

Eqs. 7.86-7.88 are then added to get

$$\lambda_w \gamma_3 \epsilon_3^{n+1} + \bar{\lambda}_w \gamma_1 \epsilon_3^n + \lambda_o \gamma_3 \epsilon_3^{n+1} + \lambda_g \gamma_3 \epsilon_3^{n+1} + \bar{\lambda}_g \gamma_2 \epsilon_2^n = 0. \quad (7.89)$$

If we solve Eq. 7.89 for $\gamma_3 \epsilon_3^{n+1}$ and substitute the result in Eqs. 7.86 and 7.87, then

$$\beta \epsilon_1^{n+1} = \left[\beta - \frac{\bar{\lambda}_w}{\lambda} (\lambda_o + \lambda_g) \gamma_1 \right] \epsilon_1^n + \left(\frac{\lambda_w \bar{\lambda}_g}{\lambda} \right) \gamma_2 \epsilon_2^n \quad (7.90)$$

$$\beta \epsilon_2^{n+1} + \beta \epsilon_2^{n+1} = \left[\beta - \frac{\lambda_o \bar{\lambda}_w}{\lambda} \gamma_1 \right] \epsilon_1^n + \left[\beta - \frac{\lambda_o \bar{\lambda}_g}{\lambda} \gamma_2 \right] \epsilon_2^n \quad (7.91)$$

where $\lambda \equiv \lambda_o + \lambda_g + \lambda_w$. These can be written in matrix form as

$$Ae^{n+1} = Be^n \text{ or } e^{n+1} = A^{-1}Be^n = Ce^n \quad (7.92)$$

where

$$\Delta^2 p_w = \Delta^2 p_o - P'_{cwo} \Delta^2 S_w \quad (7.74)$$

$$\Delta^2 p_g = \Delta^2 p_o + P'_{cgo} \Delta^2 S_g \quad (7.75)$$

Substitution of Eqs. 7.74 and 7.75 in Eqs. 7.71 and 7.73 yields

$$T_w (\Delta^2 p - P'_{cwo} \Delta^2 S_w) + B_w q_w = \frac{V_p}{\Delta t} \Delta_t S_w \quad (7.76)$$

$$T_o \Delta^2 p + B_o q_o = \frac{V_p}{\Delta t} \Delta_t S_w - \frac{V_p}{\Delta t} \Delta_t S_g \quad (7.77)$$

$$T_g (\Delta^2 p + P'_{cgo} \Delta^2 S_g) + B_g q_g = \frac{V_p}{\Delta t} \Delta_t S_g \quad (7.78)$$

where $\Delta^2 p_o \equiv \Delta^2 p$.

For the three unknowns S_w , S_g , and p assume errors are introduced with magnitudes ϵ_1 , ϵ_2 , ϵ_3 , respectively, that satisfy Eqs. 7.76-7.78. Consequently,

$$T_w \Delta^2 \epsilon_3^{n+1} + \bar{T}_w \Delta^2 \epsilon_1^n = \frac{V_p}{\Delta t} \Delta_t \epsilon_1 \quad (7.79)$$

$$T_o \Delta^2 \epsilon_3^{n+1} = -\frac{V_p}{\Delta t} \Delta_t \epsilon_1 - \frac{V_p}{\Delta t} \Delta_t \epsilon_2 \quad (7.80)$$

$$T_g \Delta^2 \epsilon_3^{n+1} + \bar{T}_g \Delta^2 \epsilon_2^n = \frac{V_p}{\Delta t} \Delta_t \epsilon_2 \quad (7.81)$$

where $\bar{T}_w \equiv -T_w P'_{cwo}$ and $\bar{T}_g \equiv T_g P'_{cgo}$.

The error component is given by

$$\epsilon_{ijk}^n = \zeta^{n_g J (\alpha_x + j \alpha_y + k \alpha_z)}, J = \sqrt{-1}. \quad (7.82)$$

Therefore,

$$\Delta^2 \epsilon_{ijk}^n = -\gamma_x \epsilon_{ijk}^n \quad (7.83)$$

$$\Delta^2 \epsilon_{ijk}^n = -\gamma_y \epsilon_{ijk}^n \quad (7.84)$$

$$\Delta^2 \epsilon_{ijk}^n = -\gamma_z \epsilon_{ijk}^n \quad (7.85)$$

where

This is the conditional stability restriction caused by explicit treatment of the P_c 's. For regions where water and oil are flowing, we get an identical expression to Eq. 7.96 except μ_o/k_{ro} is replaced by μ_w/k_{rw} , and P_{cgo} by P_{cwo} . In both cases the time-step restrictions caused by explicit transmissibilities are not reflected. For areal calculations $k_z = 0$, and if $\Delta x \approx \Delta y$ and $k_x = k_y$ then we get

$$\Delta t \leq \min \left\{ \frac{\phi \Delta x \Delta y \left(\frac{\mu_o}{k_{ro}} + \frac{\mu_g}{k_{rg}} \right)_{\min}}{4 k P_{cgo}}, \frac{\phi \Delta x \Delta y \left(\frac{\mu_o}{k_{ro}} + \frac{\mu_w}{k_{rw}} \right)_{\min}}{4 k P_{cwo}} \right\} \quad (7.97)$$

The ratio of the maximum time-step size in cross-sectional or three-dimensional studies to the maximum time-step size in two-dimensional studies is then

$$\frac{\Delta t_{3-D}}{\Delta t_{2-D}} = \frac{2 k \Delta z^2}{k_z \Delta x \Delta y} \quad (7.98)$$

This is usually of the order of 0.0010.

The stability restriction imposed by explicit treatment of the transmissibilities is dependent primarily on the fractional flow of each phase. If we assume that the fractional flow of a phase is dependent only on the saturation of that phase, then we define

$$f_i(S_i) = \frac{\lambda_i}{\lambda} \quad i = o, w, g. \quad (7.99)$$

If we assume zero P_c 's and zero compressibility we get the restriction

$$\Delta t \leq \min \left\{ f_i \left(\frac{U_x}{\Delta x} + \frac{U_y}{\Delta y} + \frac{U_z}{\Delta z} \right) \right\} \quad i = o, w, g \quad (7.100)$$

where U_x, U_y, U_z are the total Darcy velocities in each of the three directions and

$$f_i = \frac{df(S_i)}{dS_i} \quad (7.101)$$

The IMPES method with explicit transmissibilities and P_c 's will handle most reservoir problems. However, in problems involving highly unfavourable mobility ratios or rapidly converging flow, some difficulty may be encountered. This can usually be overcome by computing the

$$C = \frac{1}{\beta} \begin{bmatrix} \beta - \gamma_1 \bar{\lambda}_w \left(1 - \frac{\lambda_w}{\lambda} \right) & \frac{\lambda_w \bar{\lambda}_g \gamma_2}{\lambda} \\ \frac{\bar{\lambda}_w \lambda_g \gamma_1}{\lambda} & \frac{\bar{\lambda}_g \gamma_2 (\lambda_o + \lambda_w)}{\lambda} \end{bmatrix}$$

For stability and convergence we require that $|\rho(C)| < 1$ or for any eigenvalue σ , we require $-1 < \sigma < 1$. The characteristic equation is

$$\begin{vmatrix} C_{11} - \sigma & C_{12} \\ C_{21} & C_{22} - \sigma \end{vmatrix} = 0.$$

Thus,

$$\sigma_{1,2} = \frac{-(C_{11} + C_{22}) \pm \sqrt{(C_{11} + C_{22})^2 - 4\psi_c}}{2} \quad (7.93)$$

where $\psi_c \equiv C_{11} C_{22} - C_{12} C_{21}$.

Substituting the definitions of the C 's we get

$$\sigma_{\max} = \frac{1}{2\beta} \left\{ 2\beta - \frac{\gamma_1 \bar{\lambda}_w (\lambda_o + \lambda_w) - \gamma_2 \bar{\lambda}_g (\lambda_o + \lambda_w)}{\lambda} - \frac{1}{\lambda} [\gamma_1 \bar{\lambda}_w (\lambda_o + \lambda_w) - \gamma_2 \bar{\lambda}_g (\lambda_o + \lambda_w)]^2 + 4 \lambda_w \bar{\lambda}_g \bar{\lambda}_w \bar{\lambda}_g \gamma_1 \gamma_2 \right\}^{1/2}. \quad (7.94)$$

This implies that $\sigma_{\max} < 1$. However, for $\sigma_{\max} > -1$, we get the following restriction

$$\Delta t \leq \frac{2V_p}{(2\beta - \alpha)} \quad (7.95)$$

where α represents the last three terms in the curly brackets in Eq. 7.94. Thus we have conditional stability for the IMPES method. For a compressible system, however, this restriction may be relaxed somewhat. The conditional stability arises from explicit treatment of capillary pressure and the transmissibilities.

The restriction given by Eq. 7.95 is not important in areal (x, y) calculations where the grid block dimensions are roughly equal. It becomes important in three-dimensional calculations and cross-sectional calculations where Δz is small relative to Δx and Δy . For regions where gas and oil are flowing, then $\lambda_w = \bar{\lambda}_w = 0$, and the inequality in Eq. 7.95 becomes

$$\Delta t \leq \frac{\phi \Delta x \Delta y \Delta z \left(\frac{\mu_o}{k_{ro}} + \frac{\mu_w}{k_{rw}} \right)_{\min}}{\lambda_w \Delta z + \Delta x \Delta z + \Delta x \Delta y} \quad (7.96)$$

Another technique called the *sequential method* provides for an implicit computation of saturation.¹² The procedure involves two steps. First p^{n+1} is computed from Eq. 7.64 with semi-implicit rates and transmissibilities. This pressure is employed in a reformulation of Eqs. 7.59-7.61, where p^{n+1} is now a known, to compute two of the saturations implicitly. The second step treats flow as incompressible. Moreover, it does not preserve material balances in all phases and various artifices must be employed to correct for this. The work involved is approximately twice that of IMPES. Its principal advantage is more stability than IMPES and less computer effort than a fully implicit procedure.

7.3.3 Fully Implicit Formulation

The fully implicit formulation provides for a simultaneous solution of three unknowns. Its chief advantage is unconditional stability. However, the computer work required to obtain a solution is roughly seven times that of an IMPES model. Nevertheless, fully implicit models are widely used in black-oil systems for coning studies and, at least in one instance, for multiwell simulations.¹³ The auxiliary relationships in Eqs. 7.47-7.49 are used to eliminate three of the six unknowns (p_l and S_l , $l = w, o, g$) in Eqs. 7.44-7.46. Typically the three remaining unknowns are the phase potentials (or pressures) or two saturations and a phase pressure. If three potentials are the unknowns, then nonzero capillary pressure data are required, in particular nonzero slopes of P_c vs. saturation. Should one wish to simulate a situation where they are zero, a fictitious P_c -slope can be introduced which is sufficiently small to yield results essentially identical to the zero-slope case. The chief advantage of this formulation is that the unknowns are smoothly varying in spite of saturation discontinuities. This leads to more rapid convergence when iterative procedures are employed for solving the matrix problem. For direct methods, the requirement of nonzero capillary data is an unnecessary limitation and the alternate formulation is preferred. In this case, the three unknowns are typically p_o , S_w , and S_o . With wider usage of direct methods, the trend today is to use the alternate formulation. For our purpose here, we assume this case and sketch just the essentials.

To distinguish between changes over an iteration and changes over a time step, we employ the operator δx to mean the same as Δ_t , and δ to indicate a change from iteration level k to $k + 1$. Thus, $\delta x = x^{k+1} - x^k$ and $\delta x = x^{k+1} - x^k$. The relationship between the two is $\delta x \approx x^{k+1} - x^k = x^k + \delta x - x^k$. The fully implicit formulation of Eqs. 7.44-7.46 can be written in the compact form

$$\Delta T_1^{n+1} \Delta \Phi_1^{n+1} + q_l^{n+1} + \omega [\Delta (T_o R_o)^{n+1} \Delta \phi_o^{n+1} + (t_o R_o)^{n+1}] \\ = \frac{V_b}{\Delta t} \delta [\phi b_1 S_1 + \omega (\phi b_o R_o S_o)], \quad l = w, o, g \quad (7.107)$$

transmissibilities and/or capillary pressures implicitly or semi-implicitly. Blair and Weinaug⁹ describe a completely implicit approach for a well-bore coning model. MacDonald and Coats¹⁰ and later Nolen and Berry¹¹ give a treatment using semi-implicit approaches to the same problem. The Blair-Weinaug approach greatly relaxes the time-step restrictions but it requires considerable computational work at each time step. Essentially the same results can be obtained with less work by expressing the nonlinear coefficients semi-implicitly. Furthermore, no significant time-step restrictions are realized by this approach. We briefly illustrate the technique with regard to transmissibility.

In this formulation we use a Taylor's series expansion and neglect terms of order two and higher. Thus,

$$T_1^{n+1} \approx T_1^n + \left(\frac{dT_1}{dk_{r1}} \right)^n \left(\frac{dk_{r1}}{dS_1} \right) \Delta S_1 + O(\Delta S_1^2) \quad (7.102)$$

where

$$\Delta S_1 = S_1^{n+1} - S_1^n, \quad (7.103)$$

dT_1/dk_{r1} is evaluated analytically, and dk_{r1}/dS_1 is either evaluated at time level n as done by MacDonald and Coats¹⁰ or as a chord slope (Nolen and Berry¹¹), i.e.,

$$\frac{dk_{r1}}{dS_1} = \frac{k_{r1}(S_1^n + \delta S_1) - k_{r1}(S_1^n)}{\delta S_1} \quad (7.104)$$

where δS_1 is an increment in saturation that is specified arbitrarily. If we substitute Eq. 7.102 in the flow equations for the IMPES method we get terms of the form

$$(T_1^{n+1})_{i+1/2} (\Phi_{i+1}^{n+1} - \Phi_i^{n+1}) = [(T_1^n)_{i+1/2} + \Delta T_1] (\Phi_{i+1}^{n+1} - \Phi_i^{n+1}) \\ = [(T_1^n)_{i+1/2} + \Delta T_1] (\Phi_{i+1}^{n+1} - \Phi_i^n) + [(T_1^n)_{i+1/2} + \Delta T_1] (\Delta \Phi_{i+1} - \Delta \Phi_i) \quad (7.105)$$

$$\approx [(T_1^n)_{i+1/2} + \Delta T_1] (\Phi_{i+1}^n - \Phi_i^n) + (T_1^n)_{i+1/2} (\Delta \Phi_{i+1} - \Delta \Phi_i) \quad (7.106)$$

where $\Delta T_1 = (dT_1/dk_{r1}) / (dk_{r1}/dS_1) \Delta S_1$ and we have neglected terms of the form $\Delta T_1 \Delta \Phi_1$ to give a better linearization. If the transmissibilities are updated at each iteration level, then the transmissibilities at the converged iteration level will approximate those at time level $n + 1$. For most time steps normally used, i.e., $\Delta t \leq 90$ days, this linearization does not produce significant truncation errors. Similar techniques are used for computing P_c^{n+1}

where

$$\omega \equiv \begin{cases} 1, & \mathbf{l} = \mathbf{g} \\ 0, & \mathbf{l} = \mathbf{w}, \mathbf{o} \end{cases}$$

Writing this expression in terms of the δ -operator gives

$$\begin{aligned} \Delta(T_1^k + \delta T_0)\Delta(\Phi_1^k + \delta\Phi_1) + q_1^k + \delta q_1 \\ + \omega[\Delta(T_0 R_x)^k + \delta(T_0 R_x)]\Delta(\Phi_0^k + \delta\Phi_0) + (q_0 R_x)^k + \delta(q_0 R_x) \\ = \frac{V_b}{\Delta t} [(\phi b_1 S_1 + \omega(\phi b_0 R_x S_0))^k + \delta(\phi b_1 S_1 + \omega(\phi b_0 R_x S_0))] \\ - [\phi b_1 S_1 + \omega(\phi b_0 R_x S_0)]^k \quad \mathbf{l} = \mathbf{w}, \mathbf{o}, \mathbf{g}. \end{aligned} \quad (7.108)$$

In the expansion of Eq. 7.108, we drop second order terms of the form $\Delta(\delta x)\Delta(\delta y)$. The residual, at the k^{th} iterate is given by

$$\begin{aligned} R_1^k \equiv \Delta T_1^k \Delta \Phi_1^k + q_1^k + \omega[\Delta(T_0 R_x)^k \Delta \Phi_0^k + (q_0 R_x)^k] \\ - \frac{V_b}{\Delta t} [(\phi b_1 S_1 + \omega(\phi b_0 R_x S_0))^k - [\phi b_1 S_1 + \omega(\phi b_0 R_x S_0)]^k] \\ \quad \mathbf{l} = \mathbf{w}, \mathbf{o}, \mathbf{g}. \end{aligned} \quad (7.109)$$

Consequently, Eq. 7.108 can be written in the residual form

$$\begin{aligned} \Delta(\delta T_1)\Delta\Phi_1^k + \Delta T_1^k \Delta(\delta\Phi_1) + \delta q_1 \\ + \omega[\Delta\delta(T_0 R_x)\Delta\Phi_0^k + \Delta(T_0 R_x)^k \Delta(\delta\Phi_0) + \delta(q_0 R_x)] \\ = \frac{V_b}{\Delta t} \delta[\phi b_1 S_1 + \omega(\phi b_0 R_x S_0)] - R_1^k \\ \quad \mathbf{l} = \mathbf{w}, \mathbf{o}, \mathbf{g}. \end{aligned} \quad (7.110)$$

For a convergent process, $R_1^k \rightarrow 0$ as $k = 1, 2, \dots$ for $\mathbf{l} = \mathbf{w}, \mathbf{o}, \mathbf{g}$. Instead of expressing porosity in terms of p_w as in Eq. 7.50, we express it in terms of p_o which is considered synonymous with reservoir pressure, p . Thus, $\phi = \phi_o(1 + c_r p)$. The right-hand sides of Eq. 7.110 are expanded in terms of the three primary unknowns, δp , δS_w , and δS_g where again a consistent expansion is

$$\delta(ab) = a^{k+1}\delta b + b^k\delta a. \quad (7.111)$$

Each expansion has the form

$$RHS_1 = C_{11}\delta p + C_{12}\delta S_w + C_{13}\delta S_g - R_1^k \quad (7.112)$$

where here index $\mathbf{l} = 1, 2, 3$ corresponding to the water, oil, and gas equations, respectively.

$$C_{11} = \frac{V_b}{\Delta t} S_w^k \{\phi^{k+1} b_w + b_w^k \phi_o c_r\} \quad (7.113)$$

$$C_{12} = \frac{V_b}{\Delta t} \phi^{k+1} [b_w^{k+1} - S_w^k b_w^k P_{cwo}] \quad (7.114)$$

$$C_{13} = 0 \quad (7.115)$$

$$R_w^k = \Delta T_w^k \Delta \Phi_w^k + q_w^k - \frac{V_b}{\Delta t} [(\phi b_w S_w)^k - (\phi b_w S_w)^n] \quad (7.116)$$

For the oil equation:

$$C_{21} = \frac{V_b}{\Delta t} S_o^k \{\phi^{k+1} b_o + b_o^k \phi_o c_r\} \quad (7.117)$$

$$C_{22} = -\frac{V_b}{\Delta t} (\phi b_o)^{k+1} \quad (7.118)$$

$$C_{23} = C_{22} \quad (7.119)$$

$$R_o^k = \Delta T_o^k \Delta \Phi_o^k + q_o^k - \frac{V_b}{\Delta t} [(\phi b_o S_o)^k - (\phi b_o S_o)^n] \quad (7.120)$$

Finally, for the gas equation:

$$C_{31} = \frac{V_b}{\Delta t} [(b_g S_g + b_o R_x S_o)^k \phi_o c_r + \phi^{k+1} [S_g^k b_g^k + S_o^k (R_x^{k+1} b_o^k + b_o^k R_x^k)]] \quad (7.121)$$

$$C_{32} = -\frac{V_b}{\Delta t} (\phi b_o R_x)^{k+1} \quad (7.122)$$

$$C_{33} = \frac{V_b}{\Delta t} \phi^{k+1} [b_g^{k+1} + S_g^k b_g^k P_{cgo}] + C_{32} \quad (7.123)$$

$$\begin{aligned} R_g^k = \Delta T_g^k \Delta \Phi_g^k + \Delta(T_o R_x)^k \Delta \Phi_o^k + (q_g + q_o R_x)^k \\ - \frac{V_b}{\Delta t} [(\phi b_g S_g + \phi b_o R_x S_o)^k - \{\phi b_g S_g + \phi b_o R_x S_o\}^n] \end{aligned} \quad (7.124)$$

In these expressions, b_l^k indicates a derivative with respect to pressure and P_{cwo}^k and P_{cgo}^k denote derivatives with respect to the water and gas saturations, respectively.

If we define a vector, $\mathbf{X} \equiv [\delta p, \delta S_w, \delta S_g]^T$, then the right-hand sides

$$N_o = \Delta(\delta T_o) \Delta \Phi_o^k \tag{7.130}$$

$$N_g = \Delta(\delta T_g) \Delta \Phi_g^k + \Delta[\delta(T_o R_s)] \Delta \Phi_o^k \tag{7.131}$$

Each of the terms in Eqs. 7.126–7.131 are expanded using the second-order correct difference scheme employed in chapter 6. The expansion of the M_1 operators is straight-forward. In the N_1 operators, the δT_1 terms are to be determined and the Φ_1^k values are known. We employ $(\Delta \Phi_1^k)_{m+1} \equiv (\Phi_1^k)_{m+1} - (\Phi_1^k)_m$ and $(\Delta \Phi_1^k)_{m-1} \equiv (\Phi_1^k)_m - (\Phi_1^k)_{m-1}$, $m = i, j, k$ and $\mathbf{l} = w, o, g$. The change in the transmissibilities is given by

$$\delta T_1 = \delta \left\{ \psi \left(\frac{k_{r1} b_1}{\mu_1} \right) \right\} = \frac{\psi}{\mu_1} \delta(k_{r1} b_1) \tag{7.132}$$

where ψ is the time-invariant part and we assume T_1 is little affected by the viscosity change, hence μ_1 is taken outside the δ -operator and is evaluated at the k^{th} iteration level. Expanding further,

$$\delta T_1 = \frac{\psi}{\mu_1} (b_1^{k+1} \delta k_{r1} + k_{r1}^k \delta b_1) \tag{7.133}$$

For water this becomes

$$\delta T_w = \frac{\psi}{\mu_w} (b_w^{k+1} k'_{rw} \delta S_w + k_{rw}^k b_w^k \delta p) \tag{7.134}$$

Frequently, the second term in Eq. 7.134 is neglected since most of the nonlinearity is attributed to the water saturation. This is especially true if b_w changes slightly with pressure. For oil we write

$$\delta T_o = \frac{\psi}{\mu_o} (b_o^{k+1} (k'_{row} \delta S_w + k'_{rop} \delta S_g) + k_o^k b_o^k \delta p) \tag{7.135}$$

Here k'_{row} and k'_{rop} are the derivatives of k_{ro} with respect to the water and gas saturations, respectively. This is consistent with the assumptions regarding k_{ro} cited in chapter 3. Since k_{rw} and k_{rg} are assumed single-valued functions of their respective saturations only, their derivatives are denoted simply by k'_{rw} and k'_{rg} . The change in gas transmissibility is

$$\delta T_g = \frac{\psi}{\mu_g} (b_g^{k+1} k'_{rg} \delta S_g + k_{rg}^k b_g^k \delta p) \tag{7.136}$$

and for the second term in Eq. 7.131 we get

can be represented by CX where C is an accumulation matrix, i.e. $\mathbf{C} = [C_{ij}]$.

The left-hand sides are also expanded in terms of the components of vector X. To indicate how this is done, define two operators M_1 and N_1

$$M_1 \equiv \Delta T_1^k \Delta(\delta \Phi_1^k) + \omega[\Delta(T_o R_s)^k \Delta(\delta \Phi_o)]$$

$$N_1 \equiv \Delta(\delta T_1) \Delta \Phi_1^k + \omega[\Delta \delta(T_o R_s) \Delta \Phi_o^k]$$

such that Eq. 7.110 can be written as

$$M_1 + N_1 + \delta q_1 + \omega \delta(q_o R_s) = RHS_{\mathbf{l}}, \mathbf{l} = w, o, g. \tag{7.125}$$

The operator M_1 is expanded as follows: for water,

$$\begin{aligned} M_w &= \Delta T_w^k \Delta(\delta \Phi_w) \\ &= \Delta T_w^k \Delta[\delta(p - \gamma_w d - P_{cwo})] \\ &= \Delta T_w^k \Delta(\delta p) - \Delta T_w^k \Delta(\gamma_w d) - \Delta T_w^k \Delta(\delta P_{cwo}) \\ &\approx \Delta T_w^k \Delta(\delta p) - \Delta T_w^k \Delta(P_{cwo} \delta S_w). \end{aligned} \tag{7.126}$$

To arrive at Eq. 7.126, the term $\delta(\gamma_w d)$ is dropped since it is normally negligible over an iteration. For oil,

$$\begin{aligned} M_o &= \Delta T_o^k \Delta(\delta \Phi_o) \\ &= \Delta T_o^k \Delta[\delta(p - \gamma_o d)] \\ &\approx \Delta T_o^k \Delta(\delta p) \end{aligned} \tag{7.127}$$

where again the change in the gravity term is neglected. Finally, for gas we have

$$\begin{aligned} M_g &= \Delta T_g^k \Delta(\delta \Phi_g) + \Delta(T_o R_s)^k \Delta(\delta \Phi_o) \\ &= \Delta T_g^k \Delta[\delta(p - \gamma_g d + P_{cgo})] + \Delta(T_o R_s)^k \Delta[\delta(p - \gamma_o d)] \\ &\approx \Delta T_g^k \Delta(\delta p) + \Delta T_g^k \Delta(\delta P_{cgo}) + \Delta(T_o R_s)^k \Delta(\delta p) \\ &\approx \Delta(T_g + T_o R_s)^k \Delta(\delta p) + \Delta T_g^k \Delta(P'_{cgo} \delta S_g). \end{aligned} \tag{7.128}$$

The N_1 operators are

$$N_w = \Delta(\delta T_w) \Delta \Phi_w^k \tag{7.129}$$

where

$$D_{21} = WI\lambda_o \equiv PI_o \tag{7.143}$$

$$D_{22} = WI \frac{b_o}{\mu_o} k_{row}(p - p_{wf}) \tag{7.144}$$

$$D_{23} = WI \frac{b_o}{\mu_o} k_{rog}(p - p_{wf}) \tag{7.145}$$

In a similar manner we evaluate $\delta(q_{ij})$, $\delta(q_w)$, and $\delta(R_i q_o)$ to generate a set of D -coefficients which makes up the elements of a *well matrix*, D . The actual form these coefficients take depends upon the specifications on a well. This is treated in more detail in the next chapter.

In matrix form Eq. 7.125 becomes

$$TX + DX = CX + R \tag{7.146}$$

or simply

$$AX = R \tag{7.147}$$

where $A \equiv T + D - C$ and R is a vector containing the k^{th} iteration level residuals. Matrix A is a band matrix identical in structure to those previously discussed. However, each entry (e.g., the x 's in Figs. 6.6 and 6.7) is a 3×3 submatrix corresponding to each active grid block in the reservoir. The first, second and third rows within a submatrix contain the coefficients for δp , δS_w , and δS_g (in that order) that come from the water, oil, and gas equations, respectively. If one or more of these phases is not flowing in a particular grid block then some of the elements within the corresponding submatrix are zero. However, the main diagonal of A is always nonzero since it contains the D - and C -coefficients and darcian flow terms from T . If we consider a two-phase flow problem, then A consists of 2×2 submatrices arranged in band form. Obviously single phase flow and the matrix problem in Eq. 7.64 constitute the degenerate case where the submatrices reduce to a single element.

The matrix problem in Eq. 7.147 can be solved by direct Gaussian elimination, SIP or the block SOR methods discussed in chapter 6. For Gaussian elimination, alternate diagonal ordering is still applicable. For 3-D problems, it can be extended to alternate diagonal planes.¹⁴ For large problems, this approach loses its advantage, and an iterative procedure is advisable to minimize storage. One can use LSOR or 2LSOR or, alternatively, the multiphase flow algorithm for SIP.¹⁵ Solution of Eq. 7.147 produces the change in the primary variables over an iteration (the *outer itera-*

(7.137)

$$\delta(T_o R_i) = R_i^{k+1} \delta T_o + T_o^k \delta R_i \tag{7.138}$$

$$= \frac{\psi}{\mu_o} \{ (R_i b_o)^{k+1} (k_{row} \delta S_w + k_{rog} \delta S_g) + [R_i^{k+1} k_{ro} b_o^k + (k_{ro} b_o)^k R_i^k] \delta p \}$$

In both the M_1 and N_1 operators, terms involving k_{ri} , its derivatives, and frequently R_i , are weighted up-stream. For example, upstream weighting of Eq. 7.134 produces

$$(\delta T_w)_{in=1/2} = \frac{\psi}{\mu_w} \{ \bar{b}_w^{k+1} k_{rw} \delta S_w + k_{rw}^k \bar{b}_w^k \delta p \}_u \tag{7.139}$$

for $m = i, j$, or k . Subscript u denotes the upstream block while the bars indicate arithmetic averages between the up- and downstream blocks. All derivatives in Eqs. 7.126-7.138 are evaluated at the current iteration level, i.e. the k^{th} . It is also usually sufficient to evaluate the $(k + 1)^{st}$ -level terms at the k^{th} level since they are all pressure-dependent and change slowly from iteration to iteration.

Clearly, the sum $M_1 + N_1$ produces a set of algebraic equations in terms of the primary unknowns. In matrix form these can be written as TX where T contains all the components from M_1 and N_1 that contribute to darcian flow. We refer to T as the *transmissibility matrix* where $T \equiv [T_{ij}]$.

The well terms in Eq. 7.125 are also expanded in terms of the primary unknowns. For example, for phase l ,

$$q_l = WI(p - p_{wf})\lambda_l; \quad \lambda_l \equiv \frac{k_{rl} b_l}{\mu_l} \tag{7.140}$$

where WI is a wellbore index (in reality a shape factor) and p_{wf} is the flowing bottomhole pressure. The product of WI and λ_l is referred to as the *productivity index (PI)* for producing wells and the *injectivity index (II)* for injection wells. Since WI is independent of time, a change in q_l over an iteration is due to a shift in p , p_{wf} or λ_l (or all three). Thus, in general,

$$\delta(Zq_l) = Z^{k+1} \delta(q_l) + q_l^k \delta Z$$

$$= Z^{k+1} \left\{ \frac{\partial q_l}{\partial p} \delta p + \frac{\partial q_l}{\partial S_w} \delta S_w + \frac{\partial q_l}{\partial S_g} \delta S_g + \frac{\partial q_l}{\partial p_{wf}} \delta p_{wf} \right\} + q_l^k \delta Z. \tag{7.141}$$

To illustrate, consider an oil producing well with a fixed flowing bottomhole pressure, then $Z = 1$ and

$$\delta(q_o) = D_{21} \delta p + D_{22} \delta S_w + D_{23} \delta S_g \tag{7.142}$$

tion if an iterative technique is used to solve the matrix problem). Usually 2 to 4 iterations are required to complete the time step.

7.3.4 The Adaptive Implicit Method

There are obvious advantages to both an IMPES simulator and a fully implicit one. The former provides adequate results at minimum cost in many instances. However, it can be inadequate for problems where rapid surges in the dependent variables occur. These problems generally require an implicit calculation for those variables experiencing such surges to maintain stability. A fully implicit simulator provides such capability. However, the additional computer expense is substantial. Furthermore, time truncation errors are generally higher in implicit simulators.¹⁶ It is clear that rapid changes in pressure and saturation don't occur over an entire reservoir simultaneously. Rather this is a local phenomenon usually restricted to the proximity of wells or aquifer-reservoir boundaries. For example, consider three wells penetrating the same reservoir which are sufficiently distant that interference effects are minimal. Suppose one well is producing at such a high rate so as to locally distort the gas-oil and water-oil contacts. Another, producing at moderate rate, is perforated at a lower elevation in the oil zone causing a distortion only in the water-oil contact. The third, a low rate well, has a producing interval far removed from the fluid contacts. The first well will normally require an implicit computation of p , S_w , S_g in those grid blocks affected by the fluid contact distortions. In the blocks surrounding the second well, an implicit calculation of p and S_w would be sufficient. And in the third, one need treat only p implicitly. Moreover, in the intervening blocks, an implicit calculation of p (an IMPES model) would probably be sufficient.

A fully implicit model can adequately handle the situation described above. Clearly, however, there would be a large amount of overkill in many of the grid blocks, i.e., more implicitness is supplied than is required. On the other hand, an IMPES model would lead to underkill and instabilities for practical time step sizes in those grid blocks requiring more implicitness. Unfortunately, simulators offering a fixed level of implicitness do not recognize that only a small fraction of the total number of grid blocks experience sufficiently large surges in pressures and/or saturation to justify implicit treatment. When it is needed, it may not be required in all phases, or for long periods of time. Moreover, those cells requiring implicit calculations will change as the simulation advances.

Consideration of this problem has led to the recent development of the adaptive implicit method (AIM).^{17,18} Because some confusion exists in the technical literature with respect to the exact meaning of *fully implicit*,¹⁶

or *strongly coupled*,¹³ we prefer to speak of *degrees of implicitness*.† We identify degrees of implicitness with the number of dependent variables, computed implicitly. Thus an IMPES calculation is first degree and a fully implicit calculation is third degree. With AIM, various degrees of implicitness are invoked regionally or individually cell-by-cell, i.e. the solution is advanced with adjacent cells having different degrees of implicitness. As the calculations proceed, the degrees of implicitness locally and dynamically shift as needed—all automatically. The idea is to apply the right amount of implicitness where and when needed and only as long as needed. The advantages of this approach are obvious. First, the user need not make a choice regarding an IMPES model vs. a fully implicit one. There is also no problem of over/under-kill. Thus, in our hypothetical three well reservoir, third, second, and first degree calculations would be performed locally around the first, second, and third wells, respectively, while the intervening blocks are first degree. Finally, one achieves an optimum with regard to the stability-truncation error-cost question.

In the adaptive implicit method, the order of matrix A in Eq. 7.147 is not fixed but is permitted to change with time. The order at any moment is determined by the total number of variables to be computed implicitly. The shift in order can occur within a time step since Eq. 7.147 is 'based on changes over an iteration. The shift is accomplished by performing the *equivalent* of elementary row operations in A and then setting certain coefficients to zero.