3 OPTIMAL LINEAR STATE FEEDBACK CONTROL SYSTEMS

3.1 INTRODUCTION

In Chapter 2 we gave an exposition of the problems of linear control theory. In this chapter we begin to build a theory that can be used to solve the problems outlined in Chapter 2. The main restriction of this chapter is that we assume that the complete state $x(t)$ of the plant can be accurately measured at all times and is available for feedback. Although this is an unrealistic assumption for many practical control systems, the theory of this chapter will prove to be an important foundation for the more general case where we do not assume that $x(t)$ is completely accessible.

Much attention of this chapter is focused upon regulator problems, that is, problems where the goal is to maintain the state of the system at a desired value. We shall see that linear control theory provides powerful tools for solving such problems. Both the deterministic and the stochastic versions of the optimal linear regulator problem are studied in detail. Important extensions of the regulator problem—the nonzero set point regulator and the optimal linear tracking problem—also receive considerable attention.

Other topics dealt with are the numerical solution of Riccati equations, asymptotic properties of optimal control laws, and the sensitivity of linear optimal state feedback systems.

3.2 STABILITY IMPROVEMENT OF LINEAR SYSTEMS BY STATE FEEDBACK

3.2.1 Linear State Feedback Control

In Chapter 2 we saw that an important aspect of feedback system design is the stability of the control system. Whatever we want to achieve with the control system, its stability must be assured. Sometimes the main goal of a feedback design is actually to stabilize a system if it is initially unstable, or to improve its stability if transient phenomena do not die out sufficiently fast.

The purpose of this section is to investigate how the stability properties of linear systems can be improved by state feedback.

Consider the linear time-varying system with state differential equation

$$\dot{x}(t) = A(t)x(t) + B(t)u(t). \tag{3-1}$$

If we suppose that the complete state can be accurately measured at all times, it is possible to implement a *linear control law* of the form

$$u(t) = -F(t)x(t) + u'(t), \tag{3-2}$$

where $F(t)$ is a time-varying *feedback gain matrix* and $u'(t)$ a new input. If this control law is connected to the system **3-1**, the closed-loop system is described by the state differential equation

$$\dot{x}(t) = [A(t) - B(t)F(t)]x(t) + B(t)u'(t). \tag{3-3}$$

The stability of this system depends of course on the behavior of $A(t)$ and $B(t)$ but also on that of the gain matrix $F(t)$. It is convenient to introduce the following terminology.

Definition 3.1. *The linear control law*

$$u(t) = -F(t)x(t) + u'(t) \tag{3-4}$$

is called an ***asymptotically stable control law*** *for the system*

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{3-5}$$

if the closed-loop system

$$\dot{x}(t) = [A(t) - B(t)F(t)]x(t) + B(t)u'(t) \tag{3-6}$$

is asymptotically stable.

If the system **3-5** is *time-invariant*, and we choose a constant matrix F, the stability of the control law **3-4** is determined by the characteristic values of the matrix $A - BF$. In the next section we find that under a mildly restrictive condition (namely, the system must be completely controllable), all closed-loop characteristic values can be arbitrarily located in the complex plane by choosing F suitably (with the restriction of course that complex poles occur in complex conjugate pairs). If all the closed-loop poles are placed in the left-half plane, the system is of course asymptotically stable.

We also see in the next section that for single-input systems, that is, systems with a scalar input u, usually a unique gain matrix F is found for a given set of closed-loop poles. Melsa (1970) lists a FORTRAN computer program to determine this matrix. In the multiinput case, however, a given set of poles can usually be achieved with many different choices of F.

Example 3.1. *Stabilization of the inverted pendulum*

The state differential equation of the inverted pendulum positioning system of Example 1.1 (Section 1.2.3) is given by

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{F}{M} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{g}{L'} & 0 & \frac{g}{L'} & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ \frac{1}{M} \\ 0 \\ 0 \end{pmatrix} \mu(t). \qquad \text{3-7}$$

Let us consider the time-invariant control law

$$\mu(t) = -(\phi_1, \phi_2, \phi_3, \phi_4)x(t). \qquad \text{3-8}$$

It follows that for the system **3-7** and control law **3-8** we have

$$A - BF = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{\phi_1}{M} & -\frac{F+\phi_2}{M} & -\frac{\phi_3}{M} & -\frac{\phi_4}{M} \\ 0 & 0 & 0 & 1 \\ -\frac{g}{L'} & 0 & \frac{g}{L'} & 0 \end{pmatrix}. \qquad \text{3-9}$$

The characteristic polynomial of this matrix is

$$s^4 + s^3\frac{F+\phi_2}{M} + s^2\left(\frac{\phi_1}{M} - \frac{g}{L'}\right) - s\frac{F+\phi_2+\phi_4}{M}\frac{g}{L'} - \frac{\phi_1+\phi_3}{M}\frac{g}{L'}. \qquad \text{3-10}$$

Now suppose that we wish to assign all closed-loop poles to the location $-\alpha$. Then the closed-loop characteristic polynomial should be given by

$$(s+\alpha)^4 = s^4 + 4\alpha s^3 + 6\alpha^2 s^2 + 4\alpha^3 s + \alpha^4. \qquad \text{3-11}$$

Equating the coefficients of **3-10** and **3-11**, we find the following equations in ϕ_1, ϕ_2, ϕ_3, and ϕ_4:

$$\begin{aligned} \frac{F+\phi_2}{M} &= 4\alpha, \\ \frac{\phi_1}{M} - \frac{g}{L'} &= 6\alpha^2, \\ -\frac{F+\phi_2+\phi_4}{M}\frac{g}{L'} &= 4\alpha^3, \\ -\frac{\phi_1+\phi_3}{M}\frac{g}{L'} &= \alpha^4. \end{aligned} \qquad \text{3-12}$$

With the numerical values of Example 1.1 and with $\alpha = 3\ s^{-1}$, we find from these linear equations the following control law:

$$\mu(t) = -(65.65, 11.00, -72.60, -21.27)x(t). \tag{3-13}$$

Example 3.2. *Stirred tank*

The stirred tank of Example 1.2 (Section 1.2.3) is an example of a multi-input system. With the numerical values of Example 1.2, the linearized state differential equation of the system is

$$\dot{x}(t) = \begin{pmatrix} -0.01 & 0 \\ 0 & -0.02 \end{pmatrix} x(t) + \begin{pmatrix} 1 & 1 \\ -0.25 & 0.75 \end{pmatrix} u(t). \tag{3-14}$$

Let us consider the time-invariant control law

$$u(t) = -\begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} x(t). \tag{3-15}$$

It follows from **3-14** and **3-15** that the closed-loop characteristic polynomial is given by

$$\det(sI - A + BF) = s^2 + s(0.03 + \phi_{11} - 0.25\phi_{12} + \phi_{21} + 0.75\phi_{22})$$
$$+ (0.0002 + 0.02\phi_{11} - 0.0025\phi_{12} + 0.02\phi_{21} + 0.0075\phi_{22} + \phi_{11}\phi_{22} - \phi_{12}\phi_{21}). \tag{3-16}$$

We can see at a glance that a given closed-loop characteristic polynomial can be achieved for many different values of the gain factors ϕ_{ij}. For example, the three following feedback gain matrices

$$F_a = \begin{pmatrix} 1.1 & 3.7 \\ 0 & 0 \end{pmatrix}, \quad F_b = \begin{pmatrix} 0 & 0 \\ 1.1 & -1.2333 \end{pmatrix} \quad \text{and} \quad F_c = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix} \tag{3-17}$$

all yield the closed-loop characteristic polynomial $s^2 + 0.2050s + 0.01295$, so that the closed-loop characteristic values are $-0.1025 \pm j0.04944$. We note that in the control law corresponding to the first gain matrix the second component of the input is not used, the second feedback matrix leaves the first component untouched, while in the third control law both inputs control the system.

In Fig. 3.1 are sketched the responses of the three corresponding closed-loop systems to the initial conditions

$$\xi_1(0) = 0\ \text{m}^3, \qquad \xi_2(0) = 0.1\ \text{kmol/m}^3. \tag{3-18}$$

Note that even though the closed-loop poles are the same the differences in the three responses are very marked.

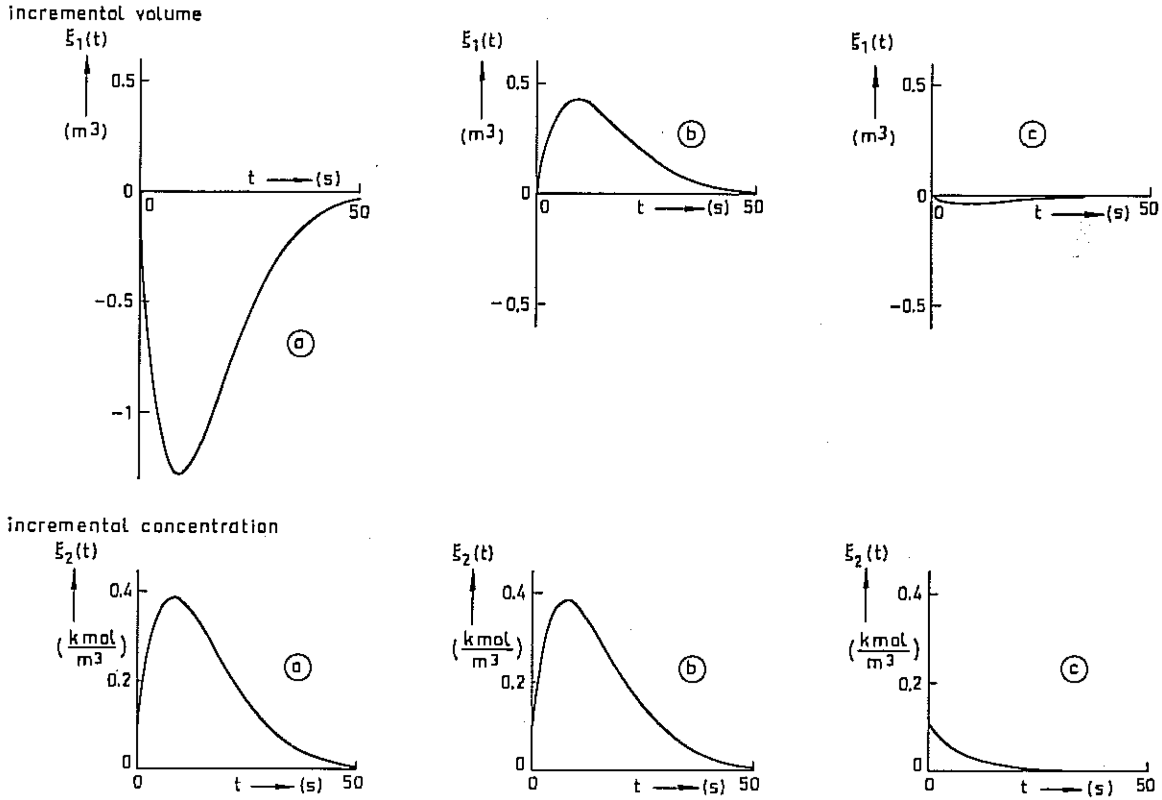


Fig. 3.1. Closed-loop responses of the stirred tank to the initial conditions $\xi_1(0) = 0\ m^3$, $\xi_2(0) = 0.1$ kmol/m^3 for the feedback gain matrices (*a*) F_a; (*b*) F_b; (*c*) F_c.

3.2.2* Conditions for Pole Assignment and Stabilization

In this section we state precisely (1) under what conditions the closed-loop poles of a time-invariant linear system can be arbitrarily assigned to any location in the complex plane by linear state feedback, and (2) under what conditions the system can be stabilized. First, we have the following result.

Theorem 3.1. *Consider the linear time-invariant system*

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{3-19}$$

with the time-invariant control law

$$u(t) = -Fx(t) + u'(t). \tag{3-20}$$

*Then the closed-loop characteristic values, that is, the characteristic values of $A - BF$, can be arbitrarily located in the complex plane (with the restriction that complex characteristic values occur in complex conjugate pairs) by choosing F suitably if and only if the system **3-19** is completely controllable.*

A complete proof of this theorem is given by Wonham (1967a), Davison (1968b), Chen (1968b), and Heymann (1968). Wolovich (1968) considers the time-varying case. We restrict our proof to single-input systems. Suppose that the system with the state differential equation

$$\dot{x}(t) = Ax(t) + b\mu(t), \tag{3-21}$$

where $\mu(t)$ is a scalar input, is completely controllable. Then we know from Section 1.9 that there exists a state transformation $x'(t) = T^{-1}x(t)$, where T is a nonsingular transformation matrix, which transforms the system **3-19** into its phase-variable canonical form:

$$\dot{x}'(t) = \begin{pmatrix} 0 & 1 & 0 & \cdots\cdots & 0 \\ 0 & 0 & 1 \quad 0 & \cdots\cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & 1 \\ -\alpha_0 & -\alpha_1 & \cdots & \cdots & -\alpha_{n-1} \end{pmatrix} x'(t) + \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 1 \end{pmatrix} \mu(t). \tag{3-22}$$

Here the numbers α_i, $i = 0, 1, \cdots, n - 1$ are the coefficients of the characteristic polynomial of the system **3-21**, that is, $\det(sI - A) = s^n + \alpha_{n-1}s^{n-1} \cdots + \alpha_1 s + \alpha_0$. Let us write **3-22** more compactly as

$$\dot{x}'(t) = A'x'(t) + b'\mu(t). \tag{3-23}$$

Consider now the linear control law

$$\mu(t) = -f'x'(t) + \mu'(t), \tag{3-24}$$

where f' is the row vector

$$f' = (\phi_1, \phi_2, \cdots, \phi_n). \tag{3-25}$$

If this control law is connected to the system, the closed-loop system is described by the state differential equation

$$\dot{x}'(t) = (A' - b'f')x'(t) + b'\mu'(t). \tag{3-26}$$

It is easily seen that the matrix $A' - b'f$ is given by

$$A' - b'f' = \begin{pmatrix} 0 & 1 & 0 \cdots\cdots\cdots\cdots 0 \\ 0 & 0 & 1 \quad 0 \cdots\cdots\cdots 0 \\ \cdots & \cdots & \cdots\cdots\cdots\cdots\cdots \\ 0 \cdots\cdots & \cdots\cdots 0 & 1 \\ -\alpha_0 - \phi_1 & -\alpha_1 - \phi_2 \cdots\cdots & -\alpha_{n-1} - \phi_n \end{pmatrix}. \tag{3-27}$$

This clearly shows that the characteristic polynomial of the matrix $A' - b'f'$ has the coefficients $(\alpha_i + \phi_{i+1})$, $i = 0, 1, \cdots, n - 1$. Since the ϕ_i, $i = 1, 2, \cdots, n$, are arbitrarily chosen real numbers, the coefficients of the closed-loop characteristic polynomial can be given any desired values, which means that the closed-loop poles can be assigned to arbitrary locations in the complex plane (provided complex poles occur in complex conjugate pairs).

Once the feedback law in terms of the transformed state variable has been chosen, it can immediately be expressed in terms of the original state variable $x(t)$ as follows:

$$\mu(t) = -f'x'(t) + \mu'(t) = -f'T^{-1}x(t) + \mu'(t) = -fx(t) + \mu'(t). \tag{3-28}$$

This proves that if **3-19** is completely controllable, the closed-loop characteristic values may be arbitrarily assigned. For the proof of the converse of this statement, see the end of the proof of Theorem 3.2. Since the proof for multiinput systems is somewhat more involved we omit it. As we have seen in Example 3.2, for multiinput systems there usually are many solutions for the feedback gain matrix F for a given set of closed-loop characteristic values.

Through Theorem 3.1 it is always possible to stabilize a completely controllable system by state feedback, or to improve its stability, by assigning the closed-loop poles to locations in the left-half complex plane. The theorem gives no guidance, however, as to where in the left-half complex plane the closed-loop poles should be located. Even more uncertainty occurs in the multiinput case where the same closed-loop pole configuration can be achieved by various control laws. This uncertainty is removed by optimal linear regulator theory, which is discussed in the remainder of this chapter.

Theorem 3.1 implies that it is always possible to stabilize a completely controllable linear system. Suppose, however, that we are confronted with a time-invariant system that is not completely controllable. From the discussion of stabilizability in Section 1.6.4, it can be shown that stabilizability, as the name expresses, is precisely the condition that allows us to stabilize a not completely controllable time-invariant system by a time-invariant linear control law (Wonham, 1967a):

Theorem 3.2. *Consider the linear time-invariant system*

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad \textbf{3-29}$$

with the time-invariant control law

$$u(t) = -Fx(t) + u'(t). \qquad \textbf{3-30}$$

Then it is possible to find a constant matrix F such that the closed-loop system is asymptotically stable if and only if the system **3-29** *is stabilizable.*

The proof of this theorem is quite simple. From Theorem 1.26 (Section 1.6.3), we know that the system can be transformed into the controllability canonical form

$$\dot{x}'(t) = \begin{pmatrix} A'_{11} & A'_{12} \\ 0 & A'_{22} \end{pmatrix} x'(t) + \begin{pmatrix} B'_1 \\ 0 \end{pmatrix} u(t), \qquad \textbf{3-31}$$

where the pair $\{A'_{11}, B'_1\}$ is completely controllable. Consider the linear control law

$$u(t) = -(F'_1, \quad F'_2)x'(t) + u'(t). \qquad \textbf{3-32}$$

For the closed-loop system we find

$$\begin{aligned} \dot{x}'(t) &= \begin{pmatrix} A'_{11} & A'_{12} \\ 0 & A'_{22} \end{pmatrix} x'(t) - \begin{pmatrix} B'_1 \\ 0 \end{pmatrix} (F'_1, \quad F'_2)x'(t) + \begin{pmatrix} B'_1 \\ 0 \end{pmatrix} u'(t) \\ &= \begin{pmatrix} A'_{11} - B'_1F'_1 & A'_{12} - B'_1F'_2 \\ 0 & A'_{22} \end{pmatrix} x'(t) + \begin{pmatrix} B'_1 \\ 0 \end{pmatrix} u'(t). \end{aligned} \qquad \textbf{3-33}$$

The characteristic values of the compound matrix in this expression are the characteristic values of $A'_{11} - B'_1F'_1$ together with those of A'_{22}. Now if the system **3-29** is stabilizable, A'_{22} is asymptotically stable, and since the pair $\{A'_{11}, B'_1\}$ is completely controllable, it is always possible to find an F'_1 such that $A'_{11} - B'_1F'_1$ is stable. This proves that if **3-29** is stabilizable it is always possible to find a feedback law that stabilizes the system. Conversely, if one can find a feedback law that stabilizes the system, A'_{22} must be asymptotically stable, hence the system is stabilizable. This proves the other direction of the theorem.

The proof of the theorem shows that, if the system is stabilizable but not completely controllable, only some of the closed-loop poles can be arbitrarily located since the characteristic values of A'_{22} are not affected by the control law. This proves one direction of Theorem 3.1.

3.3 THE DETERMINISTIC LINEAR OPTIMAL REGULATOR PROBLEM

3.3.1 Introduction

In Section 3.2 we saw that under a certain condition (complete controllability) a time-invariant linear system can always be stabilized by a linear feedback law. In fact, more can be done. Because the closed-loop poles can be located anywhere in the complex plane, the system can be stabilized; but, moreover, by choosing the closed-loop poles far to the left in the complex plane, the convergence to the zero state can be made arbitrarily fast. To make the system move fast, however, large input amplitudes are required. In any practical problem the input amplitudes must be bounded; this imposes a limit on the distance over which the closed-loop poles can be moved to the left. These considerations lead quite naturally to the formulation of an optimization problem, where we take into account both the speed of convergence of the state to zero and the magnitude of the input amplitudes.

To introduce this optimization problem, we temporarily divert our attention from the question of the pole locations, to return to it in Section 3.8.

Consider the linear time-varying system with state differential equation

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \qquad \textbf{3-34}$$

and let us study the problem of bringing this system from an arbitrary initial state to the zero state as quickly as possible (in Section 3.7 we consider the case where the desired state is not the zero state). There are many criteria that express how fast an initial state is reduced to the zero state; a very useful one is the quadratic integral criterion

$$\int_{t_0}^{t_1} x^T(t)R_1(t)x(t)\,dt. \qquad \textbf{3-35}$$

Here $R_1(t)$ is a nonnegative-definite symmetric matrix. The quantity $x^T(t)R_1(t)x(t)$ is a measure of the extent to which the state at time t deviates from the zero state; the weighting matrix $R_1(t)$ determines how much weight is attached to each of the components of the state. The integral **3-35** is a criterion for the cumulative deviation of $x(t)$ from the zero state during the interval $[t_0, t_1]$.

As we saw in Chapter 2, in many control problems it is possible to identify a controlled variable $z(t)$. In the linear models we employ, we usually have

$$z(t) = D(t)x(t). \tag{3-36}$$

If the actual problem is to reduce the controlled variable $z(t)$ to zero as fast as possible, the criterion **3-35** can be modified to

$$\int_{t_0}^{t_1} z^T(t)R_3(t)z(t)\, dt, \tag{3-37}$$

where $R_3(t)$ is a positive-definite symmetric weighting matrix. It is easily seen that **3-37** is equivalent to **3-35**, since with **3-36** we can write

$$\int_{t_0}^{t_1} z^T(t)R_3(t)z(t)\, dt = \int_{t_0}^{t_1} x^T(t)R_1(t)x(t)\, dt, \tag{3-38}$$

where

$$R_1(t) = D^T(t)R_3(t)D(t). \tag{3-39}$$

If we now attempt to find an optimal input to the system by minimizing the quantity **3-35** or **3-37**, we generally run into the difficulty that indefinitely large input amplitudes result. To prevent this we include the input in the criterion; we thus consider

$$\int_{t_0}^{t_1} [z^T(t)R_3(t)z(t) + u^T(t)R_2(t)u(t)]\, dt, \tag{3-40}$$

where $R_2(t)$ is a positive-definite symmetric weighting matrix. The inclusion of the second term in the criterion reduces the input amplitudes if we attempt to make the total value of **3-40** as small as possible. The relative importance of the two terms in the criterion is determined by the matrices R_3 and R_2.

If it is very important that the terminal state $x(t_1)$ is as close as possible to the zero state, it is sometimes useful to extend **3-40** with a third term as follows

$$\int_{t_0}^{t_1} [z^T(t)R_3(t)z(t) + u^T(t)R_2(t)u(t)]\, dt + x^T(t_1)P_1x(t_1), \tag{3-41}$$

where P_1 is a nonnegative-definite symmetric matrix.

We are now in a position to introduce the deterministic linear optimal regulator problem:

Definition 3.2. *Consider the linear time-varying system*

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \tag{3-42}$$

where

$$x(t_0) = x_0, \tag{3-43}$$

with the controlled variable

$$z(t) = D(t)x(t). \tag{3-44}$$

Consider also the criterion

$$\int_{t_0}^{t_1} [z^T(t)R_3(t)z(t) + u^T(t)R_2(t)u(t)]\, dt + x^T(t_1)P_1x(t_1), \tag{3-45}$$

where P_1 is a nonnegative-definite symmetric matrix and $R_3(t)$ and $R_2(t)$ are positive-definite symmetric matrices for $t_0 \leq t \leq t_1$. Then the problem of determining an input $u^0(t)$, $t_0 \leq t \leq t_1$, for which the criterion is minimal is called the ***deterministic linear optimal regulator problem.***

Throughout this chapter, and indeed throughout this book, it is understood that $A(t)$ is a continuous function of t and that $B(t)$, $D(t)$, $R_3(t)$, and $R_2(t)$ are piecewise continuous functions of t, and that all these matrix functions are bounded.

A special case of the regulator problem is the time-invariant regulator problem:

Definition 3.3. *If all matrices occurring in the formulation of the deterministic linear optimal regulator problem are constant, we refer to it as the* ***time-invariant deterministic linear optimal regulator problem.***

We continue this section with a further discussion of the formulation of the regulator problem. First, we note that in the regulator problem, as it stands in Definition 3.2, we consider only the *transient* situation where an arbitrary initial state must be reduced to the zero state. The problem formulation does not include disturbances or a reference variable that should be tracked; these more complicated situations are discussed in Section 3.6.

A difficulty of considerable interest is how to choose the weighting matrices R_3, R_2, and P_1 in the criterion **3-45**. This must be done in the following manner. Usually it is possible to define three quantities, the *integrated square regulating error*, the *integrated square input*, and the *weighted square terminal error*. The integrated square regulating error is given by

$$\int_{t_0}^{t_1} z^T(t)W_e(t)z(t)\, dt, \tag{3-46}$$

where $W_e(t)$, $t_0 \leq t \leq t_1$, is a weighting matrix such that $z^T(t)W_e(t)z(t)$ is properly dimensioned and has physical significance. We discussed the selection of such weighting matrices in Chapter 2. Furthermore, the integrated square input is given by

$$\int_{t_0}^{t_1} u^T(t)W_u(t)u(t)\, dt, \tag{3-47}$$

where the weighting matrix $W_u(t)$, $t_0 \leq t \leq t_1$, is similarly selected. Finally, the weighted square terminal error is given by

$$x^T(t_1)W_tx(t_1), \tag{3-48}$$

where also W_t is a suitable weighting matrix. We now consider various problems, such as:

1. Minimize the integrated square regulating error with the integrated square input and the weighted square terminal error constrained to certain maximal values.
2. Minimize the weighted square terminal error with the integrated square input and the integrated square regulating error constrained to certain maximal values.
3. Minimize the integrated square input with the integrated square regulating error and the weighted square terminal error constrained to certain maximal values.

All these versions of the problem can be studied by considering the minimization of the criterion

$$\rho_1 \int_{t_0}^{t_1} z^T(t) W_e(t) z(t)\, dt + \rho_2 \int_{t_0}^{t_1} u^T(t) W_u(t) u(t)\, dt + \rho_3 x^T(t_1) W_t x(t_1), \qquad \textbf{3-49}$$

where the constants ρ_1, ρ_2, and ρ_3 are suitably chosen. The expression **3-45** is exactly of this form. Let us, for example, consider the important case where the terminal error is unimportant and where we wish to minimize the integrated square regulating error with the integrated square input constrained to a certain maximal value. Since the terminal error is of no concern, we set $\rho_3 = 0$. Since we are minimizing the integrated square regulating error, we take $\rho_1 = 1$. We thus consider the minimization of the quantity

$$\int_{t_0}^{t_1} [z^T(t) W_e(t) z(t) + \rho_2 u^T(t) W_u(t) u(t)]\, dt. \qquad \textbf{3-50}$$

The scalar ρ_2 now plays the role of a Lagrange multiplier. To determine the appropriate value of ρ_2, we solve the problem for many different values of ρ_2. This provides us with a graph as indicated in Fig. 3.2, where the integrated square regulating error is plotted versus the integrated square input with ρ_2 as a parameter. As ρ_2 decreases, the integrated square regulating error decreases but the integrated square input increases. From this plot we can determine the value of ρ_2 that gives a sufficiently small regulating error without excessively large inputs.

From the same plot we can solve the problem where we must minimize the integrated square input with a constrained integrated square regulating error. Other versions of the problem formulation can be solved in a similar manner. We thus see that the regulator problem, as formulated in Definition 3.2, is quite versatile and can be adapted to various purposes.

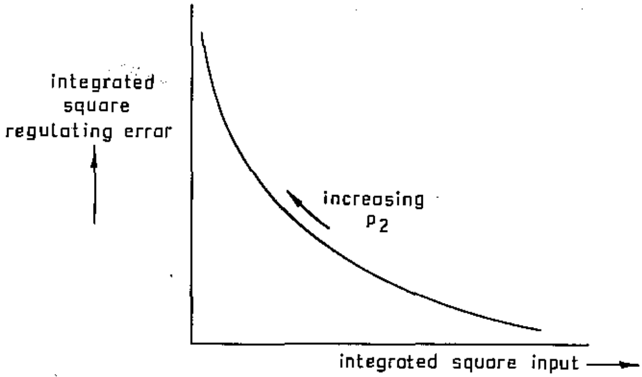


Fig. 3.2. Integrated square regulating error versus integrated square input, with $\rho_1 = 1$ and $\rho_2 = 0$.

We see in later sections that the solution of the regulator problem can be given in the form of a linear control law which has several useful properties. This makes the study of the regulator problem an interesting and practical proposition.

Example 3.3. *Angular velocity stabilization problem*

As a first example, we consider an angular velocity stabilization problem. The plant consists of a dc motor the shaft of which has the angular velocity $\xi(t)$ and which is driven by the input voltage $\mu(t)$. The system is described by the scalar state differential equation

$$\dot{\xi}(t) = -\alpha\xi(t) + \kappa\mu(t), \qquad \textbf{3-51}$$

where α and κ are given constants. We consider the problem of stabilizing the angular velocity $\xi(t)$ at a desired value ω_0. In the formulation of the general regulator problem we have chosen the origin of state space as the equilibrium point. Since in the present problem the desired equilibrium position is $\xi(t) \equiv \omega_0$, we shift the origin. Let μ_0 be the constant input voltage to which ω_0 corresponds as the steady-state angular velocity. Then μ_0 and ω_0 are related by

$$0 = -\alpha\omega_0 + \kappa\mu_0. \qquad \textbf{3-52}$$

Introduce now the new state variable

$$\xi'(t) = \xi(t) - \omega_0. \qquad \textbf{3-53}$$

Then with the aid of **3-52**, it follows from **3-51** that $\xi'(t)$ satisfies the state differential equation

$$\dot{\xi}'(t) = -\alpha\xi'(t) + \kappa\mu'(t), \qquad \textbf{3-54}$$

where

$$\mu'(t) = \mu(t) - \mu_0. \tag{3-55}$$

This shows that the problem of bringing the system **3-51** from an arbitrary initial state $\xi(t_0) = \omega_1$ to the state $\xi = \omega_0$ is equivalent to bringing the system **3-51** from the initial state $\xi(t_0) = \omega_1 - \omega_0$ to the equilibrium state $\xi = 0$. Thus, without restricting the generality of the example, we consider the problem of regulating the system **3-51** about the zero state. The controlled variable ζ in this problem obviously is the state ξ:

$$\zeta(t) = \xi(t). \tag{3-56}$$

As the optimization criterion, we choose

$$\int_{t_0}^{t_1} [\zeta^2(t) + \rho\mu^2(t)]\, dt + \pi_1 \xi^2(t_1), \tag{3-57}$$

with $\rho > 0$, $\pi_1 \geq 0$. This criterion ensures that the deviations of $\xi(t)$ from zero are restricted [or, equivalently, that $\xi(t)$ stays close to ω_0], that $\mu(t)$ does not assume too large values [or, equivalently, $\mu(t)$ does not deviate too much from μ_0], and that the terminal state $\xi(t_1)$ will be close to zero [or, equivalently, that $\xi(t_1)$ will be close to ω_0]. The values of ρ and π_1 must be determined by trial and error. For α and κ we use the following numerical values:

$$\begin{aligned} \alpha &= 0.5\ \mathrm{s}^{-1}, \\ \kappa &= 150\ \mathrm{rad/(V\ s^2)}. \end{aligned} \tag{3-58}$$

Example 3.4. *Position control*

In Example 2.4 (Section 2.3), we discussed position control by a dc motor. The system is described by the state differential equation

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ 0 & -\alpha \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ \kappa \end{pmatrix} \mu(t), \tag{3-59}$$

where $x(t)$ has as components the angular position $\xi_1(t)$ and the angular velocity $\xi_2(t)$ and where the input variable $\mu(t)$ is the input voltage to the dc amplifier that drives the motor. We suppose that it is desired to bring the angular position to a constant value ξ_{10}. As in the preceding example, we make a shift in the origin of the state space to obtain a standard regulator problem. Let us define the new state variable $x'(t)$ with components

$$\begin{aligned} \xi_1'(t) &= \xi_1(t) - \xi_{10}, \\ \xi_2'(t) &= \xi_2(t). \end{aligned} \tag{3-60}$$

A simple substitution shows that $x'(t)$ satisfies the state differential equation

$$\dot{x}'(t) = \begin{pmatrix} 0 & 1 \\ 0 & -\alpha \end{pmatrix} x'(t) + \begin{pmatrix} 0 \\ \kappa \end{pmatrix} \mu(t). \tag{3-61}$$

Note that in contrast to the preceding example we need not define a new input variable. This results from the fact that the angular position can be maintained at any constant value with a zero input. Since the system **3-61** is identical to **3-59**, we omit the primes and consider the problem of regulating **3-59** about the zero state.

For the controlled variable we choose the angular position:

$$\zeta(t) = \xi_1(t) = (1, 0)x(t). \tag{3-62}$$

An appropriate optimization criterion is

$$\int_{t_0}^{t_1} [\zeta^2(t) + \rho\mu^2(t)]\, dt. \tag{3-63}$$

The positive scalar weighting coefficient ρ determines the relative importance of each term of the integrand. The following numerical values are used for α and κ:

$$\begin{aligned} \alpha &= 4.6 \text{ s}^{-1}, \\ \kappa &= 0.787 \text{ rad/(V s}^2). \end{aligned} \tag{3-64}$$

3.3.2 Solution of the Regulator Problem

In this section we solve the deterministic optimal regulator problem using elementary methods of the calculus of variations. It is convenient to rewrite the criterion **3-45** in the form

$$\int_{t_0}^{t_1} [x^T(t)R_1(t)x(t) + u^T(t)R_2(t)u(t)]\, dt + x^T(t_1)P_1x(t_1), \tag{3-65}$$

where $R_1(t)$ is the nonnegative-definite symmetric matrix

$$R_1(t) = D^T(t)R_3(t)D(t). \tag{3-66}$$

Suppose that the input that minimizes this criterion exists and let it be denoted by $u^0(t)$, $t_0 \le t \le t_1$. Consider now the input

$$u(t) = u^0(t) + \varepsilon\tilde{u}(t), \qquad t_0 \le t \le t_1, \tag{3-67}$$

where $\tilde{u}(t)$ is an arbitrary function of time and ε is an arbitrary number. We shall check how this change in the input affects the criterion **3-65**. Owing to the change in the input, the state will change, say from $x^0(t)$ (the optimal behavior) to

$$x(t) = x^0(t) + \varepsilon\tilde{x}(t), \qquad t_0 \le t \le t_1. \tag{3-68}$$

This defines $\tilde{x}(t)$, which we now determine. The solution $x(t)$ as given by **3-68** must satisfy the state differential equation **3-42** with $u(t)$ chosen according to **3-67**. This yields

$$\dot{x}^0(t) + \varepsilon\dot{\tilde{x}}(t) = A(t)x^0(t) + \varepsilon A(t)\tilde{x}(t) + B(t)u^0(t) + \varepsilon B(t)\tilde{u}(t). \qquad \textbf{3-69}$$

Since the optimal solution must also satisfy the state differential equation, we have

$$\dot{x}^0(t) = A(t)x^0(t) + B(t)u^0(t). \qquad \textbf{3-70}$$

Subtraction of **3-69** and **3-70** and cancellation of ε yields

$$\dot{\tilde{x}}(t) = A(t)\tilde{x}(t) + B(t)\tilde{u}(t). \qquad \textbf{3-71}$$

Since the initial state does not change if the input changes from $u^0(t)$ to $u^0(t) + \varepsilon\tilde{u}(t)$, $t_0 \le t \le t_1$, we have $\tilde{x}(t_0) = 0$, and the solution of **3-71** using **1-61** can be written as

$$\tilde{x}(t) = \int_{t_0}^{t} \Phi(t, \tau)B(\tau)\tilde{u}(\tau)\, d\tau, \qquad \textbf{3-72}$$

where $\Phi(t, t_0)$ is the transition matrix of the system **3-71**. We note that $\tilde{x}(t)$ does not depend upon ε. We now consider the criterion **3-65**. With **3-67** and **3-68** we can write

$$\begin{aligned}
&\int_{t_0}^{t_1} [x^T(t)R_1(t)x(t) + u^T(t)R_2(t)u(t)]\, dt + x^T(t_1)P_1x(t_1) \\
&\quad= \int_{t_0}^{t_1} [x^{0T}(t)R_1(t)x^0(t) + u^{0T}(t)R_2(t)u^0(t)]\, dt + x^{0T}(t_1)P_1x^0(t_1) \\
&\qquad + 2\varepsilon\left\{\int_{t_0}^{t_1} [\tilde{x}^T(t)R_1(t)x^0(t) + \tilde{u}^T(t)R_2(t)u^0(t)]\, dt + \tilde{x}^T(t_1)P_1x^0(t_1)\right\} \\
&\qquad + \varepsilon^2\left\{\int_{t_0}^{t_1} [\tilde{x}^T(t)R_1(t)\tilde{x}(t) + \tilde{u}^T(t)R_2(t)\tilde{u}(t)]\, dt + \tilde{x}^T(t_1)P_1\tilde{x}(t_1)\right\}. \qquad \textbf{3-73}
\end{aligned}$$

Since $u^0(t)$ is the optimal input, changing the input from $u^0(t)$ to the input **3-67** can only increase the value of the criterion. This implies that, as a function of ε, **3-73** must have a minimum at $\varepsilon = 0$. Since **3-73** is a quadratic expression in ε, it can assume a minimum for $\varepsilon = 0$ only if its first derivative with respect to ε is zero at $\varepsilon = 0$. Thus we must have

$$\int_{t_0}^{t_1} [\tilde{x}^T(t)R_1(t)x^0(t) + \tilde{u}^T(t)R_2(t)u^0(t)]\, dt + \tilde{x}^T(t_1)P_1x^0(t_1) = 0. \qquad \textbf{3-74}$$

Substitution of **3-72** into **3-74** yields after an interchange of the order of integration and a change of variables

$$\int_{t_0}^{t_1} \tilde{u}^T(t)\left\{B^T(t)\int_{t}^{t_1} \Phi^T(\tau, t)R_1(\tau)x^0(\tau)\,d\tau + R_2(t)u^0(t) + B^T(t)\Phi^T(t_1, t)P_1x^0(t_1)\right\} dt = 0. \qquad \text{3-75}$$

Let us now abbreviate,

$$p(t) = \int_{t}^{t_1} \Phi^T(\tau, t)R_1(\tau)x^0(\tau)\,d\tau + \Phi^T(t_1, t)P_1x^0(t_1). \qquad \text{3-76}$$

With this abbreviation **3-75** can be written more compactly as

$$\int_{t_0}^{t_1} \tilde{u}^T(t)\{B^T(t)p(t) + R_2(t)u^0(t)\}\,dt = 0. \qquad \text{3-77}$$

This can be true for every $\tilde{u}(t)$, $t_0 \leq t \leq t_1$, only if

$$B^T(t)p(t) + R_2(t)u^0(t) = 0, \qquad t_0 \leq t \leq t_1. \qquad \text{3-78}$$

By the assumption that $R_2(t)$ is nonsingular for $t_0 \leq t \leq t_1$, we can write

$$u^0(t) = -R_2^{-1}(t)B^T(t)p(t), \qquad t_0 \leq t \leq t_1. \qquad \text{3-79}$$

If $p(t)$ were known, this relation would give us the optimal input at time t.

We convert the relation **3-76** for $p(t)$ into a differential equation. First, we see by setting $t = t_1$ that

$$p(t_1) = P_1x^0(t_1). \qquad \text{3-80}$$

By differentiating **3-76** with respect to t, we find

$$\dot{p}(t) = -R_1(t)x^0(t) - A^T(t)p(t), \qquad \text{3-81}$$

where we have employed the relationship [Theorem 1.2(d), Section 1.3.1]

$$\frac{d}{dt}\Phi^T(t_0, t) = -A^T(t)\Phi^T(t_0, t). \qquad \text{3-82}$$

We are now in a position to state the *variational equations*. Substitution of **3-79** into the state differential equation yields

$$\dot{x}^0(t) = A(t)x^0(t) - B(t)R_2^{-1}(t)B^T(t)p(t). \qquad \text{3-83}$$

Together with **3-81** this forms a set of $2n$ simultaneous linear differential equations in the n components of $x^0(t)$ and the n components of $p(t)$. We term $p(t)$ the *adjoint variable*. The $2n$ boundary conditions for the differential equations are

$$x^0(t_0) = x_0 \qquad \text{3-84}$$

and

$$p(t_1) = P_1x^0(t_1). \qquad \text{3-85}$$

We see that the boundary conditions hold at opposite ends of the interval $[t_0, t_1]$, which means that we are faced with a two-point boundary value problem. To solve this boundary value problem, let us write the simultaneous differential equations **3-83** and **3-81** in the form

$$\begin{pmatrix} \dot{x}^0(t) \\ \dot{p}(t) \end{pmatrix} = \begin{pmatrix} A(t) & -B(t)R_2^{-1}(t)B^T(t) \\ -R_1(t) & -A^T(t) \end{pmatrix} \begin{pmatrix} x^0(t) \\ p(t) \end{pmatrix}. \qquad \textbf{3-86}$$

Consider this the state differential equation of an $2n$-dimensional linear system with the transition matrix $\Theta(t, t_0)$. We partition this transition matrix corresponding to **3-86** as

$$\Theta(t, t_0) = \begin{pmatrix} \Theta_{11}(t, t_0) & \Theta_{12}(t, t_0) \\ \Theta_{21}(t, t_0) & \Theta_{22}(t, t_0) \end{pmatrix}. \qquad \textbf{3-87}$$

With this partitioning we can express the state at an intermediate time t in terms of the state and adjoint variable at the terminal time t_1 as follows:

$$x^0(t) = \Theta_{11}(t, t_1)x^0(t_1) + \Theta_{12}(t, t_1)p(t_1). \qquad \textbf{3-88}$$

With the terminal condition **3-85**, it follows

$$x^0(t) = [\Theta_{11}(t, t_1) + \Theta_{12}(t, t_1)P_1]x^0(t_1). \qquad \textbf{3-89}$$

Similarly, we can write for the adjoint variable

$$\begin{aligned} p(t) &= \Theta_{21}(t, t_1)x^0(t_1) + \Theta_{22}(t, t_1)p(t_1) \\ &= [\Theta_{21}(t, t_1) + \Theta_{22}(t, t_1)P_1]x^0(t_1). \end{aligned} \qquad \textbf{3-90}$$

Elimination of $x^0(t_1)$ from **3-89** and **3-90** yields

$$p(t) = [\Theta_{21}(t, t_1) + \Theta_{22}(t, t_1)P_1][\Theta_{11}(t, t_1) + \Theta_{12}(t, t_1)P_1]^{-1}x^0(t). \qquad \textbf{3-91}$$

The expression **3-91** shows that there exists a linear relation between $p(t)$ and $x^0(t)$ as follows

$$p(t) = P(t)x^0(t), \qquad \textbf{3-92}$$

where

$$P(t) = [\Theta_{21}(t, t_1) + \Theta_{22}(t, t_1)P_1][\Theta_{11}(t, t_1) + \Theta_{12}(t, t_1)P_1]^{-1}. \qquad \textbf{3-93}$$

With **3-79** we obtain for the optimal input to the system

$$u^0(t) = -F(t)x^0(t), \qquad \textbf{3-94}$$

where

$$F(t) = R_2^{-1}(t)B^T(t)P(t). \qquad \textbf{3-95}$$

This is the solution of the regulator problem, which has been derived under the assumption that an optimal solution exists. We summarize our findings as follows.

Theorem 3.3. *Consider the deterministic linear optimal regulator problem. Then the optimal input can be generated through a linear control law of the form*

$$u^0(t) = -F(t)x^0(t), \tag{3-96}$$

where

$$F(t) = R_2^{-1}(t)B^T(t)P(t). \tag{3-97}$$

The matrix $P(t)$ is given by

$$P(t) = [\Theta_{21}(t, t_1) + \Theta_{22}(t, t_1)P_1][\Theta_{11}(t, t_1) + \Theta_{12}(t, t_1)P_1]^{-1}, \tag{3-98}$$

where $\Theta_{11}(t, t_0)$, $\Theta_{12}(t, t_0)$, $\Theta_{21}(t, t_0)$, and $\Theta_{22}(t, t_0)$ are obtained by partitioning the transition matrix $\Theta(t, t_0)$ of the state differential equation

$$\begin{pmatrix} \dot{x}(t) \\ \dot{p}(t) \end{pmatrix} = \begin{pmatrix} A(t) & -B(t)R_2^{-1}(t)B^T(t) \\ -R_1(t) & -A^T(t) \end{pmatrix} \begin{pmatrix} x(t) \\ p(t) \end{pmatrix}, \tag{3-99}$$

where

$$R_1(t) = D^T(t)R_3(t)D(t). \tag{3-100}$$

This theorem gives us the solution of the regulator problem in the form of a *linear control law*. The control law automatically generates the optimal input for *any* initial state. A block diagram interpretation is given in Fig. 3.3 which very clearly illustrates the closed-loop nature of the solution.

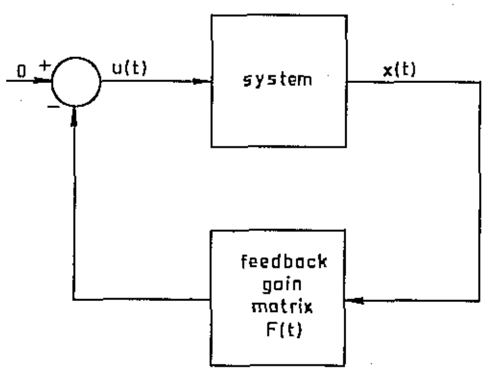


Fig. 3.3. The feedback structure of the optimal linear regulator.

The formulation of the regulator problem as given in Definition 3.2 of course does not impose this closed-loop form of the solution. We can just as easily derive an open-loop representation of the solution. At time t_0 the expression **3-89** reduces to

$$x_0 = [\Theta_{11}(t_0, t_1) + \Theta_{12}(t_0, t_1)P_1]x^0(t_1). \tag{3-101}$$

Solving **3-101** for $x^0(t_1)$ and substituting the result into **3-90**, we obtain

$$p(t) = [\Theta_{21}(t, t_1) + \Theta_{22}(t, t_1)P_1][\Theta_{11}(t_0, t_1) + \Theta_{12}(t_0, t_1)P_1]^{-1}x_0. \quad \textbf{3-102}$$

This gives us from **3-79**

$$u^0(t) = -R_2^{-1}(t)B^T(t)[\Theta_{21}(t, t_1) + \Theta_{22}(t, t_1)P_1][\Theta_{11}(t_0, t_1) + \Theta_{12}(t_0, t_1)P_1]^{-1}x_0,$$
$$t_0 \le t \le t_1. \quad \textbf{3-103}$$

For a given x_0 this yields the prescribed behavior of the input. The corresponding behavior of the state follows by substituting $x(t_1)$ as obtained from **3-101** into **3-89**:

$$x^0(t) = [\Theta_{11}(t, t_1) + \Theta_{12}(t, t_1)P_1][\Theta_{11}(t_0, t_1) + \Theta_{12}(t_0, t_1)P_1]^{-1}x_0. \quad \textbf{3-104}$$

In view of what we learned in Chapter 2 about the many advantages of closed-loop control, for practical implementation we prefer of course the closed-loop form of the solution **3-96** to the open-loop form **3-103**. In Section 3.6, where we deal with the stochastic regulator problem, it is seen that state feedback is not only preferable but in fact imperative.

Example 3.5. *Angular velocity stabilization*

The angular velocity stabilization problem of Example 3.3 (Section 3.3.1) is the simplest possible nontrivial application of the theory of this section. The combined state and adjoint variable equations **3-99** are now given by

$$\begin{pmatrix} \dot{\xi}(t) \\ \dot{\pi}(t) \end{pmatrix} = \begin{pmatrix} -\alpha & -\dfrac{\kappa^2}{\rho} \\ -1 & \alpha \end{pmatrix} \begin{pmatrix} \xi(t) \\ \pi(t) \end{pmatrix}. \quad \textbf{3-105}$$

The transition matrix corresponding to this system of differential equations can be found to be

$$\Theta(t, t_0) = \begin{pmatrix} \dfrac{\gamma - \alpha}{2\gamma} e^{\gamma(t-t_0)} + \dfrac{\gamma + \alpha}{2\gamma} e^{-\gamma(t-t_0)} & -\dfrac{\kappa^2}{2\rho\gamma}[e^{\gamma(t-t_0)} - e^{-\gamma(t-t_0)}] \\ -\dfrac{1}{2\gamma}[e^{\gamma(t-t_0)} - e^{-\gamma(t-t_0)}] & \dfrac{\gamma + \alpha}{2\gamma} e^{\gamma(t-t_0)} + \dfrac{\gamma - \alpha}{2\gamma} e^{-\gamma(t-t_0)} \end{pmatrix}, \quad \textbf{3-106}$$

where

$$\gamma = \sqrt{\alpha^2 + \frac{\kappa^2}{\rho}}. \quad \textbf{3-107}$$

To simplify the notation we write the transition matrix as

$$\Theta(t, t_0) = \begin{pmatrix} \theta_{11}(t, t_0) & \theta_{12}(t, t_0) \\ \theta_{21}(t, t_0) & \theta_{22}(t, t_0) \end{pmatrix}. \quad \textbf{3-108}$$

It follows from **3-103** and **3-104** that in open-loop form the optimal input and state are given by

$$\mu^0(t) = -\frac{\kappa}{\rho}\frac{\theta_{21}(t, t_1) + \theta_{22}(t, t_1)\pi_1}{\theta_{11}(t_0, t_1) + \theta_{12}(t_0, t_1)\pi_1}\xi_0, \qquad \textbf{3-109}$$

$$\xi^0(t) = \frac{\theta_{11}(t, t_1) + \theta_{12}(t, t_1)\pi_1}{\theta_{11}(t_0, t_1) + \theta_{12}(t_0, t_1)\pi_1}\xi_0. \qquad \textbf{3-110}$$

Figure 3.4 shows the optimal trajectories and the behavior of the optimal input for different values of the weighting factor ρ. The following numerical values have been used:

$$\begin{aligned} \alpha &= 0.5\ \text{s}^{-1}, \\ \kappa &= 150\ \text{rad/(V s}^2), \\ t_0 &= 0\ \text{s}, \qquad t_1 = 1\ \text{s}. \end{aligned} \qquad \textbf{3-111}$$

The weighting coefficient π_1 has in this case been set to zero. The figure clearly shows that as ρ decreases the input amplitude grows, whereas the settling time becomes smaller.

Figure 3.5 depicts the influence of the weighting coefficient π_1; the factor ρ is kept constant. It is seen that as π_1 increases the terminal state tends to be closer to the zero state at the expense of a slightly larger input amplitude toward the end of the interval.

Suppose now that it is known that the deviations in the initial state are usually not larger than ± 100 rad/s and that the input amplitudes should be limited to ± 3 V. Then we see from the figures that a suitable choice for ρ is about 1000. The value of π_1 affects the behavior only near the terminal time.

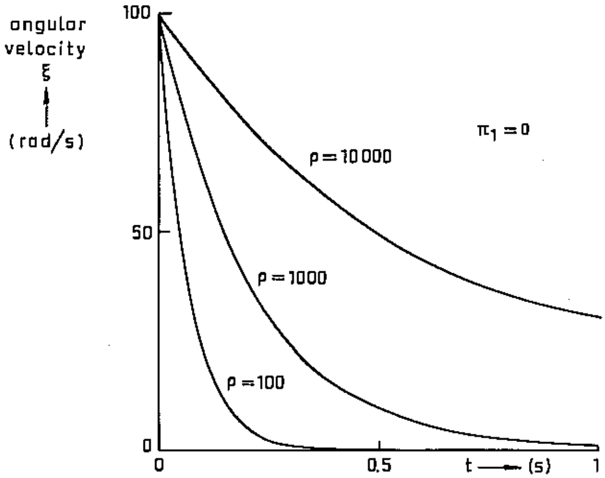
Let us now consider the feedback form of the solution. It follows from Theorem 3.3 that the optimal trajectories of Figs. 3.4 and 3.5 can be generated by the control law

$$\mu^0(t) = -F(t)\xi(t), \qquad \textbf{3-112}$$

where the time-varying scalar gain $F(t)$ is given by

$$F(t) = \frac{\kappa}{\rho}\frac{\theta_{21}(t, t_1) + \theta_{22}(t, t_1)\pi_1}{\theta_{11}(t, t_1) + \theta_{12}(t, t_1)\pi_1}. \qquad \textbf{3-113}$$

Figure 3.6 shows the behavior of the gain $F(t)$ corresponding to the various numerical values used in Figs. 3.4 and 3.5. Figure 3.6 exhibits quite clearly that in most cases the gain factor $F(t)$ is constant during almost the whole interval $[t_0, t_1]$. Only near the end do deviations occur. We also see that $\pi_1 = 0.19$ gives a constant gain factor over the entire interval. Such a gain factor would be very desirable from a practical point of view since the implementation of a time-varying gain is complicated and costly. Comparison



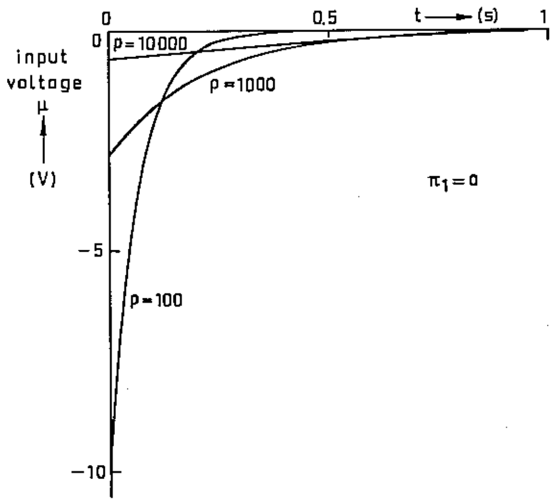


Fig. 3.4. The behavior of state and input for the angular velocity stabilization problem for different values of ρ.

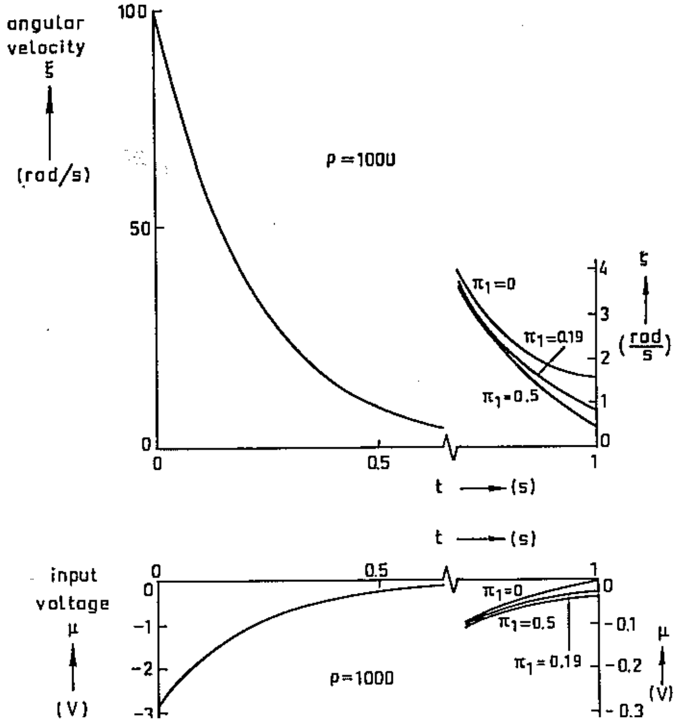


Fig. 3.5. The behavior of state and input for the angular velocity stabilization problem for different values of π_1. Note the changes in the vertical scales near the end of the interval

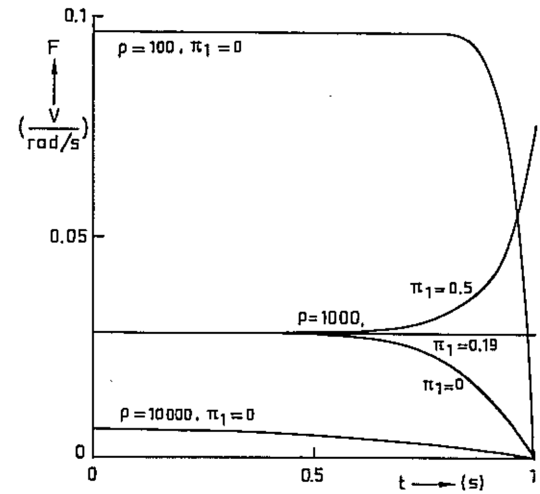


Fig. 3.6. The behavior of the optimal feedback gain factor for the angular velocity stabilization problem for various values of ρ and π_1.

of the curves for $\pi_1 = 0.19$ in Fig. 3.5 with the other curves shows that there is little point in letting F vary with time unless the terminal state is very heavily weighted.

3.3.3 Derivation of the Riccati Equation

We proceed with establishing a few more facts about the matrix $P(t)$ as given by **3-98**. In our further analysis, $P(t)$ plays a crucial role. It is possible to derive a differential equation for $P(t)$. To achieve this we differentiate $P(t)$ as given by **3-98** with respect to t. Using the rule for differentiating the inverse of a time-dependent matrix $M(t)$,

$$\frac{d}{dt} M^{-1}(t) = -M^{-1}(t)\dot{M}(t)M^{-1}(t), \qquad \textbf{3-114}$$

which can be proved by differentiating the identity $M(t)M^{-1}(t) = I$, we obtain

$$\begin{aligned}\dot{P}(t) = {} & [\dot{\Theta}_{21}(t, t_1) + \dot{\Theta}_{22}(t, t_1)P_1][\Theta_{11}(t, t_1) + \Theta_{12}(t, t_1)P_1]^{-1} \\ & - [\Theta_{21}(t, t_1) + \Theta_{22}(t, t_1)P_1][\Theta_{11}(t, t_1) + \Theta_{12}(t, t_1)P_1]^{-1} \\ & \cdot [\dot{\Theta}_{11}(t, t_1) + \dot{\Theta}_{12}(t, t_1)P_1][\Theta_{11}(t, t_1) + \Theta_{12}(t, t_1)P_1]^{-1}, \end{aligned} \qquad \textbf{3-115}$$

where a dot denotes differentiation with respect to t. Since $\Theta(t, t_0)$ is the transition matrix of **3-99**, we have

$$\begin{aligned}\dot{\Theta}_{11}(t, t_1) &= A(t)\Theta_{11}(t, t_1) - B(t)R_2^{-1}(t)B^T(t)\Theta_{21}(t, t_1), \\ \dot{\Theta}_{12}(t, t_1) &= A(t)\Theta_{12}(t, t_1) - B(t)R_2^{-1}(t)B^T(t)\Theta_{22}(t, t_1), \\ \dot{\Theta}_{21}(t, t_1) &= -R_1(t)\Theta_{11}(t, t_1) - A^T(t)\Theta_{21}(t, t_1), \\ \dot{\Theta}_{22}(t, t_1) &= -R_1(t)\Theta_{12}(t, t_1) - A^T(t)\Theta_{22}(t, t_1). \end{aligned} \qquad \textbf{3-116}$$

Substituting all this into **3-115**, we find after rearrangement the following differential equation for $P(t)$:

$$-\dot{P}(t) = R_1(t) - P(t)B(t)R_2^{-1}(t)B^T(t)P(t) + P(t)A(t) + A^T(t)P(t). \qquad \textbf{3-117}$$

The boundary condition for this differential equation is found by setting $t = t_1$ in **3-98**. It follows that

$$P(t_1) = P_1. \qquad \textbf{3-118}$$

The matrix differential equation thus derived resembles the well-known scalar differential equation

$$\frac{dy}{dx} + \alpha(x)y + \beta(x)y^2 = \gamma(x), \qquad \textbf{3-119}$$

where x is the independent and y the dependent variable, and $\alpha(x)$, $\beta(x)$, and $\gamma(x)$ are known functions of x. This equation is known as the Riccati equation (Davis, 1962). Consequently, we refer to **3-117** as a *matrix Riccati equation* (Kalman, 1960).

We note that since the matrix P_1 that occurs in the terminal condition for $P(t)$ is symmetric, and since the matrix differential equation for $P(t)$ is also symmetric, the solution $P(t)$ must be symmetric for all $t_0 \leq t \leq t_1$. This symmetry will often be used, especially when computing P.

We now find an interpretation for the matrix $P(t)$. The optimal closed-loop system is described by the state differential equation

$$\dot{x}(t) = [A(t) - B(t)F(t)]x(t). \qquad \textbf{3-120}$$

Let us consider the optimization criterion **3-65** computed over the interval $[t, t_1]$. We write

$$\int_t^{t_1} [x^T(\tau)R_1(\tau)x(\tau) + u^T(\tau)R_2(\tau)u(\tau)]\, d\tau + x^T(t_1)P_1x(t_1)$$
$$= \int_t^{t_1} x^T(\tau)[R_1(\tau) + F^T(\tau)R_2(\tau)F(\tau)]x(\tau)\, d\tau + x^T(t_1)P_1x(t_1), \qquad \textbf{3-121}$$

since

$$u(\tau) = -F(\tau)x(\tau). \qquad \textbf{3-122}$$

From the results of Section 1.11.5 (Theorem 1.54), we know that **3-121** can be written as

$$x^T(t)\tilde{P}(t)x(t), \qquad \textbf{3-123}$$

where $\tilde{P}(t)$ is the solution of the matrix differential equation

$$-\dot{\tilde{P}}(t) = R_1(t) + F^T(t)R_2(t)F(t)$$
$$+ \tilde{P}(t)[A(t) - B(t)F(t)] + [A(t) - B(t)F(t)]^T\tilde{P}(t), \qquad \textbf{3-124}$$

with

$$\tilde{P}(t_1) = P_1.$$

Substituting $F(t) = R_2^{-1}(t)B^T(t)P(t)$ into **3-124** yields

$$-\dot{\tilde{P}}(t) = R_1(t) + P(t)B(t)R_2^{-1}(t)B^T(t)P(t) + \tilde{P}(t)A(t)$$
$$- \tilde{P}(t)B(t)R_2^{-1}(t)B^T(t)P(t) + A^T(t)\tilde{P}(t)$$
$$- P(t)B(t)R_2^{-1}(t)B^T(t)\tilde{P}(t). \qquad \textbf{3-125}$$

We claim that the solution of this matrix differential equation is precisely

$$\tilde{P}(t) = P(t). \qquad \textbf{3-126}$$

This is easily seen since substitution of $P(t)$ for $\bar{P}(t)$ reduces the differential equation **3-125** to

$$-\dot{P}(t) = R_1(t) - P(t)B(t)R_2^{-1}(t)B^T(t)P(t) + P(t)A(t) + A^T(t)P(t). \quad \textbf{3-127}$$

This is the matrix Riccati equation **3-117** which is indeed satisfied by $P(t)$; also, the terminal condition is correct. This derivation also shows that $P(t)$ must be nonnegative-definite since **3-121** is a nonnegative expression because R_1, R_2, and P_1 are nonnegative-definite.

We summarize our conclusions as follows.

Theorem 3.4. *The optimal input for the deterministic optimal linear regulator is generated by the linear control law*

$$u^0(t) = -F^0(t)x^0(t), \quad \textbf{3-128}$$

where

$$F^0(t) = R_2^{-1}(t)B^T(t)P(t). \quad \textbf{3-129}$$

Here the symmetric nonnegative-definite matrix $P(t)$ satisfies the matrix Riccati equation

$$-\dot{P}(t) = R_1(t) - P(t)B(t)R_2^{-1}(t)B^T(t)P(t) + P(t)A(t) + A^T(t)P(t), \quad \textbf{3-130}$$

with the terminal condition

$$P(t_1) = P_1, \quad \textbf{3-131}$$

and where

$$R_1(t) = D^T(t)R_3(t)D(t).$$

For the optimal solution we have

$$\int_t^{t_1} [x^{0T}(\tau)R_1(\tau)x^0(\tau) + u^{0T}(\tau)R_2(\tau)u^0(\tau)]\, d\tau + x^{0T}(t_1)P_1x^0(t_1) = x^{0T}(t)P(t)x^0(t), \qquad t \le t_1. \quad \textbf{3-132}$$

We see that the matrix $P(t)$ not only gives us the optimal feedback law but also allows us to evaluate the value of the criterion for any given initial state and initial time.

From the derivation of this section, we extract the following result (Wonham, 1968a), which will be useful when we consider the stochastic linear optimal regulator problem and the optimal observer problem.

Lemma 3.1. *Consider the matrix differential equation*

$$-\dot{\bar{P}}(t) = R_1(t) + F^T(t)R_2(t)F(t) + \bar{P}(t)[A(t) - B(t)F(t)] + [A(t) - B(t)F(t)]^T\bar{P}(t), \quad \textbf{3-133}$$

with the terminal condition

$$\bar{P}(t_1) = P_1, \tag{3-134}$$

where $R_1(t)$, $R_2(t)$, $A(t)$ and $B(t)$ are given time-varying matrices of appropriate dimensions, with $R_1(t)$ nonnegative-definite and $R_2(t)$ positive-definite for $t_0 \leq t \leq t_1$, and P_1 nonnegative-definite. Let $F(t)$ be an arbitrary continuous matrix function for $t_0 \leq t \leq t_1$. Then for $t_0 \leq t \leq t_1$

$$\bar{P}(t) \geq P(t), \tag{3-135}$$

where $P(t)$ is the solution of the matrix Riccati equation

$$-\dot{P}(t) = R_1(t) - P(t)B(t)R_2^{-1}(t)B^T(t)P(t) + P(t)A(t) + A^T(t)P(t), \tag{3-136}$$

$$P(t_1) = P_1. \tag{3-137}$$

The inequality **3-135** *converts into an equality if*

$$F(\tau) = R_2^{-1}(\tau)B^T(\tau)P(\tau) \qquad \textit{for } t \leq \tau \leq t_1. \tag{3-138}$$

The lemma asserts that $\bar{P}(t)$ is "minimized" in the sense stated in **3-135** by choosing F as indicated in **3-138.** The proof is simple. The quantity

$$x^T(t)\bar{P}(t)x(t) \tag{3-139}$$

is the value of the criterion **3-121** if the system is controlled with the arbitrary linear control law

$$u(\tau) = -F(\tau)x(\tau), \qquad t \leq \tau \leq t_1. \tag{3-140}$$

The optimal control law, which happens to be linear and is therefore also the best linear control law, yields $x^T(t)P(t)x(t)$ for the criterion (Theorem 3.4), so that

$$x^T(t)\bar{P}(t)x(t) \geq x^T(t)P(t)x(t) \qquad \text{for all } x(t). \tag{3-141}$$

This proves **3-135.**

We conclude this section with a remark about the existence of the solution of the regulator problem. It can be proved that under the conditions formulated in Definition 3.2 the deterministic linear optimal regulator problem always has a unique solution. The existence of the solution of the regulator problem also guarantees (1) the existence of the inverse matrix in **3-98,** and (2) the fact that the matrix Riccati equation **3-130** with the terminal condition **3-131** has the unique solution **3-98.** Some references on the existence of the solutions of the regulator problem and Riccati equations are Kalman (1960), Athans and Falb (1966), Kalman and Englar (1966), Wonham (1968a), Bucy (1967a, b), Moore and Anderson (1968), Bucy and Joseph (1968), and Schumitzky (1968).

Example 3.6. *Angular velocity stabilization*

Let us continue Example 3.5. $P(t)$ is in this case a scalar function and satisfies the scalar Riccati equation

$$-\dot{P}(t) = 1 - \frac{\kappa^2}{\rho} P^2(t) - 2\alpha P(t), \qquad \textbf{3-142}$$

with the terminal condition

$$P(t_1) = \pi_1. \qquad \textbf{3-143}$$

In this scalar situation the Riccati equation **3-142** can be solved directly. In view of the results obtained in Example 3.5, however, we prefer to use **3-98**, and we write

$$P(t) = \frac{\theta_{21}(t, t_1) + \theta_{22}(t, t_1)\pi_1}{\theta_{11}(t, t_1) + \theta_{12}(t, t_1)\pi_1}, \qquad t \le t_1, \qquad \textbf{3-144}$$

with the θ_{ij} defined as in Example 3.5. Figure 3.7 shows the behavior of $P(t)$ for some of the cases previously considered. We note that $P(t)$, just as the gain factor $F(t)$, has the property that it is constant during almost the entire interval except near the end. (This is not surprising since $P(t)$ and $F(t)$ differ by a constant factor.)

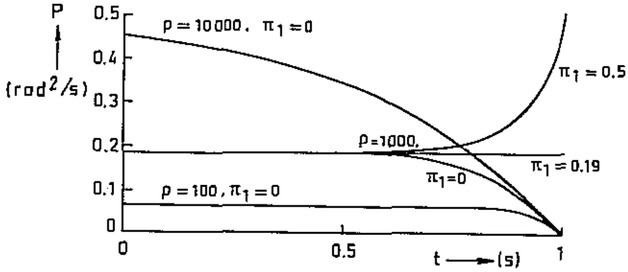


Fig. 3.7. The behavior of $P(t)$ for the angular velocity stabilization problem for various values of ρ and π_1.

3.4 STEADY-STATE SOLUTION OF THE DETERMINISTIC LINEAR OPTIMAL REGULATOR PROBLEM

3.4.1 Introduction and Summary of Main Results

In the preceding section we considered the problem of minimizing the criterion

$$\int_{t_0}^{t_1} [z^T(t)R_3(t)z(t) + u^T(t)R_2(t)u(t)]\, dt + x^T(t_1)P_1x(t_1) \qquad \textbf{3-145}$$

for the system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t),$$
$$z(t) = D(t)x(t), \qquad 3\text{-}146$$

where the terminal time t_1 is finite. From a practical point of view, it is often natural to consider very long control periods $[t_0, t_1]$. In this section we therefore extensively study the asymptotic behavior of the solution of the deterministic regulator problem as $t_1 \to \infty$.

The main results of this section can be summarized as follows.

1. *As the terminal time t_1 approaches infinity, the solution $P(t)$ of the matrix Riccati equation*

$$-\dot{P}(t) = D^T(t)R_3(t)D(t) - P(t)B(t)R_2^{-1}(t)B^T(t)P(t) + A^T(t)P(t) + P(t)A(t), \qquad 3\text{-}147$$

with the terminol condition

$$P(t_1) = P_1, \qquad 3\text{-}148$$

generally approaches a ***steady-state solution*** *$\bar{P}(t)$ that is independent of P_1.*

The conditions under which this result holds are precisely stated in Section 3.4.2. We shall also see that in the time-invariant case, that is, when the matrices A, B, D, R_3, and R_2 are constant, the steady-state solution $\bar{P}$, not surprisingly, is also constant and is a solution of the *algebraic Riccati equation*

$$0 = D^TR_3D - \bar{P}BR_2^{-1}B^T\bar{P} + A^T\bar{P} + \bar{P}A. \qquad 3\text{-}149$$

It is easily recognized that $\bar{P}$ is nonnegative-definite. We prove that in general (the precise conditions are given) the steady-state solution $\bar{P}$ is the only solution of the algebraic Riccati equation that is nonnegative-definite, so that it can be uniquely determined.

Corresponding to the steady-state solution of the Riccati equation, we obtain of course the *steady-state control law*

$$u(t) = -\bar{F}(t)x(t), \qquad 3\text{-}150$$

where

$$\bar{F}(t) = R_2^{-1}(t)B^T(t)\bar{P}(t). \qquad 3\text{-}151$$

It will be proved that this steady-state control law minimizes the criterion **3-145** with t_1 replaced with ∞. Of great importance is the following:

2. *The steady-state control law is in general asymptotically stable.*

Again, precise conditions will be given. Intuitively, it is not difficult to understand this fact. Since

$$\int_{t_0}^{\infty} [z^T(t)R_3(t)z(t) + u^T(t)R_2(t)u(t)]\, dt \qquad 3\text{-}152$$

exists for the steady-state control law, it follows that in the closed-loop system $u(t) \to 0$ and $z(t) \to 0$ as $t \to \infty$. In general, this can be true only if $x(t) \to 0$, which means that the closed-loop system is asymptotically stable.

Fact 2 is very important since we now have the means to devise linear feedback systems that are asymptotically stable and at the same time possess optimal transient properties in the sense that any nonzero initial state is reduced to the zero state in an optimal fashion. For time-invariant systems this is a welcome addition to the theory of stabilization outlined in Section 3.2. There we saw that any time-invariant system in general can be stabilized by a linear feedback law, and that the closed-loop poles can be arbitrarily assigned. The solution of the regulator problem gives us a prescription to assign these poles in a rational manner. We return to the question of the optimal closed-loop pole distribution in Section 3.8.

Example 3.7. *Angular velocity stabilization*

For the angular velocity stabilization problem of Examples 3.3, 3.5, and 3.6, the solution of the Riccati equation is given by **3-144**. It is easily found with the aid of **3-106** that as $t_1 \to \infty$,

$$P(t) \to \bar{P} = \frac{\rho}{\kappa^2}\left(-\alpha + \sqrt{\alpha^2 + \frac{\kappa^2}{\rho}}\right). \qquad \textbf{3-153}$$

$\bar{P}$ can also be found by solving the algebraic equation **3-149** which in this case reduces to

$$0 = 1 - \frac{\kappa^2}{\rho}\bar{P}^2 - 2\alpha\bar{P}. \qquad \textbf{3-154}$$

This equation has the solutions

$$\frac{\rho}{\kappa^2}\left(-\alpha \pm \sqrt{\alpha^2 + \frac{\kappa^2}{\rho}}\right). \qquad \textbf{3-155}$$

Since $\bar{P}$ must be nonnegative, it follows immediately that **3-153** is the correct solution.

The corresponding steady-state gain is given by

$$\bar{F} = \frac{1}{\kappa}\left(-\alpha + \sqrt{\alpha^2 + \frac{\kappa^2}{\rho}}\right). \qquad \textbf{3-156}$$

By substituting

$$\mu(t) = -\bar{F}\xi(t) \qquad \textbf{3-157}$$

into the system state differential equation, it follows that the closed-loop system is described by the state differential equation

$$\dot{\xi}(t) = -\sqrt{\alpha^2 + \frac{\kappa^2}{\rho}}\,\xi(t). \qquad \textbf{3-158}$$

Obviously, this system is asymptotically stable.

Example 3.8. *Position control*

As a more complicated example, we consider the position control problem of Example 3.4 (Section 3.3.1). The steady-state solution $\bar{P}$ of the Riccati equation **3-147** must now satisfy the equation

$$0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}(1, 0) - \bar{P}\begin{pmatrix} 0 \\ \kappa \end{pmatrix}\frac{1}{\rho}(0, \kappa)\bar{P} + \begin{pmatrix} 0 & 0 \\ 1 & -\alpha \end{pmatrix}\bar{P} + \bar{P}\begin{pmatrix} 0 & 1 \\ 0 & -\alpha \end{pmatrix}. \quad \textbf{3-159}$$

Let $\bar{P}_{ij}$, $i, j = 1, 2$, denote the elements of $\bar{P}$. Then using the fact that $\bar{P}_{12} = \bar{P}_{21}$, the following algebraic equations are obtained from **3-159**

$$\begin{aligned} 0 &= 1 - \frac{\kappa^2}{\rho}\bar{P}_{12}^2, \\ 0 &= -\frac{\kappa^2}{\rho}\bar{P}_{12}\bar{P}_{22} + \bar{P}_{11} - \alpha\bar{P}_{12}, \\ 0 &= -\frac{\kappa^2}{\rho}\bar{P}_{22}^2 + 2\bar{P}_{12} - 2\alpha\bar{P}_{22}. \end{aligned} \quad \textbf{3-160}$$

These equations have several solutions, but it is easy to verify that the only nonnegative-definite solution is given by

$$\begin{aligned} \bar{P}_{11} &= \frac{\sqrt{\rho}}{\kappa}\sqrt{\alpha^2 + \frac{2\kappa}{\sqrt{\rho}}}, \\ \bar{P}_{12} &= \bar{P}_{21} = \frac{\sqrt{\rho}}{\kappa}, \\ \bar{P}_{22} &= \frac{\rho}{\kappa^2}\left(-\alpha + \sqrt{\alpha^2 + \frac{2\kappa}{\sqrt{\rho}}}\right). \end{aligned} \quad \textbf{3-161}$$

The corresponding steady-state feedback gain matrix can be found to be

$$\bar{F} = \left(\frac{1}{\sqrt{\rho}}, \frac{1}{\kappa}\left(-\alpha + \sqrt{\alpha^2 + \frac{2\kappa}{\sqrt{\rho}}}\right)\right). \quad \textbf{3-162}$$

Thus the input is given by

$$\mu(t) = -\bar{F}x(t). \quad \textbf{3-163}$$

It is easily found that the optimal closed-loop system is described by the state differential equation

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ -\frac{\kappa}{\sqrt{\rho}} & -\sqrt{\alpha^2 + \frac{2\kappa}{\sqrt{\rho}}} \end{pmatrix} x(t). \quad \textbf{3-164}$$

The closed-loop characteristic polynomial can be computed to be

$$s^2 + s\sqrt{\alpha^2 + \frac{2\kappa}{\sqrt{\rho}}} + \frac{\kappa}{\sqrt{\rho}}. \tag{3-165}$$

The closed-loop characteristic values are

$$\frac{1}{2}\left(-\sqrt{\alpha^2 + \frac{2\kappa}{\sqrt{\rho}}} \pm \sqrt{\alpha^2 - \frac{2\kappa}{\sqrt{\rho}}}\right). \tag{3-166}$$

Figure 3.8 gives the loci of the closed-loop characteristic values as ρ varies. It is interesting to see that as ρ decreases the closed-loop poles go to infinity along two straight lines that make an angle of $\pi/4$ with the negative real axis. Asymptotically, the closed-loop poles are given by

$$\frac{\kappa^{1/2}}{\rho^{1/4}}\,\tfrac{1}{2}\sqrt{2}(-1 \pm j) \qquad \text{as } \rho \to 0. \tag{3-167}$$

Figure 3.9 shows the response of the steady-state optimal closed-loop system

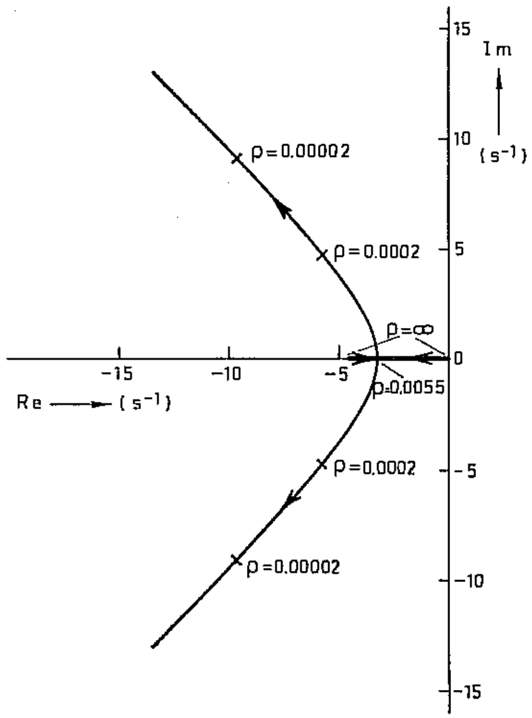


Fig. 3.8. Loci of the closed-loop roots of the position control system as a function of ρ.

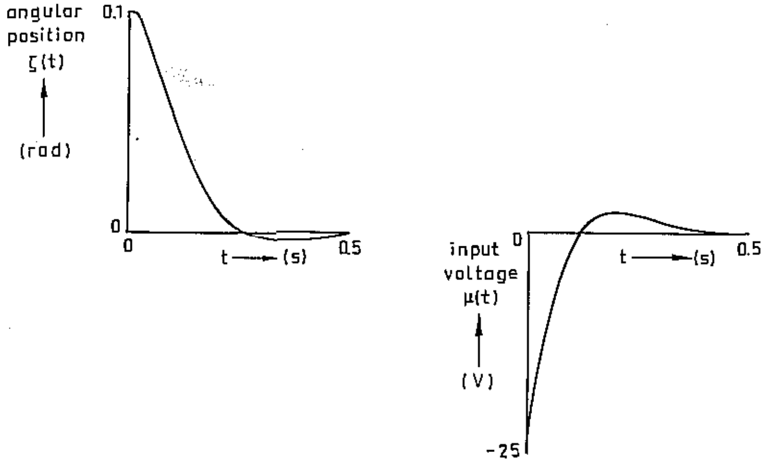


Fig. 3.9. Response of the optimal position control system to the initial state $\xi_1(0) = 0.1$ rad, $\xi_2(0) = 0$ rad/s.

corresponding to the following numerical values:

$$\begin{aligned} \kappa &= 0.787 \text{ rad/(V s}^2\text{)}, \\ \alpha &= 4.6 \text{ s}^{-1}, \\ \rho &= 0.00002 \text{ rad}^2/\text{V}^2. \end{aligned} \qquad \textbf{3-168}$$

The corresponding gain matrix is

$$\bar{F} = (223.6,\ 18.69), \qquad \textbf{3-169}$$

while the closed-loop poles can be computed to be $-9.658 \pm j9.094$. We observe that the present design is equivalent to the position and velocity feedback design of Example 2.4 (Section 2.3). The gain matrix **3-169** is optimal from the point of view of transient response. It is interesting to note that the present design method results in a second-order system with relative damping of nearly $\frac{1}{2}\sqrt{2}$, which is exactly what we found in Example 2.7 (Section 2.5.2) to be the most favorable design.

To conclude the discussion we remark that it follows from Example 3.4 that if $x(t)$ is actually the deviation of the state from a certain equilibrium state x_0 which is not the zero state, $x(t)$ in the control law **3-163** should be replaced with $x'(t)$, where

$$x'(t) = \begin{pmatrix} \xi_1(t) - \xi_{10} \\ \xi_2(t) \end{pmatrix}. \qquad \textbf{3-170}$$

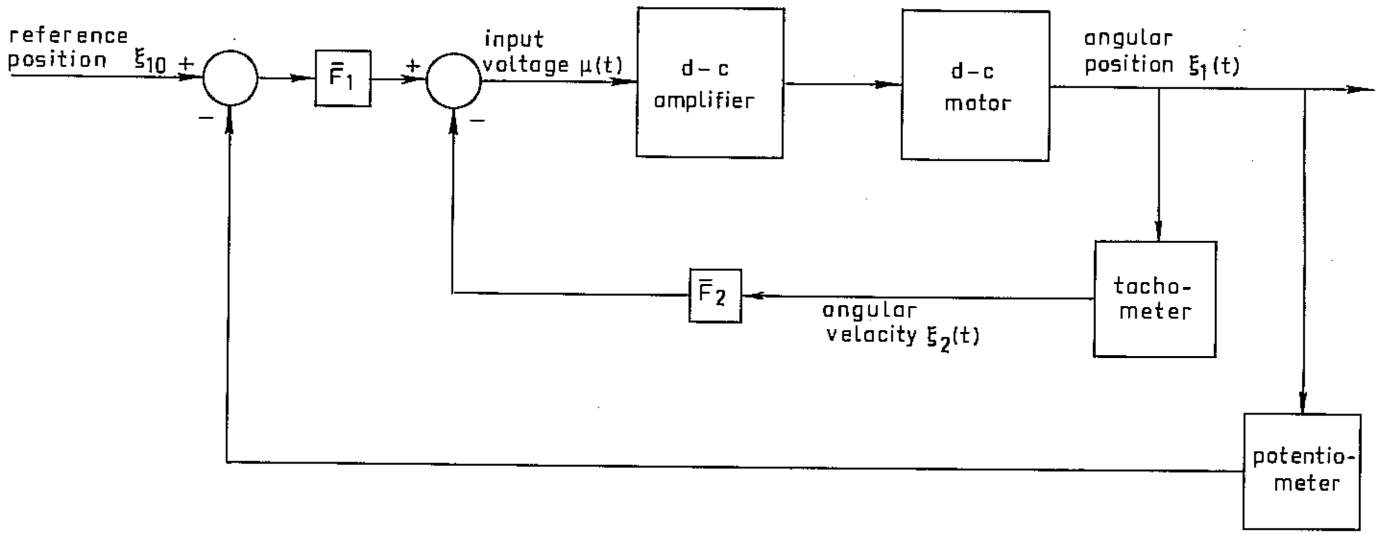


Fig. 3.10. Block diagram of the optimal position control system.

Here ξ_{10} is the desired angular position. This results in the control law

$$\mu(t) = -\bar{F}_1[\xi_1(t) - \xi_{10}] - \bar{F}_2\xi_2(t), \qquad \textbf{3-171}$$

where $\bar{F} = (\bar{F}_1, \bar{F}_2)$. The block diagram corresponding to this control law is given in Fig. 3.10.

Example 3.9. *Stirred tank*

As another example, we consider the stirred tank of Example 1.2 (Section 1.2.3). Suppose that it is desired to stabilize the outgoing flow $F(t)$ and the outgoing concentration $c(t)$. We therefore choose as the controlled variable

$$z(t) = y(t) = \begin{pmatrix} 0.01 & 0 \\ 0 & 1 \end{pmatrix} x(t), \qquad \textbf{3-172}$$

where we use the numerical values of Example 1.2. To determine the weighting matrix R_3, we follow the same argument as in Example 2.8 (Section 2.5.3). The nominal value of the outgoing flow is 0.02 m^3/s. A 10% change corresponds to 0.002 m^3/s. The nominal value of the outgoing concentration is 1.25 $kmol/m^3$. Here a 10% change corresponds to about 0.1 $kmol/m^3$. Suppose that we choose R_3 diagonal with diagonal elements σ_1 and σ_2. Then

$$z^T(t)R_3z(t) = \sigma_1\zeta_1^2(t) + \sigma_2\zeta_2^2(t), \qquad \textbf{3-173}$$

where $z(t) = \text{col}\,(\zeta_1(t), \zeta_2(t))$. Then if a 10% change in the outgoing flow is to make about the same contribution to the criterion as a 10% change in the outgoing concentration, we must have

$$\sigma_1(0.002)^2 \simeq \sigma_2(0.1)^2, \qquad \textbf{3-174}$$

or

$$\frac{\sigma_1}{\sigma_2} \simeq 2500. \qquad \textbf{3-175}$$

Let us therefore select

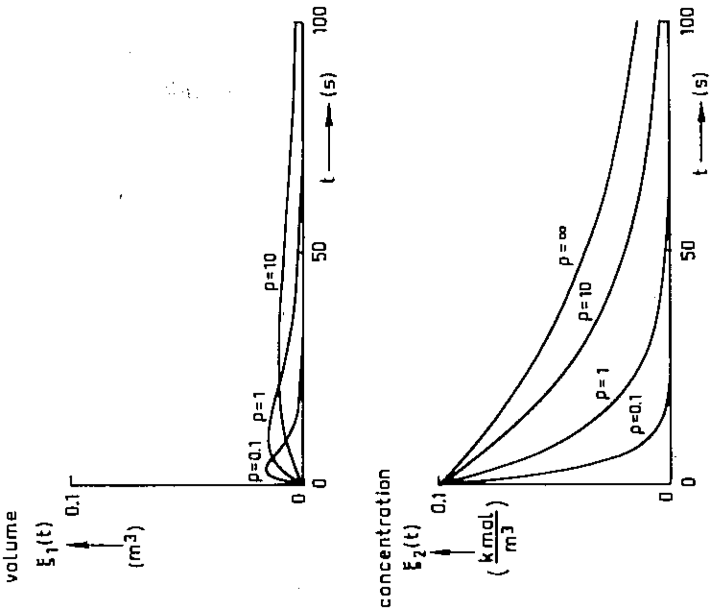
$$\sigma_1 = 50, \qquad \sigma_2 = \tfrac{1}{50}, \qquad \textbf{3-176}$$

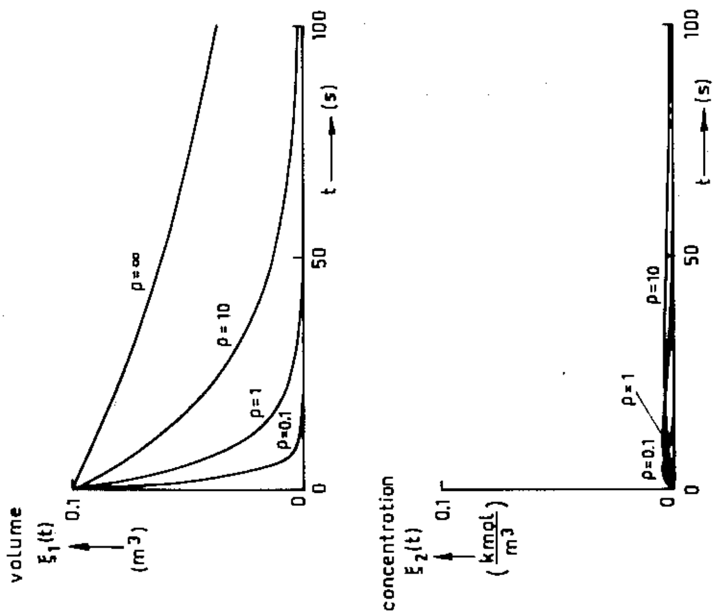
or

$$R_3 = \begin{pmatrix} 50 & 0 \\ 0 & 0.02 \end{pmatrix}. \qquad \textbf{3-177}$$

To choose R_2 we follow a similar approach. A 10% change in the feed F_1 corresponds to 0.0015 m^3/s, while a 10% change in the feed F_2 corresponds to 0.0005 m^3/s. Let us choose $R_2 = \text{diag}\,(\rho_1, \rho_2)$. Then the 10% changes in F_1 and F_2 contribute an amount of

$$\rho_1(0.0015)^2 + \rho_2(0.0005)^2 \qquad \textbf{3-178}$$


volume
$\xi_1(t)$
(m^3)
0.1
0
ρ=0.1
ρ=1
ρ=10
0
50
100
t ⟶ (s)
concentration
$\xi_2(t)$
$\left(\frac{kmol}{m^3}\right)$
0.1
0
ρ=∞
ρ=10
ρ=1
ρ=0.1
0
50
100
t ⟶ (s)


volume
$\xi_1(t)$
(m^3)
0.1
0
ρ=∞
ρ=10
ρ=1
ρ=0.1
0
50
100
t ⟶ (s)
concentration
$\xi_2(t)$
$\left(\frac{kmol}{m^3}\right)$
0.1
0
ρ=10
ρ=1
ρ=0.1
0
50
100
t ⟶ (s)

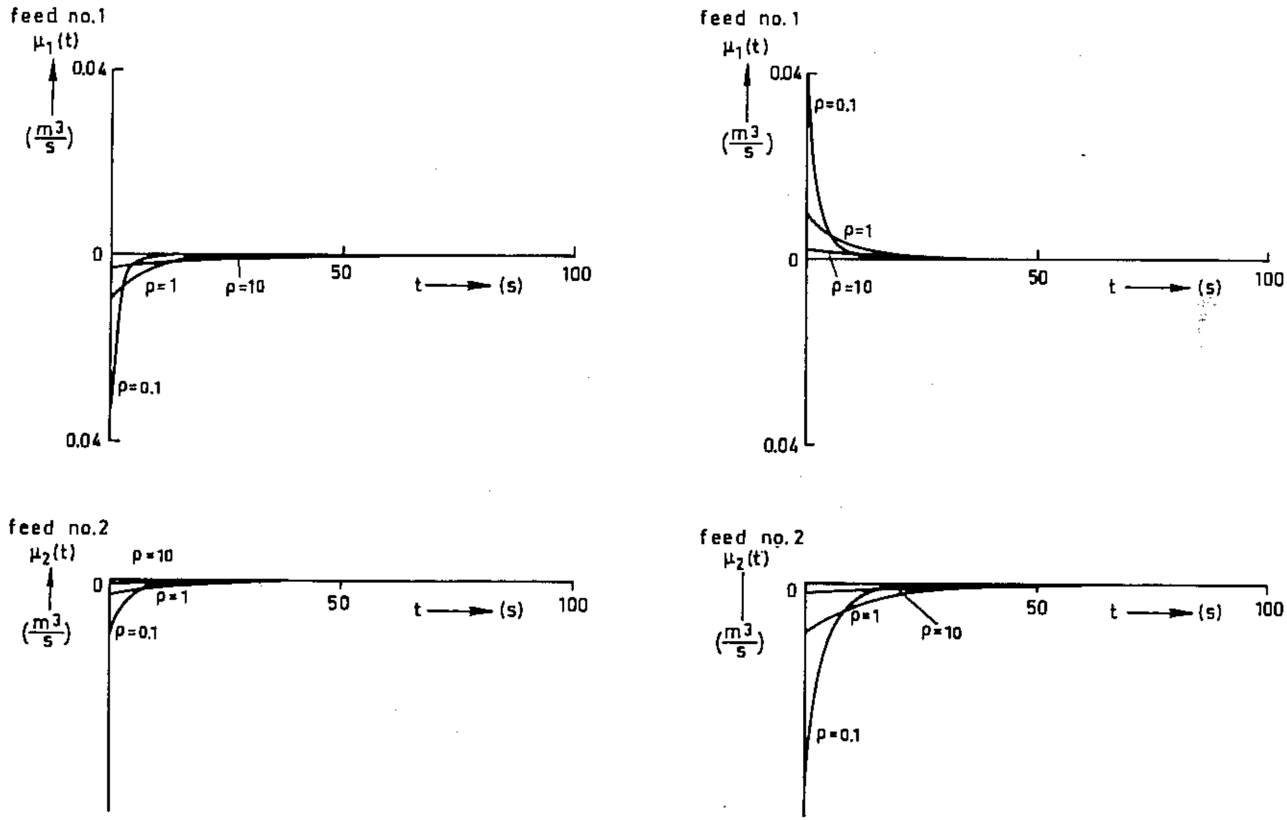


Fig. 3.11. Closed-loop responses of the regulated stirred tank for various values of the weighting factor ρ. Left column: Responses of incremental volume, concentration, feed no. 1, and feed no. 2 to the initial state $x(0) = \text{col}(0.1, 0)$. Right column: Responses of incremental volume, concentration, feed no. 1, and feed no. 2 to the initial state $x(0) = \text{col}(0, 0.1)$.

to the criterion. Both terms contribute equally if

$$\frac{\rho_1}{\rho_2} = \frac{1}{9}. \tag{3-179}$$

We therefore select

$$R_2 = \rho \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 3 \end{pmatrix}, \tag{3-180}$$

where ρ is a scalar constant to be determined.

Figure 3.11 depicts the behavior of the optimal steady-state closed-loop system for $\rho = \infty$, 10, 1, and 0.1. The case $\rho = \infty$ corresponds to the open-loop system (no control at all). We see that as ρ decreases a faster and faster response is obtained at the cost of larger and larger input amplitudes. Table 3.1 gives the closed-loop characteristic values as a function of ρ. We see that in all cases a system is obtained with closed-loop poles that are well inside the left-half complex plane.

Table 3.1 Locations of the Steady-State Optimal Closed-Loop Poles as a Function of ρ for the Regulated Stirred Tank

ρ	Optimal closed-loop poles (s^{-1})	
∞	−0.01	−0.02
10	−0.02952,	−0.04523
1	−0.07517,	−0.1379
0.1	−0.2310,	−0.4345

We do not list here the gain matrices $\bar{F}$ found for each value of ρ, but it turns out that they are not diagonal, as opposed to what we considered in Example 2.8. The feedback schemes obtained in the present example are optimal in the sense that they are the best compromises between the requirement of maximal speed of response and the limitations on the input amplitudes.

Finally, we observe from the plots of Fig. 3.11 that the closed-loop system shows relatively little interaction, that is, the response to an initial disturbance in the concentration hardly affects the tank volume, and vice versa.

3.4.2* Steady-State Properties of Optimal Regulators

In this subsection and the next we give precise results concerning the steady-state properties of optimal regulators. This section is devoted to the general, time-varying case; in the next section the time-invariant case is investigated in much more detail. Most of the results in the present section are due to Kalman (1960). We more or less follow his exposition.

We first state the following result.

Theorem 3.5. *Consider the matrix Riccati equation*

$$-\dot{P}(t) = D^T(t)R_3(t)D(t) - P(t)B(t)R_2^{-1}(t)B^T(t)P(t) + A^T(t)P(t) + P(t)A(t). \qquad \textbf{3-181}$$

Suppose that $A(t)$ is continuous and bounded, that $B(t)$, $D(t)$, $R_3(t)$, and $R_2(t)$ are piecewise continuous and bounded on $[t_0, \infty)$, and furthermore that

$$R_3(t) \geq \alpha I, \qquad R_2(t) \geq \beta I, \qquad \textit{for all } t, \qquad \textbf{3-182}$$

where α and β are positive constants.

(i) *Then if the system*

$$\dot{x}(t) = A(t)x(t) + B(t)u(t),$$
$$z(t) = D(t)x(t), \qquad \textbf{3-183}$$

is either

(a) *completely controllable, or*

(b) *exponentially stable,*

the solution $P(t)$ of the Riccati equation **3-181** *with the terminal condition $P(t_1) = 0$ converges to a nonnegative-definite matrix function $\bar{P}(t)$ as $t_1 \to \infty$. $\bar{P}(t)$ is a solution of the Riccati equation* **3-181**.

(ii) *Moreover, if the system* **3-183** *is either*

(c) *both uniformly completely controllable and uniformly completely reconstructible, or*

(d) *exponentially stable,*

the solution $P(t)$ of the Riccati equation **3-181** *with the terminal condition $P(t_1) = P_1$ converges to $\bar{P}(t)$ as $t_1 \to \infty$ for any $P_1 \geq 0$.*

The proof of the first part of this theorem is not very difficult. From Theorem 3.4 (Section 3.3.3), we know that for finite t_1

$$x^T(t)P(t)x(t) = \min_{\substack{u(\tau),\\ t \leq \tau \leq t_1}} \left\{ \int_t^{t_1} [z^T(\tau)R_3(\tau)z(\tau) + u^T(\tau)R_2(\tau)u(\tau)]\, d\tau \right\}. \qquad \textbf{3-184}$$

Of course this expression is a function of the terminal time t_1. We first establish that as a function of t_1 this expression has an upper bound. If the system is completely controllable [assumption (a)], there exists an input that transfers the state $x(t)$ to the zero state at some time t_1'. For this input we can compute the criterion

$$\int_t^{t_1'} [z^T(\tau)R_3(\tau)z(\tau) + u^T(\tau)R_2(\tau)u(\tau)]\, d\tau. \qquad \textbf{3-185}$$

This number is an upper bound for **3-184**, since obviously we can take $u(t) = 0$ for $t \geq t_1'$.

If the system is exponentially stable (Section 1.4.1), $x(t)$ converges exponentially to zero if we let $u(t) \equiv 0$. Then

$$\int_t^{t_1} [z^T(\tau)R_3(\tau)z(\tau) + u^T(\tau)R_2(\tau)u(\tau)]\, d\tau = \int_t^{t_1} z^T(\tau)R_3(\tau)z(\tau)\, d\tau \qquad \textbf{3-186}$$

converges to a finite number as $t_1 \to \infty$, since $D(t)$ and $R_3(t)$ are assumed to be bounded. This number is an upper bound for **3-184**.

Thus we have shown that as a function of t_1 the expression **3-184** has an upper bound under either assumption (a) or (b). Furthermore, it is reasonably obvious that as a function of t_1 this expression is monotonically nondecreasing. Suppose that this were not true. Then there must exist a t_1' and t_1'' with $t_1'' > t_1'$ such that for $t_1 = t_1''$ the criterion is smaller than for $t_1 = t_1'$. Now apply the input that is optimal for t_1'' over the interval $[t_0, t_1']$. Since the integrand of the criterion is nonnegative, the criterion for this smaller interval must give a value that is less than or equal to the criterion for the larger interval $[t_0, t_1'']$. This is a contradiction, hence **3-184** must be a monotonically nondecreasing function of t_1.

Since as a function of t_1 the expression **3-184** is bounded from above and monotonically nondecreasing, it must have a limit as $t_1 \to \infty$. Since $x(t)$ is arbitrary, each of the elements of $P(t)$ has a limit, hence $P(t)$ has a limit that we denote as $\bar{P}(t)$. That $\bar{P}(t)$ is nonnegative-definite and symmetric is obvious. That $\bar{P}(t)$ is a solution of the matrix Riccati equation follows by the continuity of the solutions of the Riccati equation with respect to initial conditions. Following Kalman (1960), let $\Pi(t; P_1, t_1)$ denote the solution of the matrix Riccati equation with the terminal condition $P_1(t_1) = P_1$. Then

$$\begin{aligned}\bar{P}(t) = \lim_{t_2 \to \infty} \Pi(t; 0, t_2) &= \lim_{t_2 \to \infty} \Pi[t; \Pi(t_1; 0, t_2), t_1] \\ &= \Pi[t; \lim_{t_2 \to \infty} \Pi(t_1; 0, t_2), t_1] \\ &= \Pi[t, \bar{P}(t_1), t_1], \end{aligned} \qquad \textbf{3-187}$$

which shows that $\bar{P}_1(t)$ is indeed a solution of the Riccati equation. The proof of the remainder of Theorem 3.5 will be deferred for a moment.

We refer to $\bar{P}(t)$ as the *steady-state* solution of the Riccati equation. To this steady-state solution corresponds the *steady-state optimal control law*

$$u(t) = -\bar{F}(t)x(t), \qquad \textbf{3-188}$$

where

$$\bar{F}(t) = R_2^{-1}(t)B^T(t)\bar{P}(t). \qquad \textbf{3-189}$$

Concerning the stability of the steady-state control law, we have the following result.

Theorem 3.6. *Consider the deterministic linear optimal regulator problem and suppose that the assumptions of Theorem 3.5 concerning A, B, D, R_3 and R_2 are satisfied. Then if the system*

$$\dot{x}(t) = A(t)x(t) + B(t)u(t),$$
$$z(t) = D(t)x(t), \qquad \text{3-190}$$

is either

(a) *both uniformly completely controllable and uniformly completely reconstructible, or*

(b) *exponentially stable,*

the following facts hold:

(i) *The steady-state optimal control law*

$$u(t) = -R_2^{-1}(t)B^T(t)\bar{P}(t)x(t) \qquad \text{3-191}$$

is exponentially stable.

(ii) *The steady-state control law* **3-191** *minimizes*

$$\lim_{t_1\to\infty}\left\{\int_{t_0}^{t_1}[z^T(t)R_3(t)z(t) + u^T(t)R_2(t)u(t)]\,dt + x^T(t_1)P_1x(t_1)\right\} \qquad \text{3-192}$$

for all $P_1 \geq 0$. The minimal value of the criterion **3-192**, *which is achieved by the steady-state control law, is given by*

$$x^T(t_0)\bar{P}(t_0)x(t_0). \qquad \text{3-193}$$

A rigorous proof of these results is given by Kalman (1960). We only make the theorem plausible. If condition (a) or (b) of Theorem 3.6 is satisfied, also condition (a) or (b) of Theorem 3.5 holds. It follows that the solution of the Riccati equation **3-181** with $P(t_1) = 0$ converges to $\bar{P}(t)$ as $t_1 \to \infty$. For the corresponding steady-state control law, we have

$$\int_{t_0}^{\infty}[z^T(t)R_3(t)z(t) + u^T(t)R_2(t)u(t)]\,dt = x^T(t_0)\bar{P}(t_0)x(t_0). \qquad \text{3-194}$$

Since the integral converges and $R_3(t)$ and $R_2(t)$ satisfy the conditions **3-182**, both $z(t)$ and $u(t)$ must converge to zero as $t \to \infty$. Suppose now that the closed-loop system is not asymptotically stable. Then there exists an initial state such that $x(t)$ does not approach zero while $z(t) \to 0$ and $u(t) \to 0$. This is clearly in conflict with the complete reconstructibility of the system if (a) holds, or with the assumption of exponential stability of the system if (b) holds. Hence the closed-loop system must be asymptotically stable. That it moreover is exponentially stable follows from the uniformity properties.

This settles part (i) of the theorem. Part (ii) can be shown as follows. Suppose that there exists another control law that yields a smaller value for

3-192. Because the criterion **3-192** yields a finite value when the steady-state optimal control law is used, this other control law must also yield a finite value. Then, by the same argument as for the steady-state control law, this other control law must be asymptotically stable. This means that for this control law

$$\lim_{t_1\to\infty}\left\{\int_{t_0}^{t_1}[z^T(t)R_3(t)z(t)+u^T(t)R_2(t)u(t)]\,dt+x^T(t_1)P_1x(t_1)\right\}$$
$$=\int_{t_0}^{\infty}[z^T(t)R_3(t)z(t)+u^T(t)R_2(t)u(t)]\,dt. \quad \textbf{3-195}$$

But since the right-hand side of this expression is minimized by the steady-state control law, there cannot be another control law that yields a smaller value for the left-hand side. This proves part (ii) of Theorem 3.6. This moreover proves the second part of Theorem 3.5, since under assumptions (c) or (d) of this theorem the steady-state feedback law minimizes the criterion **3-192** for all $P_1 \geq 0$, which implies that the Riccati equation converges to $\bar{P}(t)$ for all $P_1 \geq 0$.

We illustrate the results of this section as follows.

Example 3.10. *Reel-winding mechanism*

As an example of a simple time-varying system, consider the reel-winding mechanism of Fig. 3.12. A dc motor drives a reel on which a wire is being

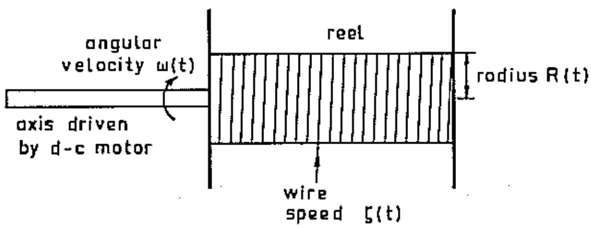


Fig. 3.12. Schematic representation of a reel-winding mechanism.

wound. The speed at which the wire runs on to the reel is to be kept constant. Because of the increasing diameter of the reel, the moment of inertia increases; moreover, to keep the wire speed constant, the angular velocity must decrease. Let $\omega(t)$ be the angular velocity of the reel, $J(t)$ the moment of inertia of reel and motor armature, and $\mu(t)$ the input voltage to the power amplifier that drives the dc motor. Then we have

$$\frac{d}{dt}[J(t)\omega(t)] = \kappa\mu(t) - \phi\omega(t), \quad \textbf{3-196}$$

where κ is a constant which expresses the proportionality of the torque of the motor and the input voltage, and where ϕ is a friction coefficient. Furthermore, let $R(t)$ denote the radius of the reel; then the speed $\zeta(t)$ at which the wire is wound is given by

$$\zeta(t) = R(t)\omega(t). \tag{3-197}$$

Let us introduce the state variable

$$\xi(t) = J(t)\omega(t). \tag{3-198}$$

The system is then described by the equations

$$\begin{aligned} \dot{\xi}(t) &= -\frac{\phi}{J(t)}\,\xi(t) + \kappa\mu(t), \\ \zeta(t) &= \frac{R(t)}{J(t)}\,\xi(t). \end{aligned} \tag{3-199}$$

We assume that the reel speed is so controlled that the wire speed is kept constant at the value ζ_0. The time dependence of J and R can then be established as follows. Suppose that during a short time dt the radius increases from R to $R + dR$. The increase in the volume of wire wound upon the reel is proportional to $R\,dR$. The volume is also proportional to dt, since the wire is wound with a supposedly constant speed. Thus we have

$$R\,dR = c\,dt, \tag{3-200}$$

where c is a constant. This yields after integration

$$R(t) = \sqrt{R^2(0) + ht}, \tag{3-201}$$

where h is another constant. However, if the radius increases from R to $R + dR$, the moment of inertia increases with an amount that is proportional to $R\,dR\,R^2 = R^3\,dR$. Thus we have

$$dJ = c'R^3\,dR, \tag{3-202}$$

where c' is a constant. This yields after integration

$$J(t) = J(0) + h'[R^4(t) - R^4(0)], \tag{3-203}$$

where h' is another constant.

Let us now consider the problem of regulating the system such that the wire speed is kept at the constant value ζ_0. The nominal solution $\zeta_0(t)$, $\mu_0(t)$ that corresponds to this situation can be found as follows. If $\zeta_0(t) \equiv \zeta_0$, we have

$$\xi_0(t) = \frac{J(t)}{R(t)}\,\zeta_0. \tag{3-204}$$

The nominal input is found from the state differential equation:

$$\mu_0(t) = \frac{1}{\kappa}\left[\dot{\xi}_0(t) + \frac{\phi}{J(t)}\,\xi_0(t)\right] = \frac{1}{\kappa}\left[\frac{d}{dt}\left\{\frac{J(t)}{R(t)}\right\} + \frac{\phi}{R(t)}\right]\zeta_0. \qquad \textbf{3-205}$$

Let us now define the shifted state, input, and controlled variables:

$$\begin{aligned} \xi'(t) &= \xi(t) - \xi_0(t),\\ \mu'(t) &= \mu(t) - \mu_0(t),\\ \zeta'(t) &= \zeta(t) - \zeta_0(t). \end{aligned} \qquad \textbf{3-206}$$

These variables satisfy the equations

$$\begin{aligned} \dot{\xi}'(t) &= -\frac{\phi}{J(t)}\,\xi'(t) + \kappa\mu'(t),\\ \zeta'(t) &= \frac{R(t)}{J(t)}\,\xi'(t). \end{aligned} \qquad \textbf{3-207}$$

Let us choose the criterion

$$\int_{t_0}^{t_1}[\zeta'^2(t) + \rho\mu'^2(t)]\,dt. \qquad \textbf{3-208}$$

Then the Riccati equation takes the form

$$-\dot{P}(t) = \frac{R^2(t)}{J^2(t)} - P^2(t)\frac{\kappa^2}{\rho} - 2\,\frac{\phi}{J(t)}\,P(t), \qquad \textbf{3-209}$$

with the terminal condition

$$P(t_1) = 0. \qquad \textbf{3-210}$$

$P(t)$ is in this case a scalar function. The scalar feedback gain factor is given by

$$F(t) = \frac{\kappa}{\rho}\,P(t). \qquad \textbf{3-211}$$

We choose the following numerical values:

$$\begin{aligned} J(t) &= 0.02 + 66.67[R^4(t) - R^4(0)]\ \text{kg m}^2,\\ R(t) &= \sqrt{0.01 + 0.0005t}\ \text{m},\\ \phi &= 0.01\ \text{kg m}^2/\text{s},\\ \kappa &= 0.1\ \text{kg m}^2\ \text{rad}/(\text{V s}^2),\\ \rho &= 0.06\ \text{m}^2/(\text{V}^2\text{s}^2). \end{aligned} \qquad \textbf{3-212}$$

Figure 3.13 shows the behavior of the optimal gain factor $F(t)$ for the terminal

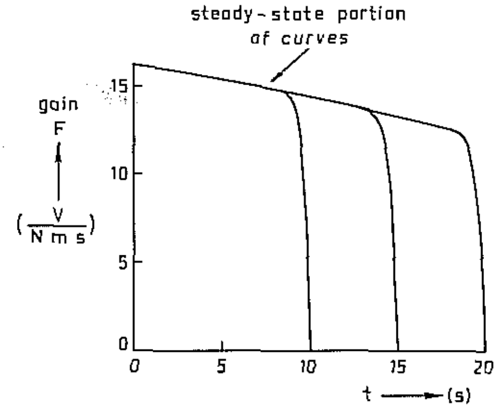


Fig. 3.13. Behavior of the optimal gain factor for the reel-winding problem for various values of the terminal time t_1.

times $t_1 = 10, 15$, and 20 s. We note that for each value of t_1 the gain exhibits an identical steady-state behavior; only near the terminal time do deviations occur. It is clearly shown that the steady-state gain is time-varying. It is not convenient to implement such a time-varying gain. In the present case a practically adequate performance might probably just as well be obtained through a time-invariant feedback gain.

3.4.3* Steady-State Properties of the Time-Invariant Optimal Regulator

In this section we study the steady-state properties of the time-invariant optimal linear regulator. We are able to state sufficient and necessary conditions under which the Riccati equation has a steady-state solution and under which the steady-state optimal closed-loop system is stable. Most of these facts have been given by Wonham (1968a), Lukes (1968), and Mårtensson (1971).

Our results can be summarized as follows.

Theorem 3.7. *Consider the time-invariant regulator problem for the system*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ z(t) &= Dx(t), \end{aligned} \qquad \textbf{3-213}$$

and the criterion

$$\int_{t_0}^{t_1} [z^T(t)R_3z(t) + u^T(t)R_2u(t)]\, dt + x^T(t_1)P_1x(t_1), \qquad \textbf{3-214}$$

with $R_3 > 0$, $R_2 > 0$, $P_1 \geq 0$. *The associated Riccati equation is given by*

$$-\dot{P}(t) = D^T R_3 D - P(t) B R_2^{-1} B^T P(t) + A^T P(t) + P(t) A, \qquad \textbf{3-215}$$

with the terminal condition

$$P(t_1) = P_1. \qquad \textbf{3-216}$$

(a) *Assume that* $P_1 = 0$. *Then as* $t_1 \rightarrow \infty$ *the solution of the Riccati equation approaches a constant steady-state value* $\bar{P}$ *if and only if the system possesses no poles that are at the same time unstable, uncontrollable, and reconstructible.*
(b) *If the system* **3-213** *is both stabilizable and detectable, the solution of the Riccati equation* **3-215** *approaches the unique value* $\bar{P}$ *as* $t_1 \rightarrow \infty$ *for every* $P_1 \geq 0$.
(c) *If* $\bar{P}$ *exists, it is a nonnegative-definite symmetric solution of the algebraic Riccati equation*

$$0 = D^T R_3 D - P B R_2^{-1} B^T P + A^T P + P A. \qquad \textbf{3-217}$$

If the system **3-213** *is stabilizable and detectable,* $\bar{P}$ *is the unique nonnegative-definite symmetric solution of the algebraic Riccati equation* **3-217**.
(d) *If* $\bar{P}$ *exists, it is strictly positive-definite if and only if the system* **3-213** *is completely reconstructible.*
(e) *If* $\bar{P}$ *exists, the steady-state control law*

$$u(t) = -\bar{F} x(t), \qquad \textbf{3-218}$$

where

$$\bar{F} = R_2^{-1} B^T \bar{P}, \qquad \textbf{3-219}$$

is asymptotically stable if and only if the system **3-213** *is stabilizable and detectable.*
(f) *If the system* **3-213** *is stabilizable and detectable, the steady-state control law minimizes*

$$\lim_{t_1 \rightarrow \infty} \left\{ \int_{t_0}^{t_1} [z^T(t) R_3 z(t) + u^T(t) R_2 u(t)]\, dt + x^T(t_1) P_1 x(t_1) \right\} \qquad \textbf{3-220}$$

for all $P_1 \geq 0$. *For the steady-state control law, the criterion* **3-220** *takes the value*

$$x^T(t_0) \bar{P} x(t_0). \qquad \textbf{3-221}$$

We first prove part (a) of this theorem. Suppose that the system is not completely reconstructible. Then it can be transformed into reconstructibility canonical form as follows.

$$\dot{x}(t) = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix} x(t) + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u(t), \qquad \textbf{3-222}$$

$$z(t) = (D_1, \quad 0) x(t),$$

where the pair $\{A_{11}, D_1\}$ is completely reconstructible. Partitioning the solution $P(t)$ of the Riccati equation **3-215** according to the partitioning in **3-222** as

$$P(t) = \begin{pmatrix} P_{11}(t) & P_{12}(t) \\ P_{12}^T(t) & P_{22}(t) \end{pmatrix}, \qquad \textbf{3-223}$$

it is easily found that the Riccati equation **3-215** reduces to the following three matrix equations

$$\begin{aligned} -\dot{P}_{11}(t) = {} & D_1^T R_3 D_1 - [P_{11}(t)B_1 + P_{12}(t)B_2]R_2^{-1} \\ & \cdot [B_1^T P_{11}(t) + B_2^T P_{12}^T(t)] + A_{11}^T P_{11}(t) \\ & + A_{21}^T P_{12}^T(t) + P_{11}(t)A_{11} + P_{12}(t)A_{21}, \end{aligned} \qquad \textbf{3-224}$$

$$\begin{aligned} -\dot{P}_{12}(t) = {} & -[P_{11}(t)B_1 + P_{12}(t)B_2]R_2^{-1}[B_1^T P_{12}(t) + B_2^T P_{22}(t)] \\ & + A_{11}^T P_{12}(t) + A_{21}^T P_{22}(t) + P_{12}(t)A_{22}, \end{aligned} \qquad \textbf{3-225}$$

$$\begin{aligned} -\dot{P}_{22}(t) = {} & -[P_{12}^T(t)B_1 + P_{22}(t)B_2]R_2^{-1}[B_1^T P_{12}(t) + B_2^T P_{22}(t)] \\ & + A_{22}^T P_{22}(t) + P_{22}(t)A_{22}. \end{aligned} \qquad \textbf{3-226}$$

It is easily seen that with the terminal conditions $P_{11}(t_1) = 0$, $P_{12}(t_1) = 0$, and $P_{22}(t_1) = 0$ Eqs. **3-225** and **3-226** are satisfied by

$$P_{12}(t) = 0, \qquad P_{22}(t) = 0, \qquad t \le t_1. \qquad \textbf{3-227}$$

With these identities **3-224** reduces to

$$\begin{aligned} & -\dot{P}_{11}(t) = D_1^T R_3 D_1 - P_{11}(t)B_1 R_2^{-1} B_1^T P_{11}(t) + A_{11}^T P_{11}(t) + P_{11}(t)A_{11}, \\ & P_{11}(t_1) = 0. \end{aligned} \qquad \textbf{3-228}$$

It follows from this that the unreconstructible poles of the system, that is, the characteristic values of A_{22}, do not affect the convergence of $P_{11}(t)$ as $t_1 \to \infty$, hence that the convergence of $P(t)$ is also not affected by the unreconstructible poles. To investigate the convergence of $P(t)$, we can therefore as well assume for the time being that the system **3-213** is completely reconstructible.

Let us now transform the system **3-213** into controllability canonical form and thus represent it as follows:

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} x(t) + \begin{pmatrix} B_1 \\ 0 \end{pmatrix} u(t), \\ z(t) &= (D_1, \quad D_2)x(t), \end{aligned} \qquad \textbf{3-229}$$

where the pair $\{A_{11}, B_1\}$ is completely controllable. Suppose now that the system is not stabilizable so that A_{22} is not asymptotically stable. Then obviously there exist initial states of the form col $(0, x_{20})$ such that $x(t) \to \infty$ no matter how $u(t)$ is chosen. By the assumed complete reconstructibility, for such initial states

$$\int_{t_0}^{t_1} [z^T(t)R_3z(t) + u^T(t)R_2u(t)]\, dt \tag{3-230}$$

will never converge to a finite number as $t_1 \to \infty$. This proves that $P(t)$ also will not converge to a finite value as $t_1 \to \infty$ if the system **3-213** is not stabilizable. However, if **3-213** is stabilizable, we can always find a feedback law that makes the closed-loop system stable. For this feedback law **3-230** converges to a finite number as $t_1 \to \infty$; this number is an upper bound for the minimal value of the criterion. As in Section 3.4.2, we can argue that the minimal value of **3-230** is a monotonically nondecreasing function of t_1. This proves that the minimal value of **3-230** has a limit as $t_1 \to \infty$, hence that $P(t)$ as solved from **3-215** with $P(t_1) = 0$ has a limit $\bar{P}$ as $t_1 \to \infty$. This terminates the proof of part (a) of the theorem.

We defer the proof of parts (b) and (c) for a moment. Part (d) is easily recognized to be valid. Suppose that the system is not completely reconstructible. Then, as we have seen in the beginning of the proof of (a), when the system is represented in reconstructibility canonical form, and $P_1 = 0$, $P(t)$ can be represented in the form

$$\begin{pmatrix} P_{11}(t) & 0 \\ 0 & 0 \end{pmatrix}, \tag{3-231}$$

which very clearly shows that $\bar{P}$, if it exists, is singular. This proves that if $\bar{P}$ is strictly positive-definite the system must be completely reconstructible. To prove the converse assume that the system is completely reconstructible and that $\bar{P}$ is singular. Then there exists a nonzero initial state such that

$$\int_{t_0}^{\infty} [z^T(t)R_3z(t) + u^T(t)R_2u(t)]\, dt = 0. \tag{3-232}$$

Since $R_3 > 0$ and $R_2 > 0$, this implies that

$$u(t) = 0 \qquad \text{and} \qquad z(t) = 0 \qquad \text{for } t \geq t_0. \tag{3-233}$$

But this would mean that there is a nonzero initial state that causes a zero input response of $z(t)$ that is zero for all t. This is in contradiction to the assumption of complete reconstructibility, and therefore the assumption that $\bar{P}$ is singular is false. This terminates the proof of part (d).

We now consider the proof of part (e). We assume that $\bar{P}$ exists. This means that the system has no unstable, uncontrollable poles that are reconstructible.

We saw in the proof of (a) that in the reconstructibility canonical representation of the system $\bar{P}$ is given in the form

$$\begin{pmatrix} \bar{P}_{11} & 0 \\ 0 & 0 \end{pmatrix}. \tag{3-234}$$

This shows that the steady-state feedback gain matrix is of the form

$$\bar{F} = R_2^{-1}(B_1^T, \quad B_2^T)\begin{pmatrix} \bar{P}_{11} & 0 \\ 0 & 0 \end{pmatrix} = (R_2^{-1}B_1^T\bar{P}_{11}, \quad 0). \tag{3-235}$$

This in turn means that the steady-state feedback gain matrix leaves the unreconstructible part of the system completely untouched, which implies that if the steady-state control law is to make the closed-loop system asymptotically stable, the unreconstructible part of the system must be asymptotically stable, that is, the open-loop system must be detectable. Moreover, if the closed-loop system is to be asymptotically stable, the open-loop system must be stabilizable, otherwise no control law, hence not the steady-state control law either, can make the closed-loop system stable. Thus we see that stabilizability and detectability are necessary conditions for the steady-state control law to be asymptotically stable.

Stabilizability and detectability are also sufficient to guarantee asymptotic stability. We have already seen that the steady-state control law does not affect and is not affected by the unreconstructible part of the system; therefore, if the system is detectable, we may as well omit the unreconstructible part and assume that the system is completely reconstructible. Let us represent the system in controllability canonical form as in **3-229**. Partitioning the matrix $P(t)$ according to the partitioning of **3-229**, we write:

$$P(t) = \begin{pmatrix} P_{11}(t) & P_{12}(t) \\ P_{12}^T(t) & P_{22}(t) \end{pmatrix}. \tag{3-236}$$

It is not difficult to find from the Riccati equation **3-215** that $P_{11}(t)$ is the solution of

$$-\dot{P}_{11}(t) = D_1^TR_3D_1 - P_{11}(t)B_1R_2^{-1}B_1^TP_{11}(t) + A_{11}^TP_{11}(t) + P_{11}(t)A_{11},$$
$$P_{11}(t_1) = 0. \tag{3-237}$$

We see that this is the usual Riccati-type equation. Now since the pair $\{A_{11}, B_1\}$ is completely controllable, we know from Theorem 3.5 that $P_{11}(t)$ has an asymptotic solution $\bar{P}_{11}$ as $t_1 \to \infty$ such that $A_{11} - B_1\bar{F}_1$, where $\bar{F}_1 = R_2^{-1}B_1^T\bar{P}_{11}$, is asymptotically stable. The control law for the whole system **3-229** is given by

$$\bar{F} = (\bar{F}_1, \bar{F}_2) = R_2^{-1}(B_1^T, \quad 0)\begin{pmatrix} \bar{P}_{11} & \bar{P}_{12} \\ \bar{P}_{12}^T & \bar{P}_{22} \end{pmatrix} = (R_2^{-1}B_1^T\bar{P}_{11}, \quad R_2^{-1}B_1^T\bar{P}_{12}). \tag{3-238}$$

With this control law the closed-loop system is described by

$$\dot{x}(t) = \begin{pmatrix} A_{11} - B_1\bar{F}_1 & A_{12} - B_1\bar{F}_2 \\ 0 & A_{22} \end{pmatrix} x(t). \tag{3-239}$$

Clearly, if the open-loop system is stabilizable, the closed-loop system is asymptotically stable since both $A_{11} - B_1\bar{F}_1$ and A_{22} are asymptotically stable. This proves that detectability and stabilizability are sufficient conditions to guarantee that the closed-loop steady-state control law will be asymptotically stable. This terminates the proof of (e).

Consider now part (f) of the theorem. Obviously, the steady-state control law minimizes

$$\int_{t_0}^{\infty} [z^T(t)R_3z(t) + u^T(t)R_2u(t)]\, dt, \tag{3-240}$$

and the minimal value of this criterion is given by $x^T(t_0)\bar{P}x(t_0)$. Let us now consider the criterion

$$\lim_{t_1\to\infty}\left\{\int_{t_0}^{t_1} [z^T(t)R_3z(t) + u^T(t)R_2u(t)]\, dt + x^T(t_1)P_1x(t_1)\right\}, \tag{3-241}$$

with $P_1 \geq 0$. If the system is stabilizable and detectable, for the steady-state control law the criterion **3-241** is equal to

$$\int_{t_0}^{\infty} [\bar{z}^T(t)R_3\bar{z}(t) + \bar{u}^T(t)R_2\bar{u}(t)]\, dt = x^T(t_0)\bar{P}x(t_0), \tag{3-242}$$

where $\bar{z}$ and $\bar{u}$ are the controlled variable and input generated by the steady-state control law. We claim that the steady-state control law not only minimizes **3-240**, but also **3-241**. Suppose that there exists another control law that gives a smaller value of **3-241**, so that for this control law

$$\int_{t_0}^{\infty} [z^T(t)R_3z(t) + u^T(t)R_2u(t)]\, dt + \lim_{t_1\to\infty} x^T(t_1)P_1x(t_1) < x^T(t_0)\bar{P}x(t_0). \tag{3-243}$$

Because $P_1 \geq 0$ this would imply that for this feedback law

$$\int_{t_0}^{\infty} [z^T(t)R_3z(t) + u^T(t)R_2u(t)]\, dt < x_0^T(t_0)\bar{P}x(t_0). \tag{3-244}$$

But since we know that the left-hand side of this expression is minimized by the steady-state control law, and no value of the criterion less than

$x^T(t_0)\bar{P}x(t_0)$ can be achieved, this is a contradiction, which means that **3-241** is also minimized by the steady-state control law. This terminates the proof of part (f).

We now return to part (b) of the theorem. The fact stated in (b) immediately follows from (f). Consider now part (c). In general, the algebraic Riccati equation has many solutions (see Problem 3.8). If $\bar{P}$ exists, it is a nonnegative-definite solution of the algebraic Riccati equation because $\bar{P}$ must be a solution of the Riccati differential equation **3-215**. Suppose that the system **3-213** is stabilizable and detectable, and let P' be any nonnegative-definite solution of the algebraic Riccati equation. Consider the Riccati differential equation **3-215** with the terminal condition $P_1 = P'$. Obviously, the solution of the Riccati equation is $P(t) = P'$, $t \leq t$. Then the steady-state solution $\bar{P}$ must also be given by P'. This proves that any nonnegative-definite solution P' of the algebraic Riccati equation is the steady-state solution $\bar{P}$, hence that the steady-state value $\bar{P}$ is the unique nonnegative-definite solution of the algebraic Riccati equation. This terminates the proof of (c), and also the proof of the whole theorem.

Comments. We conclude this section with the following comments. Parts (b) and (c) state that stabilizability and detectability are sufficient conditions for the Riccati equation to converge to a unique $\bar{P}$ for all $P_1 \geq 0$ and for the algebraic Riccati equation to have a unique nonnegative-definite solution. That these conditions are not necessary can be seen from simple examples.

Furthermore, it may very well happen that although $\bar{P}$ does not exist,

$$\bar{F} = \lim_{t_1 \to \infty} R_2^{-1} B^T P(t) \qquad \textbf{3-245}$$

does exist.

It is not difficult to conclude that the steady-state control law $u(t) = -\bar{F}x(t)$, if it exists, changes only the locations of those open-loop poles that are both controllable and reconstructible. Therefore an unfavorable situation may arise when a system possesses uncontrollable or unreconstructible poles, in particular if these poles are unstable. Unfortunately, it is usually impossible to change the structure of the system so as to make uncontrollable poles controllable. If a system possesses unreconstructible poles with undesirable locations, it is often possible, however, to redefine the controlled variable such that the system no longer has unreconstructible poles.

3.4.4* Solution of the Time-Invariant Regulator Problem by Diagonalization

In this section we further investigate the steady-state solution of the time-invariant regulator problem. This first of all provides us with a method for computing the steady-state solution $\bar{P}$ of the Riccati equation, and moreover puts us into a position to derive information about the closed-loop regulator poles and the closed-loop behavior of the regulator. Throughout the section we assume that the open-loop system is both stabilizable and detectable.

In Section 3.3.2 we saw that the regulator problem can be solved by considering the linear differential equation

$$\begin{pmatrix} \dot{x}(t) \\ \dot{p}(t) \end{pmatrix} = Z \begin{pmatrix} x(t) \\ p(t) \end{pmatrix}, \qquad \textbf{3-246}$$

where Z is the constant matrix

$$Z = \begin{pmatrix} A & -BR_2^{-1}B^T \\ -R_1 & -A^T \end{pmatrix}. \qquad \textbf{3-247}$$

Here $R_1 = D^T R_3 D$. Correspondingly, we have the boundary conditions

$$x(t_0) = x_0, \qquad \textbf{3-248a}$$

$$p(t_1) = P_1 x(t_1). \qquad \textbf{3-248b}$$

From Sections 3.3.2 and 3.3.3 (Eq. **3-92**), we know that $p(t)$ and $x(t)$ are related by

$$p(t) = P(t)x(t), \qquad \textbf{3-249}$$

where $P(t)$ is the solution of the matrix Riccati equation with the terminal condition $P(t_1) = P_1$. Suppose now that we choose

$$P_1 = \bar{P}, \qquad \textbf{3-250}$$

where $\bar{P}$ is the steady-state solution of the Riccati equation. Then the Riccati equation obviously has the solution

$$P(t) = \bar{P}, \qquad t_0 \leq t \leq t_1. \qquad \textbf{3-251}$$

This shows that the steady-state solution can be obtained by replacing the *terminal* condition **3-248b** with the *initial* condition

$$p(t_0) = \bar{P}x(t_0). \qquad \textbf{3-252}$$

Solving the differential equation **3-246** with the initial conditions **3-248a** and **3-252** gives us the steady-state behavior of the state and adjoint variable.

We study the solution of this initial value problem by diagonalization of the matrix Z. It can be shown by elementary determinant manipulations that

$$\det(-sI - Z) = \det(sI - Z). \qquad \textbf{3-253}$$

Consequently, $\det(sI - Z)$ is a polynomial in s^2 which shows that, if λ is a characteristic value of Z, $-\lambda$ is also a characteristic value. Let us for simplicity assume that the characteristic values of Z are all distinct (for the more general case, see Problem 3.9). This allows us to diagonalize Z as follows:

$$Z = W\begin{pmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{pmatrix} W^{-1}. \qquad \textbf{3-254}$$

Here Λ is a diagonal matrix which is constructed as follows. If a characteristic value λ of Z has a strictly positive real part, it is a diagonal element of Λ; $-\lambda$ is automatically placed in $-\Lambda$. If λ has zero real part, one of the pair λ, $-\lambda$ is arbitrarily assigned to Λ and the other to $-\Lambda$. The matrix W is composed of the characteristic vectors of Z; the ith column vector of W is the characteristic vector of Z corresponding to the characteristic value in the ith diagonal position of diag $(\Lambda, -\Lambda)$.

Let us now consider the differential equation

$$\begin{pmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{pmatrix} = \begin{pmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{pmatrix} \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix}, \qquad \textbf{3-255}$$

where

$$\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = W^{-1} \begin{pmatrix} x(t) \\ p(t) \end{pmatrix}. \qquad \textbf{3-256}$$

We partition W^{-1} as follows:

$$W^{-1} = V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}. \qquad \textbf{3-257}$$

Then we can write

$$\begin{aligned} z_1(t) &= V_{11}x(t) + V_{12}p(t) \\ &= (V_{11} + V_{12}\bar{P})x(t). \end{aligned} \qquad \textbf{3-258}$$

We know that the steady-state solution is stable, that is, $x(t) \to 0$ as $t \to \infty$. This also implies that $z_1(t) \to 0$ as $t \to \infty$. From **3-255**, however, we see that

$$z_1(t) = e^{\Lambda(t-t_0)}z_1(t_0). \qquad \textbf{3-259}$$

Since the characteristic values of Λ all have zero or positive real parts, $z_1(t)$ can converge to zero only if $z_1(t_0) = 0$. According to **3-258**, this can be the case for all x_0 if and only if $\bar{P}$ satisfies the relation

$$V_{11} + V_{12}\bar{P} = 0. \qquad \textbf{3-260}$$

If V_{12} is nonsingular, we can solve for $\bar{P}$ as follows:

$$\bar{P} = -V_{12}^{-1}V_{11}. \qquad \textbf{3-261}$$

In any case $\bar{P}$ must satisfy **3-260**. Let us suppose that **3-260** does not have a unique nonnegative-definite solution for $\bar{P}$ and let P' be any nonnegative-definite solution. Consider now the differential equation **3-246** with the terminal condition

$$p(t_1) = P'x(t_1). \tag{3-262}$$

We can write the solution in the form

$$\begin{pmatrix} x(t) \\ p(t) \end{pmatrix} = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \begin{pmatrix} e^{\Lambda(t-t_1)} & 0 \\ 0 & e^{-\Lambda(t-t_1)} \end{pmatrix} \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \begin{pmatrix} x(t_1) \\ p(t_1) \end{pmatrix}, \tag{3-263}$$

where W has also been partitioned. Substitution of **3-262** gives

$$\begin{pmatrix} x(t) \\ p(t) \end{pmatrix} = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \begin{pmatrix} e^{\Lambda(t-t_1)} & 0 \\ 0 & e^{-\Lambda(t-t_1)} \end{pmatrix} \begin{pmatrix} (V_{11} + V_{12}P')x(t_1) \\ (V_{21} + V_{22}P')x(t_1) \end{pmatrix}. \tag{3-264}$$

By using the fact that P' is a solution of **3-260**, this can be further worked out; we obtain

$$x(t) = W_{12}e^{-\Lambda(t-t_1)}(V_{21} + V_{22}P')x(t_1), \tag{3-265a}$$

$$p(t) = W_{22}e^{-\Lambda(t-t_1)}(V_{21} + V_{22}P')x(t_1). \tag{3-265b}$$

For $t = t_0$ the first of these equations reduces to

$$x_0 = W_{12}e^{-\Lambda(t_0-t_1)}(V_{21} + V_{22}P')x(t_1). \tag{3-266}$$

Since the two-point boundary value problem must have a solution for all x_0, the matrix that relates x_0 and $x(t_1)$ must be nonsingular (otherwise this equation would not have a solution if x_0 is not in the range of this matrix). In fact, since any $t \le t_1$ can be considered as the initial time for the interval $[t, t_1]$, the matrix

$$W_{12}e^{-\Lambda(t-t_1)}(V_{21} + V_{22}P') \tag{3-267}$$

must be nonsingular for all $t \le t_1$. Solving **3-265a** for $x(t_1)$ and substituting this into **3-265b** yields

$$p(t) = W_{22}e^{-\Lambda(t-t_1)}(V_{21} + V_{22}P')(V_{21} + V_{22}P')^{-1}e^{\Lambda(t-t_1)}W_{12}^{-1}x(t), \tag{3-268}$$

or (O'Donnell, 1966)

$$p(t) = W_{22}W_{12}^{-1}x(t). \tag{3-269}$$

Apparently, solving the two-point boundary value problem with the terminal condition $P(t_1) = P'$ yields a solution of the form

$$p(t) = \hat{P}x(t), \tag{3-270}$$

where $\hat{P}$ is constant. Since this solution is independent of the terminal time t_1, $\hat{P}$ is also the steady-state solution $\bar{P}$ of the Riccati equation as $t_1 \to \infty$. Since, as we know from Theorem 3.7, this steady-state solution is unique, we cannot but conclude that

$$\bar{P} = W_{22}W_{12}^{-1}. \qquad \textbf{3-271}$$

This argument shows that W_{12} is nonsingular and that $\bar{P}$ can be represented in the form **3-271**. Since the partitioned blocks of V and W have a special relationship, it can also be shown that V_{12} is nonsingular, hence also that **3-261** is a valid expression (Problem 3.12).

In addition to these results, we can obtain the following interesting conclusion. By solving **3-266** for $x(t_1)$ and substituting the result into **3-265a**, we find

$$x(t) = W_{12}e^{-\Lambda(t-t_0)}W_{12}^{-1}x_0. \qquad \textbf{3-272}$$

This shows very explicitly that the characteristic values of the steady-state closed-loop system are precisely the diagonal elements of $-\Lambda$ (O'Donnell, 1966). Since the closed-loop system is known to be asymptotically stable, it follows that the diagonal elements of $-\Lambda$ have strictly negative real parts. Since these characteristic values are obtained from the characteristic values of Z, this means that Z cannot have any characteristic values with zero real parts, and that the steady-state closed-loop characteristic values are precisely those characteristic values of Z that have negative real parts (Letov, 1960).

We summarize these conclusions as follows.

Theorem 3.8. *Consider the time-invariant deterministic linear optimal regulator problem and suppose that the pair $\{A, B\}$ is stabilizable and the pair $\{A, D\}$ detectable. Define the $2n \times 2n$ matrix*

$$Z = \begin{pmatrix} A & -BR_2^{-1}B^T \\ -D^TR_3D & -A^T \end{pmatrix}, \qquad \textbf{3-273}$$

and assume that Z has $2n$ distinct characteristic values. Then

(a) *If λ is a characteristic value of Z, $-\lambda$ also is a characteristic value. Z has no characteristic values with zero real parts.*

(b) *The characteristic values of the steady-state closed-loop optimal regulator are those characteristic values of Z that have negative real parts.*

(c) *If Z is diagonalized in the form*

$$Z = W\begin{pmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{pmatrix}W^{-1}, \qquad \textbf{3-274}$$

where the diagonal matrix Λ has as diagonal elements the characteristic values of Z with positive real parts, the steady-state solution of the Riccati equation **3-215** *can be written as*

$$\bar{P} = W_{22}W_{12}^{-1} = -V_{12}^{-1}V_{11}, \qquad \textbf{3-275}$$

where the W_{ij} and V_{ij}, $i, j = 1, 2$, are obtained by partitioning W and $V = W^{-1}$, respectively. The inverse matrix in both expressions exists.

(d) *The response of the steady-state closed-loop optimal regulator can be written as*

$$x(t) = W_{12} e^{-\Lambda(t-t_0)} W_{12}^{-1} x_0. \qquad \textbf{3-276}$$

The diagonalization approach discussed in this section is further pursued in Problems 3.8 through 3.12.

3.5 NUMERICAL SOLUTION OF THE RICCATI EQUATION

3.5.1 Direct Integration

In this section we discuss various methods for the numerical solution of the Riccati equation, which is of fundamental importance for the regulator problem and, as we see in Chapter 4, also for state reconstruction problems. The matrix Riccati equation is given by

$$-\dot{P}(t) = R_1(t) - P(t)B(t)R_2^{-1}(t)B^T(t)P(t) + A^T(t)P(t) + P(t)A(t), \qquad \textbf{3-277}$$

with the terminal condition

$$P(t_1) = P_1.$$

A direct approach results from considering **3-277** a set of n^2 simultaneous nonlinear first-order differential equations (assuming that $P(t)$ is an $n \times n$ matrix) and using any standard numerical technique to integrate these equations backward from t_1. The most elementary method is Euler's method, where we write

$$P(t - \Delta t) \simeq P(t) - \dot{P}(t)\,\Delta t, \qquad \textbf{3-278}$$

and compute $P(t)$ for $t = t_1 - \Delta t,\ t_1 - 2\Delta t, \cdots$. If the solution converges to a constant value, such as usually occurs in the time-invariant case, some stopping rule is needed. A disadvantage of this approach is that for sufficient accuracy usually a quite small value of Δt is required, which results in a large number of steps. Also, the symmetry of $P(t)$ tends to be destroyed because of numerical errors. This can be remedied by symmetrizing after each step, that is, replacing $P(t)$ with $\frac{1}{2}[P(t) + P^T(t)]$. Alternatively, the symmetry of $P(t)$ can be exploited by reducing **3-277** to a set of $\frac{1}{2}n(n+1)$ simultaneous first-order differential equations, which results in an appreciable saving of computer time. A further discussion of the method of direct integration may be found in Bucy and Joseph (1968).

The method of direct integration is applicable to both the time-varying and the time-invariant case. If only steady-state solutions for time-invariant problems are required, the methods presented in Sections 3.5.3 and 3.5.4 are more effective.

We finally point out the following. In order to realize a time-varying control law, the entire behavior of $F(t)$ for $t_0 \leq t \leq t_1$ must be stored. It seems attractive to circumvent this as follows. By off-line integration $P(t_0)$ can be computed. Then the Riccati equation **3-277** is integrated on-line, with the correct initial value $P(t_0)$, and the feedback gain matrix is obtained, on-line, from $F(t) = R_2^{-1}(t)B^T(t)P(t)$. This method usually leads to unsatisfactory results, however, since in the forward direction the Riccati equation **3-277** is unstable, which causes computational inaccuracies that increase with t (Kalman, 1960).

3.5.2 The Kalman–Englar Method

When a complete solution is required of the *time-invariant* Riccati equation, a convenient approach (Kalman and Englar, 1966) is based upon the following expression, which derives from **3-98**:

$$P(t_{i+1}) = [\Theta_{21}(t_{i+1}, t_i) + \Theta_{22}(t_{i+1}, t_i)P(t_i)][\Theta_{11}(t_{i+1}, t_i) + \Theta_{12}(t_{i+1}, t_i)P(t_i)]^{-1}, \qquad \textbf{3-279}$$

where

$$t_{i+1} = t_i - \Delta t. \qquad \textbf{3-280}$$

The matrices $\Theta_{ij}(t, t_0)$ are obtained by partitioning the transition matrix $\Theta(t, t_0)$ of the system

$$\begin{pmatrix} \dot{x}(t) \\ \dot{p}(t) \end{pmatrix} = Z \begin{pmatrix} x(t) \\ p(t) \end{pmatrix}, \qquad \textbf{3-281}$$

where

$$Z = \begin{pmatrix} A & -BR_2^{-1}B^T \\ -D^TR_3D & -A^T \end{pmatrix}. \qquad \textbf{3-282}$$

We can compute $\Theta(t_{i+1}, t_i)$ once and for all as

$$\Theta(t_{i+1}, t_i) = e^{-Z\Delta t}, \qquad \textbf{3-283}$$

which can be evaluated according to the power series method of Section 1.3.2. The solution of the Riccati equation is then found by repeated application of **3-279**. It is advantageous to symmetrize after each step.

Numerical difficulties occur when Δt is chosen too large. Vaughan (1969) discusses these difficulties in some detail. They manifest themselves in near-singularity of the matrix to be inverted in **3-279**. It has been shown by Vaughan that a very small Δt is required when the real parts of the characteristic values of Z have a large spread. For most problems there exists a Δt small enough to obtain accurate results. Long computing times may result, however, especially when the main interest is in the steady-state solution.

3.5.3* Solution by Diagonalization

In order to obtain the steady-state solution of the time-invariant Riccati equation, the results derived in Section 3.4.4 by diagonalizing the $2n \times 2n$ matrix Z are useful. Here the asymptotic solution is expressed as

$$\bar{P} = W_{22}W_{12}^{-1}, \tag{3-284}$$

where W_{22} and W_{12} are obtained by partitioning a matrix W as follows:

$$W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}. \tag{3-285}$$

The matrix W consists of the characteristic vectors of the matrix Z so arranged that the first n columns of W correspond to the characteristic values of Z with positive real parts, and the last n columns of W to the characteristic values of Z with negative real parts.

Generally, some or all of the characteristic vectors of Z may be complex so that W_{22} and W_{12} may be complex matrices. Complex arithmetic can be avoided as follows. Since if e is a characteristic vector of Z corresponding to a characteristic value λ with negative real part, its complex conjugate $\bar{e}$ is also a characteristic vector corresponding to a characteristic value $\bar{\lambda}$ with a negative real part, the last n columns of W will contain besides real column vectors only complex conjugate pairs of column vectors. Then it is always possible to perform a nonsingular linear transformation

$$\begin{pmatrix} W_{12}' \\ W_{22}' \end{pmatrix} = \begin{pmatrix} W_{12} \\ W_{22} \end{pmatrix} U, \tag{3-286}$$

such that every pair of complex conjugate column vectors e and $\bar{e}$ in col (W_{12}, W_{22}) is replaced with two real vectors Re (e) and Im (e) in col (W_{12}', W_{22}'). Then

$$W_{22}'W_{12}'^{-1} = (W_{22}U)(W_{12}U)^{-1} = W_{22}W_{12}^{-1}, \tag{3-287}$$

which shows that W_{22}' and W_{12}' can be used to compute $\bar{P}$ instead of W_{22} and W_{12}.

Let us summarize this method of obtaining $\bar{P}$:

(a) Form the matrix Z and use any standard numerical technique to compute those characteristic vectors that correspond to characteristic values witb negative real parts.

(b) Form from these n characteristic vectors a $2n \times n$ matrix

$$\begin{pmatrix} W_{12}' \\ W_{22}' \end{pmatrix}, \tag{3-288}$$

where W'_{21} and W'_{22} are $n \times n$ submatrices, as follows. If e is a real characteristic vector, let e be one of the columns of **3-288**. If e and $\bar{e}$ form a complex conjugate pair, let Re (e) be one column of **3-288** and Im (e) another.

(c) Compute $\bar{P}$ as

$$\bar{P} = W'_{22}W'^{-1}_{12}. \qquad \textbf{3-289}$$

The efficiency of this method depends upon the efficiency of the subprogram that computes the characteristic vectors of Z. Van Ness (1969) has suggested a characteristic vector algorithm that is especially suitable for problems of this type. The algorithm as outlined above has been successfully applied for solving high-order Riccati equations (Freestedt, Webber, and Bass, 1968; Blackburn and Bidwell, 1968; Hendricks and Haynes, 1968). Fath (1969) presents a useful modification of the method.

The diagonalization approach can also be employed to obtain not only the asymptotic solution of the Riccati equation but the complete behavior of $P(t)$ by the formulas of Problem 3.11.

A different method for computing the asymptotic solution $\bar{P}$ is to use the identity (see Problem 3.10)

$$\phi(Z)\begin{pmatrix} I \\ \bar{P} \end{pmatrix} = 0, \qquad \textbf{3-290}$$

where $\phi(s)$ is obtained by factoring

$$\det\,(sI - Z) = \phi(s)\phi(-s), \qquad \textbf{3-291}$$

such that the roots of $\phi(s)$ are precisely the characteristic values of Z with negative real parts. Clearly, $\phi(s)$ is the characteristic polynomial of the steady-state closed-loop optimal system. Here det $(sI - Z)$ can be obtained by the Leverrier algorithm of Section 1.5.1, or by any standard technique for obtaining characteristic values of matrices. Both favorable (Freestedt, Webber, and Bass, 1968) and unfavorable (Blackburn and Bidwell, 1968; Hendricks and Haynes, 1968) experiences with this method have been reported.

3.5.4* Solution by the Newton–Raphson Method

In this subsection a method is discussed for computing the steady-state solution of the time-invariant Riccati equation, which is quite different from the previous methods. It is based upon repeated solution of a linear matrix equation of the type

$$0 = A^TP + PA + R, \qquad \textbf{3-292}$$

which has been discussed in Section 1.11.3.

The steady-state solution $\bar{P}$ of the Riccati equation must satisfy the algebraic Riccati equation

$$0 = R_1 - PSP + A^T P + PA, \tag{3-293}$$

where

$$S = BR_2^{-1}B^T. \tag{3-294}$$

Consider the matrix function

$$F(P) = R_1 - PSP + A^T P + PA. \tag{3-295}$$

The problem is to find the nonnegative-definite symmetric matrix $\bar{P}$ that satisfies

$$F(\bar{P}) = 0. \tag{3-296}$$

We derive an iterative procedure. Suppose that at the k-th stage a solution P_k has been obtained, which is not much different from the desired solution $\bar{P}$, and let us write

$$\bar{P} = P_k + \tilde{P}. \tag{3-297}$$

If $\tilde{P}$ is small we can approximate $F(\bar{P})$ by omitting quadratic terms in $\tilde{P}$ and we obtain

$$F(\bar{P}) \simeq R_1 - P_k S P_k - P_k S \tilde{P} - \tilde{P} S P_k - A^T(P_k + \tilde{P}) + (P_k + \tilde{P})A. \tag{3-298}$$

The basic idea of the Newton–Raphson method is to estimate $\tilde{P}$ by setting the right-hand side of **3-298** equal to zero. If the estimate of $\tilde{P}$ so obtained is denoted as $\tilde{P}_k$ and we let

$$P_{k+1} = P_k + \tilde{P}_k, \tag{3-299}$$

then we find by setting the right-hand side of **3-298** equal to zero:

$$0 = R_1 + P_k S P_k + P_{k+1} A_k + A_k{}^T P_{k+1}, \tag{3-300}$$

where

$$A_k = A - SP_k. \tag{3-301}$$

Equation **3-300** is of the type **3-292**, for which efficient methods of solution exist (see Section 1.11.3). We have thus obtained the following algorithm.

(a) Choose a suitable P_0 and set the iteration index k equal to 0.
(b) Solve P_{k+1} from **3-300**.
(c) If convergence is obtained, stop; otherwise, increase k by one and return to (b).

Kleinman (1968) and McClamroch (1969) have shown that if the algebraic Riccati equation has a unique nonnegative-definite solution, P_k and P_{k+1} satisfy

$$P_{k+1} \leq P_k, \qquad k = 0, 1, 2, \ldots, \tag{3-302}$$

and

$$\lim_{k\to\infty} P_k = \bar{P}, \tag{3-303}$$

provided P_0 is so chosen that

$$A_0 = A - SP_0 \tag{3-304}$$

is asymptotically stable. This means that the convergence of the scheme is assured if the initial estimate is suitably chosen. If the initial estimate is incorrectly selected, however, convergence to a different solution of the algebraic Riccati equation may occur, or no convergence at all may result. If A is asymptotically stable, a safe choice is $P_0 = 0$. If A is not asymptotically stable, the initial choice may present difficulties. Wonham and Cashman (1968), Man and Smith (1969), and Kleinman (1970b) give methods for selecting P_0 when A is not asymptotically stable.

The main problem with this approach is **3-292**, which must be solved many times over. Although it is linear, the numerical effort may still be rather formidable, since the number of linear equations that must be solved at each iteration increases rapidly with the dimension of the problem (for $n = 15$ this number is 120). In Section 1.11.3 several numerical approaches to solving **3-292** are referenced. In the literature favorable experiences using the Newton–Raphson method to solve Riccati equations has been reported with up to 15-dimensional problems (Blackburn, 1968; Kleinman, 1968, 1970a).

3.6 STOCHASTIC LINEAR OPTIMAL REGULATOR AND TRACKING PROBLEMS

3.6.1 Regulator Problems with Disturbances—The Stochastic Regulator Problem

In the preceding sections we discussed the deterministic linear optimal regulator problem. The solution of this problem allows us to tackle purely transient problems where a linear system has a disturbed initial state, and it is required to return the system to the zero state as quickly as possible while limiting the input amplitude. There exist practical problems that can be formulated in this manner, but much more common are problems where there are disturbances that act uninterruptedly upon the system, and that tend to drive the state away from the zero state. The problem is then to design a feedback configuration through which initial offsets are reduced as quickly as possible, but which also counteracts the effects of disturbances as much as possible in the steady-state situation. The solution of this problem will bring us into a position to synthesize the controllers that have been asked for in

Chapter 2. For the time being we maintain the assumption that the complete state of the system can be accurately observed at each instant of time.

The effect of the disturbances can be accounted for by suitably extending the system description. We consider systems described by

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) + v(t), \\ z(t) &= D(t)x(t), \end{aligned} \tag{3-305}$$

where $u(t)$ is the input variable, $z(t)$ is the controlled variable, and $v(t)$ represents disturbances that act upon the system. We mathematically represent the disturbances as a stochastic process, which we model as the output of a linear system driven by white noise. Thus we assume that $v(t)$ is given by

$$v(t) = D_d(t)x_d(t). \tag{3-306}$$

Here $x_d(t)$ is the solution of

$$\dot{x}_d(t) = A_d(t)x_d(t) + w(t), \tag{3-307}$$

where $w(t)$ is white noise. We furthermore assume that both $x(t_0)$ and $x_d(t_0)$ are stochastic variables.

We combine the description of the system and the disturbances by defining an augmented state vector $\tilde{x}(t) = \text{col}\,[x(t), x_d(t)]$, which from **3-305**, **3-306**, and **3-307** can be seen to satisfy

$$\dot{\tilde{x}}(t) = \begin{pmatrix} A(t) & D_d(t) \\ 0 & A_d(t) \end{pmatrix}\tilde{x}(t) + \begin{pmatrix} B(t) \\ 0 \end{pmatrix}u(t) + \begin{pmatrix} 0 \\ w(t) \end{pmatrix}. \tag{3-308}$$

In terms of the augmented state, the controlled variable is given by

$$z(t) = (D(t), 0)\tilde{x}(t). \tag{3-309}$$

We note in passing that **3-308** represents a system that is not completely controllable (from u).

We now turn our attention to the optimization criterion. In the deterministic regulator problem, we considered the quadratic integral criterion

$$\int_{t_0}^{t_1} [z^T(t)R_3(t)z(t) + u^T(t)R_2(t)u(t)]\,dt + x^T(t_1)P_1x(t_1). \tag{3-310}$$

For a given input $u(t)$, $t_0 \le t \le t_1$, and a given realization of the disturbances $v(t)$, $t_0 \le t \le t_1$, this criterion is a measure for the deviations $z(t)$ and $u(t)$ from zero. *A priori*, however, this criterion cannot be evaluated because of the stochastic nature of the disturbances. We therefore average over all possible realizations of the disturbances and consider the criterion

$$E\left\{\int_{t_0}^{t_1} [z^T(t)R_3(t)z(t) + u^T(t)R_2(t)u(t)]\,dt + x^T(t_1)P_1x(t_1)\right\}. \tag{3-311}$$

In terms of the augmented state $\tilde{x}(t) = \text{col}\,[x(t), x_d(t)]$, this criterion can be expressed as

$$E\left\{\int_{t_0}^{t_1}[z^T(t)R_3(t)z(t) + u^T(t)R_2(t)u(t)]\,dt + \tilde{x}^T(t_1)\tilde{P}_1\tilde{x}(t)\right\}, \qquad \textbf{3-312}$$

where

$$\tilde{P}_1 = \begin{pmatrix} P_1 & 0 \\ 0 & 0 \end{pmatrix}. \qquad \textbf{3-313}$$

It is obvious that the problem of minimizing **3-312** for the system **3-308** is nothing but a special case of the general problem of minimizing

$$E\left\{\int_{t_0}^{t_1}[z^T(t)R_3(t)z(t) + u^T(t)R_2(t)u(t)]\,dt + x^T(t_1)P_1x(t_1)\right\} \qquad \textbf{3-314}$$

for the system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + w(t), \qquad \textbf{3-315}$$

where $w(t)$ is white noise and where $x(t_0)$ is a stochastic variable. We refer to this problem as the stochastic linear optimal regulator problem:

Definition 3.4. *Consider the system described by the state differential equation*

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + w(t) \qquad \textbf{3-316}$$

with initial state

$$x(t_0) = x_0 \qquad \textbf{3-317}$$

and controlled variable

$$z(t) = D(t)x(t). \qquad \textbf{3-318}$$

In **3-316** *$w(t)$ is white noise with intensity $V(t)$. The initial state x_0 is a stochastic variable, independent of the white noise w, with*

$$E\{x_0x_0^T\} = Q_0. \qquad \textbf{3-319}$$

Consider the criterion

$$E\left\{\int_{t_0}^{t_1}[z^T(t)R_3(t)z(t) + u^T(t)R_2(t)u(t)]\,dt + x^T(t_1)P_1x(t_1)\right\}, \qquad \textbf{3-320}$$

where $R_3(t)$ and $R_2(t)$ are positive-definite symmetric matrices for $t_0 \leq t \leq t_1$ and P_1 is nonnegative-definite symmetric. Then the problem of determining for each t, $t_0 \leq t \leq t_1$, the input $u(t)$ as a function of all information from the past such that the criterion is minimized is called the ***stochastic linear optimal regulator problem.*** *If all matrices in the problem formulation are constant, we refer to it as the* ***time-invariant stochastic linear optimal regulator problem.***

The solution of this problem is discussed in Section 3.6.3.

Example 3.11. *Stirred tank*

In Example 1.37 (Section 1.11.4), we considered an extension of the model of the stirred tank where disturbances in the form of fluctuations in the concentrations of the feeds are incorporated. The extended system model is given by

$$\dot{x}(t) = \begin{pmatrix} -\frac{1}{2\theta} & 0 & 0 & 0 \\ 0 & -\frac{1}{\theta} & \frac{F_{10}}{V_0} & \frac{F_{20}}{V_0} \\ 0 & 0 & -\frac{1}{\theta_1} & 0 \\ 0 & 0 & 0 & -\frac{1}{\theta_2} \end{pmatrix} x(t) + \begin{pmatrix} 1 & 1 \\ \frac{c_{10} - c_0}{V_0} & \frac{c_{20} - c_0}{V_0} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} u(t) + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} w(t), \qquad \text{3-321}$$

where $w(t)$ is white noise with intensity

$$V = \begin{pmatrix} \frac{2\sigma_1^2}{\theta_1} & 0 \\ 0 & \frac{2\sigma_2^2}{\theta_2} \end{pmatrix}. \qquad \text{3-322}$$

Here the components of the state are, respectively, the incremental volume of fluid, the incremental concentration in the tank, the incremental concentration of the feed F_1, and the incremental concentration of the feed F_2. Let us consider as previously the incremental outgoing flow and the incremental outgoing concentration as the components of the controlled variable. Thus we have

$$z(t) = \begin{pmatrix} \frac{1}{2\theta} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} x(t). \qquad \text{3-323}$$

The stochastic optimal regulator problem now consists in determining the input $u(t)$ such that a criterion of the form

$$E\left\{\int_{t_0}^{t_1} [z^T(t)R_3z(t) + u^T(t)R_2u(t)]\,dt + x^T(t_1)P_1x(t_1)\right\} \qquad \text{3-324}$$

is minimized. We select the weighting matrices R_3 and R_2 in exactly the same manner as in Example 3.9 (Section 3.4.1), while we choose P_1 to be the zero matrix.

3.6.2 Stochastic Tracking Problems

We have introduced the stochastic optimal regulator problem by considering regulator problems with disturbances. Stochastic regulator problems also arise when we formulate *stochastic optimal tracking problems.* Consider the linear system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \qquad \textbf{3-325}$$

with the controlled variable

$$z(t) = D(t)x(t). \qquad \textbf{3-326}$$

Suppose we wish the controlled variable to follow as closely as possible a *reference variable* $z_r(t)$ which we model as the output of a linear differential system driven by white noise:

$$z_r(t) = D_r(t)x_r(t), \qquad \textbf{3-327}$$

with

$$\dot{x}_r(t) = A_r(t)x_r(t) + w(t). \qquad \textbf{3-328}$$

Here $w(t)$ is white noise with given intensity $V(t)$. The system equations and the reference model equations can be combined by defining the augmented state $\tilde{x}(t) = \text{col}\,[x(t), x_r(t)]$, which satisfies

$$\dot{\tilde{x}}(t) = \begin{pmatrix} A(t) & 0 \\ 0 & A_r(t) \end{pmatrix}\tilde{x}(t) + \begin{pmatrix} B(t) \\ 0 \end{pmatrix}u(t) + \begin{pmatrix} 0 \\ w(t) \end{pmatrix}. \qquad \textbf{3-329}$$

In passing, we note that this system (just as that of **3-308**) is not completely controllable from u.

To obtain an *optimal* tracking system, we consider the criterion

$$E\left\{\int_{t_0}^{t_1}\{[z(t) - z_r(t)]^T R_3(t)[z(t) - z_r(t)] + u^T(t)R_2(t)u(t)\}\,dt\right\}, \qquad \textbf{3-330}$$

where $R_3(t)$ and $R_2(t)$ are suitable weighting matrices. This criterion expresses that the controlled variable should be close to the reference variable, while the input amplitudes should be restricted. In fact, for $R_3(t) = W_e(t)$ and $R_2(t) = \rho W_u(t)$, the criterion reduces to

$$\int_{t_0}^{t_1}[C_e(t) + \rho C_u(t)]\,dt, \qquad \textbf{3-331}$$

where $C_e(t)$ and $C_u(t)$ denote the mean square tracking error and the mean square input, respectively, as defined in Chapter 2 (Section 2.3):

$$C_e(t) = E\{e^T(t)W_e(t)e(t)\},$$
$$C_u(t) = E\{u^T(t)W_u(t)u(t)\}. \qquad 3\text{-}332$$

Here $e(t)$ is the tracking error

$$e(t) = z(t) - z_r(t). \qquad 3\text{-}333$$

The weighting coefficient ρ must be adjusted so as to obtain the smallest possible mean square tracking error for a given value of the mean square input.

The criterion **3-330** can be expressed in terms of the augmented state $\tilde{x}(t)$ as follows:

$$E\left\{\int_{t_0}^{t_1} [\tilde{z}^T(t)R_3(t)\tilde{z}(t) + u^T(t)R_2(t)u(t)]\, dt\right\}, \qquad 3\text{-}334$$

where

$$\tilde{z}(t) = (D(t), -D_r(t))\tilde{x}(t). \qquad 3\text{-}335$$

Obviously, the problem of minimizing the criterion **3-334** for the system **3-329** is a special case of the stochastic linear optimal regulator problem of Definition 3.4.

Without going into detail we point out that tracking problems with disturbances also can be converted into stochastic regulator problems by the state augmentation technique.

In conclusion, we note that the approach of this subsection is entirely in line with the approach of Chapter 2, where we represented reference variables as having a variable part and a constant part. In the present section we have set the constant part equal to zero; in Section 3.7.1 we deal with nonzero constant references.

Example 3.12. *Angular velocity tracking system*

Consider the angular velocity control system of Example 3.3 (Section 3.3.1). Suppose we wish that the angular velocity, which is the controlled variable $\zeta(t)$, follows as accurately as possible a reference variable $\zeta_r(t)$, which may be described as exponentially correlated noise with time constant θ and rms value σ. Then we can model the reference process as (see Example 1.36, Section 1.11.4)

$$\zeta_r(t) = \xi_r(t), \qquad 3\text{-}336$$

where $\xi_r(t)$ is the solution of

$$\dot{\xi}_r(t) = -\frac{1}{\theta}\,\xi_r(t) + w(t). \qquad 3\text{-}337$$

The white noise $w(t)$ has intensity $2\sigma^2/\theta$. Since the system state differential equation is

$$\dot{\xi}(t) = -\alpha\xi(t) + \kappa\mu(t), \qquad 3\text{-}338$$

the augmented state differential equation is given by

$$\dot{\tilde{x}}(t) = \begin{pmatrix} -\alpha & 0 \\ 0 & -\dfrac{1}{\theta} \end{pmatrix} \tilde{x}(t) + \begin{pmatrix} \kappa \\ 0 \end{pmatrix} \mu(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} w(t), \qquad \textbf{3-339}$$

with $\tilde{x}(t) = \text{col}\,[\xi(t), \xi_r(t)]$. For the optimization criterion we choose

$$E\left\{\int_{t_0}^{t_1} [[\zeta(t) - \zeta_r(t)]^2 + \rho\mu^2(t)]\, dt\right\}, \qquad \textbf{3-340}$$

where ρ is a suitable weighting factor. This criterion can be rewritten as

$$E\left\{\int_{t_0}^{t_1} [\tilde{\zeta}^2(t) + \rho\mu^2(t)]\, dt\right\}, \qquad \textbf{3-341}$$

where

$$\tilde{\zeta}(t) = (1, -1)\tilde{x}(t). \qquad \textbf{3-342}$$

The problem of minimizing **3-341** for the system described by **3-339** and **3-342** constitutes a stochastic optimal regulator problem.

3.6.3 *Solution of the Stochastic Linear Optimal Regulator Problem*

In Section 3.6.1 we formulated the stochastic linear optimal regulator problem. This problem (Definition 3.4) exhibits an essential difference from the deterministic regulator problem because the white noise makes it impossible to predict exactly how the system is going to behave. Because of this, the best policy is obviously not to determine the input $u(t)$ over the control period $[t_0, t_1]$ *a priori*, but to reconsider the situation at each intermediate instant t on the basis of all available information.

At the instant t the further behavior of the system is entirely determined by the present state $x(t)$, the input $u(\tau)$ for $\tau \geq t$, and the white noise $w(\tau)$ for $\tau \geq t$. All the information from the past that is relevant for the future is contained in the state $x(t)$. Therefore we consider control laws of the form

$$u(t) = g[x(t), t], \qquad \textbf{3-343}$$

which prescribe an input corresponding to each possible value of the state at time t.

The use of such control laws presupposes that each component of the state can be accurately measured at all times. As we have pointed out before, this is an unrealistic assumption. This is even more so in the stochastic case where the state in general includes components that describe the disturbances or the reference variable; it is very unlikely that these components can be easily measured. We postpone the solution of this difficulty until after

Chapter 4, however, where the reconstruction of the state from incomplete and inaccurate measurements is discussed.

In preceding sections we have obtained the solution of the deterministic regulator problem in the feedback form **3-343**. For the stochastic version of the problem, we have the surprising result that the presence of the white noise term $w(t)$ in the system equation **3-316** does not alter the solution except to increase the minimal value of the criterion. We first state this fact and then discuss its proof:

Theorem 3.9. *The optimal* ***linear*** *solution of the stochastic linear optimal regulator problem is to choose the input according to the linear control law*

$$u(t) = -F^0(t)x(t), \qquad \textbf{3-344}$$

where

$$F^0(t) = R_2^{-1}(t)B^T(t)P(t). \qquad \textbf{3-345}$$

Here $P(t)$ is the solution of the matrix Riccati equation

$$-\dot{P}(t) = R_1(t) - P(t)B(t)R_2^{-1}(t)B^T(t)P(t) + A^T(t)P(t) + P(t)A(t) \qquad \textbf{3-346}$$

with the terminal condition

$$P(t_1) = P_1. \qquad \textbf{3-347}$$

Here we abbreviate as usual

$$R_1(t) = D^T(t)R_3(t)D(t). \qquad \textbf{3-348}$$

The minimal value of the criterion is given by

$$\operatorname{tr}\left[P(t_0)Q_0 + \int_{t_0}^{t_1} P(t)V(t)\,dt\right]. \qquad \textbf{3-349}$$

It is observed that this theorem gives only the best *linear* solution of the stochastic regulator problem. Since we limit ourselves to linear systems, this is quite satisfactory. It can be proved, however, that the linear feedback law is optimal (without qualification) when the white noise $w(t)$ is Gaussian (Kushner, 1967, 1971; Åström, 1970).

To prove the theorem let us suppose that the system is controlled through the linear control law

$$u(t) = -F(t)x(t). \qquad \textbf{3-350}$$

Then the closed-loop system is described by the differential equation

$$\dot{x}(t) = [A(t) - B(t)F(t)]x(t) + w(t) \qquad \textbf{3-351}$$

and we can write for the criterion **3-320**

$$E\left\{\int_{t_0}^{t_1} x^T(t)[R_1(t) + F^T(t)R_2(t)F(t)]x(t)\,dt + x^T(t_1)P_1x(t_1)\right\}. \qquad \textbf{3-352}$$

We know from Theorem 1.54 (Section 1.11.5) that the criterion can be expressed as

$$\operatorname{tr}\left[\bar{P}(t_0)Q_0 + \int_{t_0}^{t_1} \bar{P}(t)V(t)\,dt\right], \tag{3-353}$$

where $\bar{P}(t)$ is the solution of the matrix differential equation

$$-\dot{\bar{P}}(t) = [A(t) - B(t)F(t)]^T\bar{P}(t) + \bar{P}(t)[A(t) - B(t)F(t)] + R_1(t) + F^T(t)R_2(t)F(t), \tag{3-354}$$

with the terminal condition

$$\bar{P}(t_1) = P_1. \tag{3-355}$$

Now Lemma 3.1 (Section 3.3.3) states that $\bar{P}(t)$ satisfies the inequality

$$\bar{P}(t) \geq P(t) \tag{3-356}$$

for all $t_0 \leq t \leq t_1$, where $P(t)$ is the solution of the Riccati equation **3-346** with the terminal condition **3-347**. The inequality **3-356** converts into an equality if F is chosen as

$$F^0(\tau) = R_2^{-1}(\tau)B^T(\tau)P(\tau), \qquad t \leq \tau \leq t_1. \tag{3-357}$$

The inequality **3-356** implies that

$$\operatorname{tr}\,[\bar{P}(t)\Gamma] \geq \operatorname{tr}\,[P(t)\Gamma] \tag{3-358}$$

for any nonnegative-definite matrix Γ. This shows very clearly that **3-353** is minimized by choosing F according to **3-357**. For this choice of F, the criterion **3-353** is given by **3-349**. This terminates the proof that the control law **3-345** is the optimal linear control law.

Theorem 3.9 puts us into a position to solve various types of problems. In Sections 3.6.1 and 3.6.2, we showed that the stochastic linear optimal regulator problem may originate from regulator problems for disturbed systems, or from optimal tracking problems. In both cases the problem has a special structure. We now briefly discuss the properties of the solutions that result from these special structures.

In the case of a regulator with disturbances, the system state differential and output equations take the partitioned form **3-308, 3-309**. Suppose that we partition the solution $P(t)$ of the Riccati equation **3-346** according to the partitioning $\tilde{x}(t) = \operatorname{col}\,[x(t), x_d(t)]$ as

$$P(t) = \begin{pmatrix} P_{11}(t) & P_{12}(t) \\ P_{12}^T(t) & P_{22}(t) \end{pmatrix}. \tag{3-359}$$

If, accordingly, the optimal feedback gain matrix is partitioned as

$$F^0(t) = (F_1(t), F_2(t)), \tag{3-360}$$

it is not difficult to see that

$$
\begin{aligned}
F_1(t) &= R_2^{-1}(t)B^T(t)P_{11}(t), \\
F_2(t) &= R_2^{-1}(t)B^T(t)P_{12}(t).
\end{aligned}
\qquad \text{3-361}
$$

Furthermore, it can be found by partitioning the Riccati equation that P_{11}, P_{12}, and P_{22} are the solutions of the matrix differential equations

$$
\begin{cases}
-\dot{P}_{11}(t) = D^T(t)R_3(t)D(t) - P_{11}(t)B(t)R_2^{-1}(t)B^T(t)P_{11}(t) \\
\qquad\qquad + A^T(t)P_{11}(t) + P_{11}(t)A(t), \\
P_{11}(t) = P_1,
\end{cases}
\qquad \text{3-362}
$$

$$
\begin{cases}
-\dot{P}_{12}(t) = P_{11}(t)D_d(t) + [A(t) - B(t)F_1(t)]^T P_{12}(t) + P_{12}(t)A_d(t), \\
P_{12}(t) = 0,
\end{cases}
\qquad \text{3-363}
$$

$$
\begin{cases}
-\dot{P}_{22}(t) = -P_{12}^T(t)B(t)R_2^{-1}(t)B^T(t)P_{12}(t) + D_d^T(t)P_{12}(t) + P_{12}^T(t)D_d(t) \\
\qquad\qquad + A_d^T(t)P_{22}(t) + P_{22}(t)A_d(t), \\
P_{22}(t_1) = 0.
\end{cases}
\qquad \text{3-364}
$$

We observe that P_{11}, and therefore also F_1, is completely independent of the properties of the disturbances, and is in fact obtained by solving the deterministic regulator problem with the disturbances omitted. Once P_{11} and F_1 have been found, **3-363** can be solved to determine P_{12} and from this F_2. The control system structure is given in Fig. 3.14. Apparently, the *feedback link*,

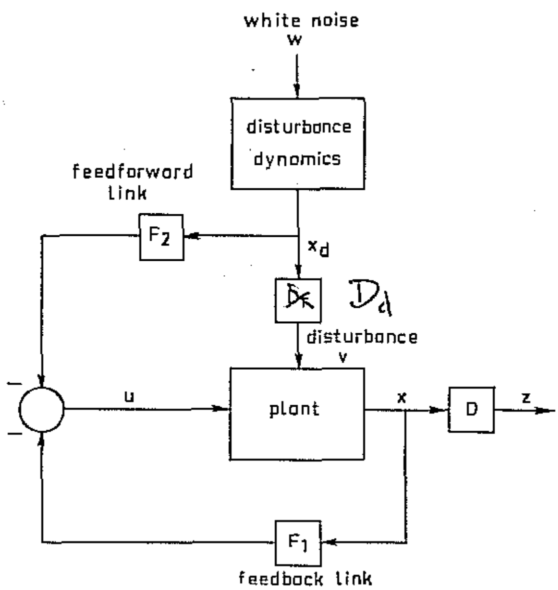


Fig. 3.14. Structure of the optimal state feedback regulator with disturbances.

that is, the link from the state x to the input u *is completely independent of the properties of the disturbances*. The *feedforward link*, that is, the link from the state of the disturbances x_d to the input u, is of course dependent upon the properties of the disturbances.

A similar conclusion can be reached for optimal tracking problems. Here it turns out that with the structures **3-329** and **3-335** of the state differential and output equations the feedback gain matrix can be partitioned as

$$F^0(t) = (F_1(t), -F_2(t)) \tag{3-365}$$

(note the minus sign that has been introduced), where

$$\begin{aligned} F_1(t) &= R_2^{-1}(t)B^T(t)P_{11}(t), \\ F_2(t) &= -R_2^{-1}(t)B^T(t)P_{12}(t). \end{aligned} \tag{3-366}$$

Here the matrices P_{11}, P_{12}, and P_{22} are obtained by partitioning the matrix P according to the partitioning $\tilde{x}(t) = \text{col}\,[x(t), x_r(t)]$; they satisfy the matrix differential equations

$$\begin{cases} -\dot{P}_{11}(t) = D^T(t)R_3(t)D(t) - P_{11}(t)B(t)R_2^{-1}(t)B^T(t)P_{11}(t) \\ \qquad\qquad + A^T(t)P_{11}(t) + P_{11}(t)A(t), \\ P(t_1) = 0, \end{cases} \tag{3-367}$$

$$\begin{cases} -\dot{P}_{12}(t) = -D^T(t)R_3(t)D_r(t) + [A(t) - B(t)F_1(t)]^T P_{12}(t) \\ \qquad\qquad + P_{12}(t)A_r(t), \\ P_{12}(t_1) = 0, \end{cases} \tag{3-368}$$

$$\begin{cases} -\dot{P}_{22}(t) = D_r^T(t)R_3(t)D_r(t) - P_{12}^T(t)B(t)R_2^{-1}(t)B^T(t)P_{12}(t) \\ \qquad\qquad + A_r^T(t)P_{22}(t) + P_{22}(t)A_r(t), \\ P_{22}(t_1) = 0. \end{cases} \tag{3-369}$$

We conclude that for the optimal tracking system as well *the feedback link is independent of the properties of the reference variable*, while the *feedforward link* is of course influenced by the properties of the reference variable. A schematic representation of the optimal tracking system is given in Fig. 3.15.

Let us now return to the general stochastic optimal regulator problem. In practice we are usually confronted with control periods that are very long, which means that we are interested in the case where $t_1 \to \infty$. In the deterministic regulator problem, we saw that normally the Riccati equation **3-346** has a steady-state solution $\bar{P}(t)$ as $t_1 \to \infty$, and that the corresponding steady-state control law $\bar{F}(t)$ is optimal for half-infinite control periods. It is not difficult to conjecture (Kushner, 1971) that the steady-state control law

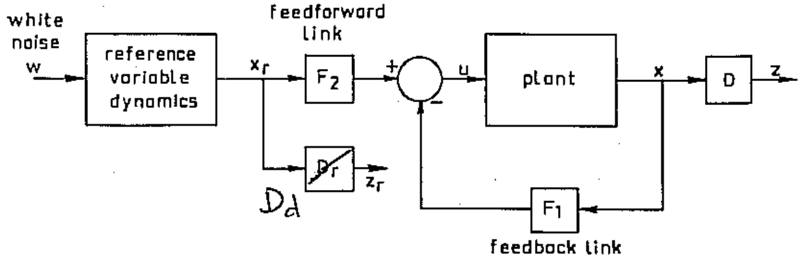


Fig. 3.15. Structure of the optimal state feedback tracking system.

is optimal for the stochastic regulator in the sense that it minimizes

$$\lim_{t_1\to\infty}\frac{1}{t_1-t_0}E\left\{\int_{t_0}^{t_1}[z^T(t)R_3(t)z(t)+u^T(t)R_2(t)u(t)]\,dt\right\}, \qquad \textbf{3-370}$$

if this expression exists for the steady-state control law, with respect to all other control laws for which **3-370** exists. For the steady-state optimal control law, the criterion **3-370** is given by

$$\lim_{t_1\to\infty}\frac{1}{t_1-t_0}\int_{t_0}^{t_1}\text{tr}\,[\bar{P}(t)V(t)]\,dt, \qquad \textbf{3-371}$$

if it exists (compare **3-349**). Moreover, it is recognized that for a time-invariant stochastic regulator problem and an asymptotically stable time-invariant control law the expression **3-370** is equal to

$$\lim_{t\to\infty}E\{z^T(t)R_3z(t)+u^T(t)R_2u(t)\}. \qquad \textbf{3-372}$$

From this it immediately follows that the steady-state optimal control law minimizes **3-372** with respect to all other time-invariant control laws. We see from **3-371** that the minimal value of **3-372** is given by

$$\text{tr}\,(\bar{P}V). \qquad \textbf{3-373}$$

We observe that if $R_3 = W_e$ and $R_2 = \rho W_u$, where W_e and W_u are the weighting matrices in the mean square tracking error and the mean square input (as introduced in Section 2.5.1), the expression **3-372** is precisely

$$C_{e\infty} + \rho C_{u\infty}. \qquad \textbf{3-374}$$

Here $C_{e\infty}$ is the steady-state mean square tracking error and $C_{u\infty}$ the steady-state mean square input. To compute $C_{e\infty}$ and $C_{u\infty}$ *separately*, as usually is required, it is necessary to set up the complete closed-loop system equations and derive from these the differential equation for the variance matrix of the state. From this variance matrix all mean square quantities of interest can be obtained.

Example 3.13. *Stirred tank regulator*

In Example 3.11 we described a stochastic regulator problem arising from the stirred tank problem. Let us, in addition to the numerical values of Example 1.2 (Section 1.2.3), assume the following values:

$$\begin{aligned} \theta_1 &= 40 \text{ s}, \\ \theta_2 &= 50 \text{ s}, \\ \sigma_1 &= 0.1 \text{ kmol/m}^3, \\ \sigma_2 &= 0.2 \text{ kmol/m}^3. \end{aligned} \qquad \text{3-375}$$

Just as in Example 3.9 (Section 3.4.1), we choose the weighting matrices R_3 and R_2 as follows.

$$R_3 = \begin{pmatrix} 50 & 0 \\ 0 & 0.02 \end{pmatrix}, \qquad R_2 = \rho \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 3 \end{pmatrix}, \qquad \text{3-376}$$

where ρ is to be selected. The optimal control law has been computed for $\rho = 10$, 1, and 0.1, as in Example 3.9, but the results are not listed here. It turns out, of course, that the feedback gains from the plant state variables are not affected by the inclusion of the disturbances in the system model. This means that the closed-loop poles are precisely those listed in Table 3.1.

In order to evaluate the detailed performance of the system, the steady-state variance matrix

$$\bar{Q} = \lim_{t \to \infty} E\{x(t)x^T(t)\} \qquad \text{3-377}$$

has been computed from the matrix equation

$$0 = (A - B\bar{F})\bar{Q} + \bar{Q}(A - B\bar{F})^T + V. \qquad \text{3-378}$$

The steady-state variance matrix of the input can be found as follows:

$$\lim_{t \to \infty} E\{u(t)u^T(t)\} = \lim_{t \to \infty} E\{\bar{F}x(t)x^T(t)\bar{F}^T\} = \bar{F}\bar{Q}\bar{F}^T. \qquad \text{3-379}$$

From these variance matrices the rms values of the components of the controlled variable and the input variable are easily obtained. Table 3.2 lists the results. The table shows very clearly that as ρ decreases the fluctuations in the outgoing concentration become more and more reduced. The fluctuations in the outgoing flow caused by the control mechanism also eventually decrease with ρ. All this happens of course at the expense of an increase in the fluctuations in the incoming feeds. Practical considerations must decide which value of ρ is most suitable.

Table 3.2 Rms Values for Stirred-Tank Regulator

	Steady-state rms values of			
			Incremental feed	
ρ	Incremental outgoing flow (m^3/s)	Incremental concentration ($kmol/m^3$)	No. 1 (m^3/s)	No. 2 (m^3/s)
∞	0	0.06124	0	0
10	0.0001038	0.03347	0.0008957	0.0006980
1	0.00003303	0.008238	0.001567	0.001487
0.1	0.000004967	0.001127	0.001769	0.001754

Example 3.14. *Angular velocity tracking system*

Let us consider the angular velocity tracking problem as outlined in Example 3.12. To solve this problem we exploit the special structure of the tracking problem. It follows from **3-365** that the optimal tracking law is given by

$$\mu(t) = -F_1(t)\xi(t) + F_2(t)\xi_r(t). \qquad \textbf{3-380}$$

The feedback gain $F_1(t)$ is independent of the properties of the reference variable and in fact has already been computed in previous examples where we considered the angular velocity regulation problem. From Example 3.7 (Section 3.4.1), it follows that the steady-state value of the feedback gain is given by

$$\bar{F}_1 = \frac{1}{\kappa}\left(-\alpha + \sqrt{\alpha^2 + \frac{\kappa^2}{\rho}}\right), \qquad \textbf{3-381}$$

while the steady-state value of P_{11} is

$$\bar{P}_{11} = \frac{\rho}{\kappa^2}\left(-\alpha + \sqrt{\alpha^2 + \frac{\kappa^2}{\rho}}\right). \qquad \textbf{3-382}$$

By using **3-368**, it follows that the steady-state value of P_{12} can be solved from

$$0 = -1 - \sqrt{\alpha^2 + \frac{\kappa^2}{\rho}}\,\bar{P}_{12} - \frac{1}{\theta}\,\bar{P}_{12}. \qquad \textbf{3-383}$$

Solution yields

$$\bar{P}_{12} = -\frac{1}{\dfrac{1}{\theta} + \sqrt{\alpha^2 + \dfrac{\kappa^2}{\rho}}}, \qquad \textbf{3-384}$$

so that

$$\bar{F}_2 = \frac{\dfrac{\kappa}{\rho}}{\dfrac{1}{\theta} + \sqrt{\alpha^2 + \dfrac{\kappa^2}{\rho}}}. \qquad \textbf{3-385}$$

Finally, solution of **3-369** for $\bar{P}_{22}$ gives

$$\bar{P}_{22} = \frac{\theta}{2} \frac{\alpha^2 + \dfrac{1}{\theta^2} + \dfrac{2}{\theta}\sqrt{\alpha^2 + \dfrac{\kappa^2}{\rho}}}{\left(\dfrac{1}{\theta} + \sqrt{\alpha^2 + \dfrac{\kappa^2}{\rho}}\right)^2}. \qquad \textbf{3-386}$$

Let us choose the following numerical values:

$$\begin{aligned} \alpha &= 0.5 \text{ s}^{-1}, \\ \kappa &= 150 \text{ rad/(V s}^2), \\ \theta &= 1 \text{ s}, \\ \sigma &= 30 \text{ rad/s}, \\ \rho &= 1000 \text{ rad}^2/(\text{V}^2 \text{ s}^2). \end{aligned} \qquad \textbf{3-387}$$

This yields the following numerical results:

$$\bar{F}_1 = 0.02846, \qquad \bar{F}_2 = 0.02600, \qquad \textbf{3-388}$$

$$\bar{P} = \begin{pmatrix} 0.1897 & -0.1733 \\ -0.1733 & 0.1621 \end{pmatrix}. \qquad \textbf{3-389}$$

From **3-373** it follows that

$$\lim_{t\to\infty} [E\{\tilde{\zeta}^2(t)\} + \rho E\{\mu^2(t)\}] = \text{tr}(\bar{P}V), \qquad \textbf{3-390}$$

where $\tilde{\zeta}(t) = \xi(t) - \xi_r(t)$. Since in the present problem

$$V = \begin{pmatrix} 0 & 0 \\ 0 & \dfrac{2\sigma^2}{\theta} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1800 \end{pmatrix}, \qquad \textbf{3-391}$$

we find that

$$\lim_{t\to\infty} [E\{\tilde{\zeta}^2(t)\} + \rho E\{\mu^2(t)\}] = 291.8 \text{ rad}^2/\text{s}^2. \qquad \textbf{3-392}$$

We can use **3-392** to obtain rough estimates of the rms tracking error and rms input voltage as follows. First, we have from **3-392**

$$\lim_{t\to\infty} E\{\tilde{\zeta}^2(t)\} < 291.8 \text{ rad}^2/\text{s}^2. \qquad \textbf{3-393}$$

It follows that

$$\text{steady-state rms tracking error} < 17.08 \text{ rad/s}. \qquad \textbf{3-394}$$

Similarly, it follows from **3-392**

$$\lim_{t\to\infty} E\{\mu^2(t)\} < \frac{291.8}{\rho} = 0.2918 \text{ V}^2. \qquad \textbf{3-395}$$

We conclude that

$$\text{steady-state rms input voltage} < 0.5402 \text{ V}. \qquad \textbf{3-396}$$

Exact values for the rms tracking error and rms input voltage can be found by computing the steady-state variance matrix of the state $\tilde{x}(t)$ of the closed-loop augmented system. This system is described by the equation

$$\dot{\tilde{x}}(t) = \begin{pmatrix} -\alpha & 0 \\ 0 & -\frac{1}{\theta} \end{pmatrix} \tilde{x}(t) + \begin{pmatrix} \kappa \\ 0 \end{pmatrix} (-\bar{F}_1, \bar{F}_2)\tilde{x}(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} w(t), \qquad \textbf{3-397}$$

or

$$\dot{\tilde{x}}(t) = \begin{pmatrix} -\alpha - \kappa \bar{F}_1 & \kappa \bar{F}_2 \\ 0 & -\frac{1}{\theta} \end{pmatrix} \tilde{x}(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} w(t). \qquad \textbf{3-398}$$

As a result, the steady-state variance matrix $\bar{Q}$ of $\tilde{x}(t)$, is the solution of the matrix equation

$$0 = \begin{pmatrix} -\alpha - \kappa \bar{F}_1 & \kappa \bar{F}_2 \\ 0 & -\frac{1}{\theta} \end{pmatrix} \bar{Q} + \bar{Q} \begin{pmatrix} -\alpha - \kappa \bar{F}_1 & 0 \\ \kappa \bar{F}_2 & -\frac{1}{\theta} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \frac{2\sigma^2}{\theta} \end{pmatrix}.$$

3-399

Numerical solution yields

$$\bar{Q} = \begin{pmatrix} 497.5 & 608.4 \\ 608.4 & 900.0 \end{pmatrix}. \qquad \textbf{3-400}$$

The steady-state mean square tracking error can be expressed as

$$\begin{aligned} \lim_{t\to\infty} E\{[\xi(t) - \xi_r(t)]^2\} &= \bar{Q}_{11} - 2\bar{Q}_{12} + \bar{Q}_{22} \\ &= 180.7 \text{ rad}^2/\text{s}^2, \end{aligned} \qquad \textbf{3-401}$$

where the $\bar{Q}_{ij}$ are the entries of $\bar{Q}$. Similarly, the mean square input is given by

$$\begin{aligned} \lim_{t\to\infty} E\{[-\bar{F}_1\xi(t) + \bar{F}_2\xi_r(t)]^2\} &= \bar{F}_1^2\bar{Q}_{11} - 2\bar{F}_1\bar{F}_2\bar{Q}_{12} + \bar{F}_2^2\bar{Q}_{22} \\ &= 0.1110 \text{ V}^2. \end{aligned} \qquad \textbf{3-402}$$

In Table 3.3 the estimated and actual rms values are compared. Also given are the open-loop rms values, that is, the rms values without any control at all. It is seen that the estimated rms tracking error and input voltage are a little on the large side, but that they give a very good indication of the orders of magnitude. We moreover see that the control is not very good since the rms tracking error of 13.44 rad/s is not small as compared to the rms value of the

Table 3.3 Numerical Results for the Angular Velocity Tracking System

	Steady-state rms tracking error (rad/s)	Steady-state rms input voltage (V)
Open-loop	30	0
Estimated closed-loop	<17.08	<0.5402
Actual closed-loop	13.44	0.3333

reference variable of 30 rad/s. Since the rms input is quite small, however, there seems to be room for considerable improvement. This can be achieved by choosing the weighting coefficient ρ much smaller (see Problem 3.5).

Let us check the reference variable and closed-loop system bandwidths for the present example. The reference variable break frequency is $1/\theta = 1$ rad/s. Substituting the control law into the system equation, we find for the closed-loop system equation

$$\dot{\xi}(t) = -\sqrt{\alpha^2 + \frac{\kappa^2}{\rho}}\,\xi(t) + F_2\kappa\xi_r(t). \qquad \textbf{3-403}$$

This is a first-order system with break frequency

$$\sqrt{\alpha^2 + \frac{\kappa^2}{\rho}} = 4.769 \text{ rad/s}. \qquad \textbf{3-404}$$

Since the power spectral density of the reference variable, which is exponentially correlated noise, decreases relatively slowly with increasing frequency, the difference in break frequencies of the reference variable and the closed-loop system is not large enough to obtain a sufficiently small tracking error.

3.7 REGULATORS AND TRACKING SYSTEMS WITH NONZERO SET POINTS AND CONSTANT DISTURBANCES

3.7.1 Nonzero Set Points

In our discussion of regulator and tracking problems, we have assumed up to this point that the zero state is always the desired equilibrium state of the system. In practice, it is nearly always true, however, that the desired equilibrium state, which we call the *set point* of the state, is a constant point in state space, different from the origin. This kind of discrepancy can be removed by shifting the origin of the state space to this point, and this is what we have always done in our examples. This section, however, is devoted to the case where the set point may be variable; that is, we assume that the set point is constant over long periods of time but that from time to time it is shifted. This is a common situation in practice.

We limit our discussion to the time-invariant case. Consider the linear time-invariant system with state differential equation

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{3-405}$$

where the controlled variable is given by

$$z(t) = Dx(t). \tag{3-406}$$

Let us suppose that the set point of the controlled variable is given by z_0. Then in order to maintain the system at this set point, a constant input u_0 must be found (diCaprio and Wang, 1969) that holds the state at a point x_0 such that

$$z_0 = Dx_0. \tag{3-407}$$

It follows from the state differential equation that x_0 and u_0 must be related by

$$0 = Ax_0 + Bu_0. \tag{3-408}$$

Whether or not the system can be maintained at the given set point depends on whether **3-407** and **3-408** can be solved for u_0 for the given value of z_0. We return to this question, but let us suppose for the moment that a solution exists. Then we define the *shifted input*, the *shifted state*, and the *shifted controlled variable*, respectively, as

$$\begin{aligned} u'(t) &= u(t) - u_0, \\ x'(t) &= x(t) - x_0, \\ z'(t) &= z(t) - z_0. \end{aligned} \tag{3-409}$$

It is not difficult to find, by solving these equations for u, x, and z, substituting the result into the state differential equation **3-405** and the output equation **3-406**, and using **3-407** and **3-408**, that the shifted variables satisfy the equations

$$\begin{aligned} \dot{x}'(t) &= Ax'(t) + Bu'(t), \\ z'(t) &= Dx'(t). \end{aligned} \tag{3-410}$$

Suppose now that at a given time the set point is suddenly shifted from one value to another. Then in terms of the shifted system equations **3-410**, the system suddenly acquires a nonzero initial state. In order to let the system achieve the new set point in an orderly fashion we propose to effect the transition such that an optimization criterion of the form

$$\int_{t_0}^{t_1} [z'^T(t)R_3 z'(t) + u'^T(t)R_2 u'(t)]\, dt + x'^T(t_1)P_1 x'(t_1) \tag{3-411}$$

is minimized. Let us assume that this shifted regulator problem possesses a steady-state solution in the form of the time-invariant asymptotically stable steady-state control law

$$u'(t) = -\bar{F}x'(t). \tag{3-412}$$

Application of this control law ensures that, in terms of the original system variables, the system is transferred to the new set point as quickly as possible without excessively large transient input amplitudes.

Let us see what form the control law takes in terms of the original system variables. We write from **3-412** and **3-409**:

$$u(t) = -\bar{F}x(t) + u_0 + \bar{F}x_0. \tag{3-413}$$

This shows that the control law is of the form

$$u(t) = -\bar{F}x(t) + u_0', \tag{3-414}$$

where the constant vector u_0' is to be determined such that in the steady-state situation the controlled variable $z(t)$ assumes the given value z_0. We now study the question under what conditions u_0' can be found.

Substitution of **3-414** into the system state differential equation yields

$$\dot{x}(t) = (A - B\bar{F})x(t) + Bu_0'. \tag{3-415}$$

Since the closed-loop system is asymptotically stable, as $t \rightarrow \infty$ the state reaches a steady-state values x_0 that satisfies

$$0 = \tilde{A}x_0 + Bu_0'. \tag{3-416}$$

Here we have abbreviated

$$\tilde{A} = A - B\bar{F}. \tag{3-417}$$

Since the closed-loop system is asymptotically stable, $\bar{A}$ has all of its characteristic values in the left-half complex plane and is therefore nonsingular; consequently, we can solve **3-416** for x_0:

$$x_0 = (-\bar{A})^{-1}Bu_0'. \tag{3-418}$$

If the set point z_0 of the controlled variable is to be achieved, we must therefore have

$$z_0 = D(-\bar{A})^{-1}Bu_0'. \tag{3-419}$$

When considering the problem of solving this equation for u_0' for a given value of z_0, three cases must be distinguished:

(a) *The dimension of z is greater than that of u:* Then **3-419** has a solution for special values of z_0 only; in general, no solution exists. In this case we attempt to control the variable $z(t)$ with an input $u(t)$ of smaller dimension; since we have too few degrees of freedom, it is not surprising that no solution can generally be found.

(b) *The dimensions of u and z are the same*, that is, a sufficient number of degrees of freedom is available to control the system. In this case **3-419** can be solved for u_0' provided $D(-\bar{A})^{-1}B$ is nonsingular; assuming this to be the case (we shall return to this), we find

$$u_0' = [D(-\bar{A})^{-1}B]^{-1}z_0, \tag{3-420}$$

which yields for the optimal input to the tracking system

$$u(t) = -\bar{F}x(t) + [D(-\bar{A})^{-1}B]^{-1}z_0. \tag{3-421}$$

(c) *The dimension of z is less than that of u:* In this case there are too many degrees of freedom and **3-419** has many solutions. We can choose one of these solutions, but it is more advisable to reformulate the tracking problem by adding components to the controlled variable.

On the basis of these considerations, we henceforth assume that

$$\dim(z) = \dim(u), \tag{3-422}$$

so that case (b) applies. We see that

$$D(-\bar{A})^{-1}B = H_c(0), \tag{3-423}$$

where

$$H_c(s) = D(sI - \bar{A})^{-1}B. \tag{3-424}$$

We call $H_c(s)$ the *closed-loop transfer matrix*, since it is the transfer matrix from $u'(t)$ to $z(t)$ for the system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ z(t) &= Dx(t), \\ u(t) &= -\bar{F}x(t) + u'(t). \end{aligned} \tag{3-425}$$

In terms of $H_c(0)$ the optimal control law **3-421** can be written as

$$u(t) = -\bar{F}x(t) + H_c^{-1}(0)z_0. \tag{3-426}$$

As we have seen, this control law has the property that after a step change in the set point z_0 the system is transferred to the new set point as quickly as possible without excessively large transient input amplitudes. Moreover, this control law of course makes the system return to the set point from any initial state in an optimal manner. We call **3-426** the *nonzero set point optimal control law*. It has the property that it statically decouples the control system, that is, the transmission $T(s)$ of the control system (the transfer matrix from the set point z_0 to the controlled variable z) has the property that $T(0) = I$.

We now study the question under what conditions $H_c(0)$ has an inverse. It will be proved that this property can be directly ascertained from the open-loop system equations

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ z(t) &= Dx(t). \end{aligned} \tag{3-427}$$

Consider the following string of equalities

$$\begin{aligned} \det[H_c(s)] &= \det[D(sI - A + B\bar{F})^{-1}B] \\ &= \det[D(sI - A)^{-1}\{I + B\bar{F}(sI - A)^{-1}\}^{-1}B] \\ &= \det[D(sI - A)^{-1}\{I - B\bar{F}[I + (sI - A)^{-1}B\bar{F}]^{-1}(sI - A)^{-1}\}B] \\ &= \det[D(sI - A)^{-1}B]\det[I - \bar{F}(sI - A + B\bar{F})^{-1}B] \\ &= \det[D(sI - A)^{-1}B]\det[I - (sI - A + B\bar{F})^{-1}B\bar{F}] \\ &= \det[D(sI - A)^{-1}B]\det[(sI - A + B\bar{F})^{-1}]\det(sI - A) \\ &= \frac{\det[D(sI - A)^{-1}B]\det(sI - A)}{\det(sI - A + B\bar{F})} \\ &= \frac{\psi(s)}{\phi_c(s)}. \end{aligned} \tag{3-428}$$

Here we have used Lemma 1.1 (Section 1.5.3) twice. The polynomial $\psi(s)$ is defined by

$$\det[H(s)] = \frac{\psi(s)}{\phi(s)}, \tag{3-429}$$

where $H(s)$ is the open-loop transfer matrix

$$H(s) = D(sI - A)^{-1}B, \tag{3-430}$$

and $\phi(s)$ the open-loop characteristic polynomial

$$\phi(s) = \det (sI - A). \tag{3-431}$$

Finally, $\phi_c(s)$ is the closed-loop characteristic polynomial

$$\phi_c(s) = \det (sI - A + B\bar{F}). \tag{3-432}$$

We see from **3-428** that the zeroes of the closed-loop transfer matrix are the same as those of the open-loop transfer matrix. We also see that

$$\det [D(-\bar{A})^{-1}B] = \det [H_c(0)] = \frac{\psi(0)}{\phi_c(0)} \tag{3-433}$$

is zero if and only if $\psi(0) = 0$. Thus the condition $\psi(0) \neq 0$ guarantees that $D(-\bar{A})^{-1}B$ is nonsingular, hence that the nonzero set point control law exists. These results can be summarized as follows.

Theorem 3.10. *Consider the time-invariant system*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ z(t) &= Dx(t), \end{aligned} \tag{3-434}$$

where z and u have the same dimensions. Consider any asymptatically stable time-invariant control law

$$u(t) = -Fx(t) + u'(t). \tag{3-435}$$

Let $H(s)$ be the open-loop transfer matrix

$$H(s) = D(sI - A)^{-1}B, \tag{3-436}$$

and $H_c(s)$ the closed-loop transfer matrix

$$H_c(s) = D(sI - A + BF)^{-1}B. \tag{3-437}$$

Then $H_c(0)$ is nonsingular and the controlled variable $z(t)$ can under steady-state conditions be maintained at any constant value z_0 by choosing

$$u'(t) = H_c^{-1}(0)z_0 \tag{3-438}$$

if and only if $H(s)$ has a nonzero numerator polynomial that has no zeroes at the origin.

It is noted that the theorem is stated for any asymptotically stable control law and not only for the steady-state optimal control law.

The discussion of this section has been confined to deterministic regulators. Of course stochastic regulators (including tracking problems) can also have nonzero set points. The theory of this section applies to stochastic regulators without modification; the nonzero set point optimal control law for the stochastic regulator is also given by

$$u(t) = -Fx(t) + H_c^{-1}(0)z_0. \qquad \text{3-439}$$

Example 3.15. *Position control system*

Let us consider the position control system of Example 3.4 (Section 3.3.1). In Example 3.8 (Section 3.4.1), we found the optimal steady-state control law. It is not difficult to find from the results of Example 3.8 that the closed-loop transfer function is given by

$$H_c(s) = \frac{\kappa}{s^2 + s\sqrt{\alpha^2 + \dfrac{2\kappa}{\sqrt{\rho}}} + \dfrac{\kappa}{\sqrt{\rho}}}. \qquad \text{3-440}$$

If follows from **3-435** and **3-438** that the nonzero set point optimal control law is given by

$$\begin{aligned} \mu(t) &= -Fx(t) + \frac{1}{H_c(0)}\zeta_0 \\ &= -\frac{1}{\sqrt{\rho}}\xi_1(t) - \frac{1}{\kappa}\left(-\alpha + \sqrt{\alpha^2 + \frac{2\kappa}{\sqrt{\rho}}}\right)\xi_2(t) + \frac{1}{\sqrt{\rho}}\zeta_0, \end{aligned} \qquad \text{3-441}$$

where ζ_0 is the set point for the angular position. This is precisely the control law **3-171** that we found in Example 3.8 from elementary considerations.

Example 3.16. *Stirred tank*

As an example of a multivariable system, we consider the stirred-tank regulator problem of Example 3.9 (Section 3.4.1). For $\rho = 1$ (where ρ is defined as in Example 3.9), the regulator problem yields the steady-state feedback gain matrix

$$F = \begin{pmatrix} 0.1009 & -0.09708 \\ 0.01681 & 0.05475 \end{pmatrix}. \qquad \text{3-442}$$

It is easily found that the corresponding closed-loop transfer matrix is given by

$$\begin{aligned} H_c(s) &= D(sI - A + BF)^{-1}B \\ &= \frac{1}{s^2 + 0.2131s + 0.01037} \\ &\quad \cdot \begin{pmatrix} 0.01s + 0.0007475 & 0.01s + 0.001171 \\ -0.25s - 0.01931 & 0.75s + 0.1084 \end{pmatrix}. \end{aligned} \qquad \text{3-443}$$

From this the nonzero set point optimal control law can be found to be

$$u(t) = -Fx(t) + \begin{pmatrix} 10.84 & -0.1171 \\ 1.931 & 0.07475 \end{pmatrix} z_0. \qquad \textbf{3-444}$$

Figure 3.16 gives the response of the closed-loop system to step changes in the components of the set point z_0. Here the set point of the outgoing flow is

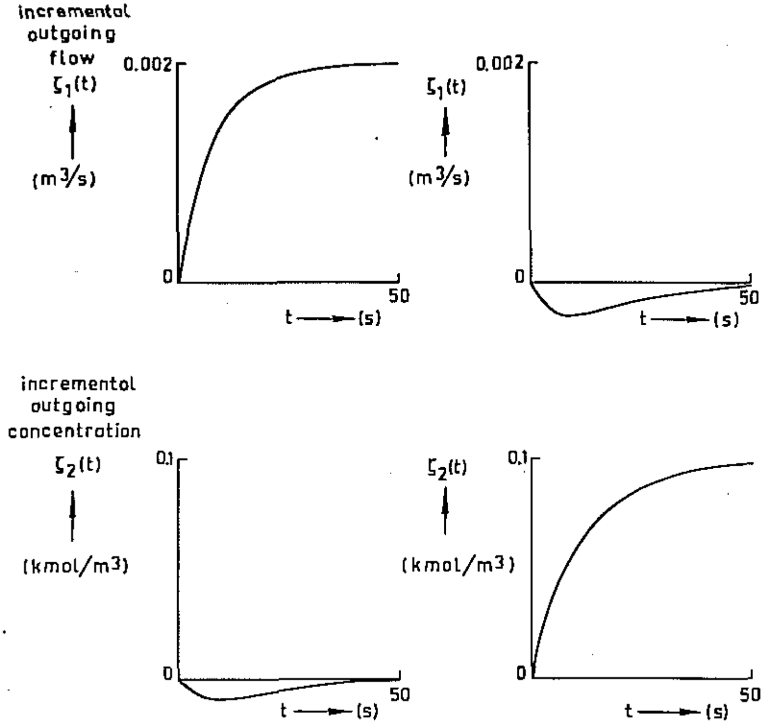


Fig. 3.16. The responses of the stirred tank as a nonzero set point regulating system. Left column: Responses of the incremental outgoing flow and concentration to a step of $0.002\ m^3/s$ in the set point of the flow. Right column: Responses of the incremental outgoing flow and concentration to a step of $0.1\ kmol/m^3$ in the set point of the concentration.

changed by $0.002\ m^3/s$, which amounts to 10% of the nominal value, while the set point of the outgoing concentration is changed by $0.1\ kmol/m^3$, which is 8% of the nominal value. We note that the control system exhibits a certain amount of dynamic *coupling* or *interaction*, that is, a change in the set point of one of the components of the controlled variable transiently affects the other component. The effect is small, however.

3.7.2* Constant Disturbances

In this subsection we discuss a method for counteracting the effect of constant disturbances in time-invariant regulator systems. As we saw in Chapter 2, in regulators and tracking systems where high precision is required, it is important to eliminate the effect of constant disturbances completely. This can be done by the application of integrating action. We introduce integrating action in the context of state feedback control by first extending the usual regulator problem, and then consider the effect of constant disturbances in the corresponding modified closed-loop control system configuration.

Consider the time-invariant system with state differential equation

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad \textbf{3-445}$$

with $x(t_0)$ given and with the controlled variable

$$z(t) = Dx(t). \qquad \textbf{3-446}$$

We add to the system variables the "integral state" $q(t)$ (Newell and Fisher, 1971; Shih, 1970; Porter, 1971), defined by

$$\dot{q}(t) = z(t), \qquad \textbf{3-447}$$

with $q(t_0)$ given. One can now consider the problem of minimizing a criterion of the form

$$\int_{t_0}^{\infty} [z^T(t)R_3z(t) + q^T(t)R_3'q(t) + u^T(t)R_2u(t)]\, dt, \qquad \textbf{3-448}$$

where R_3, R_3', and R_2 are suitably chosen weighting matrices. The first term of the integrand forces the controlled variable to zero, while the second term forces the integral state, that is, the total area under the response of the controlled variable, to go to zero. The third term serves, as usual, to restrict the input amplitudes.

Let us assume that by minimizing an expression of the form **3-448**, or by any other method, a time-invariant control law

$$u(t) = -F_1x(t) - F_2q(t) \qquad \textbf{3-449}$$

is determined that stabilizes the augmented system described by **3-445**, **3-446**, and **3-447**. (We defer for a moment the question under which conditions such an asymptotically stable control law exists.) Suppose now that a constant disturbance occurs in the system, so that we must replace the state differential equation **3-445** with

$$\dot{x}(t) = Ax(t) + Bu(t) + v_0, \qquad \textbf{3-450}$$

where v_0 is a constant vector. Since the presence of the constant disturbance does not affect the asymptotic stability of the system, we have

$$\lim_{t\to\infty} \dot{q}(t) = 0, \tag{3-451}$$

or, from **3-447**,

$$\lim_{t\to\infty} z(t) = 0. \tag{3-452}$$

This means that *the control system with the asymptotically stable control law* **3-449** *has the property that the effect of constant disturbances on the controlled variable eventually vanishes.* Since this is achieved by the introduction of the integral state q, this control scheme is a form of integral control. Figure 3.17 depicts the integral control scheme.

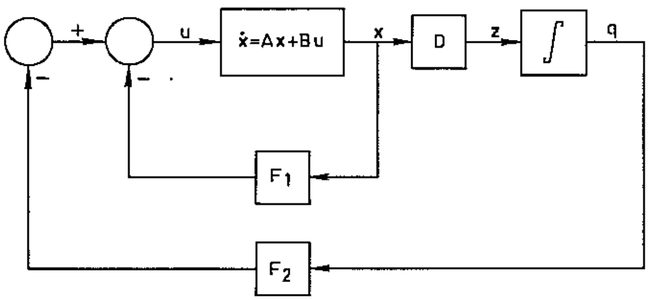


Fig. 3.17. State feedback integral control.

Let us now consider the mechanism that effects the suppression of the constant disturbance. The purpose of the multivariable integration of **3-447** is to generate a constant contribution u_0 to the input that counteracts the effect of the constant disturbance on the controlled variable. Thus let us consider the response of the system **3-450** to the input

$$u(t) = -F_1 x(t) + u_0. \tag{3-453}$$

Substitution of this expression into the state differential equation **3-450** yields

$$\dot{x}(t) = (A - BF_1)x(t) + Bu_0 + v_0. \tag{3-454}$$

In equilibrium conditions the state assumes a constant value x_0 that must satisfy the relation

$$0 = \bar{A}x_0 + Bu_0 + v_0, \tag{3-455}$$

where

$$\bar{A} = A - BF_1. \tag{3-456}$$

Solution for x_0 yields

$$x_0 = (-\bar{A})^{-1}Bu_0 + (-\bar{A})^{-1}v_0, \tag{3-457}$$

provided $\bar{A}$ is nonsingular. The corresponding equilibrium value z_0 of the controlled variable is given by

$$z_0 = Dx_0 = D(-\bar{A})^{-1}Bu_0 + D(-\bar{A})^{-1}v_0. \qquad \text{3-458}$$

When we now consider the question whether or not a value of u_0 exists that makes $z_0 = 0$, we obviously obtain the same conditions as in Section 3.7.1, broken down to the three following cases.

(a) *The dimension of z is greater than that of u:* In this case the equation

$$0 = D(-\bar{A})^{-1}Bu_0 + D(-\bar{A})^{-1}v_0 \qquad \text{3-459}$$

represents more equations than variables, which means that in general no solution exists. The number of degrees of freedom is too small, and the steady-state error in z cannot be eliminated.

(b) *The dimension of z equals that of u:* In this case a solution exists if and only if

$$D(-\bar{A})^{-1}B = H_c(0) \qquad \text{3-460}$$

is nonsingular, where

$$H_c(s) = D(sI - \bar{A})^{-1}B \qquad \text{3-461}$$

is the closed-loop transfer matrix. As we saw in Theorem 3.10, $H_c(0)$ is nonsingular if and only if the open-loop transfer matrix $H(s) = D(sI - A)^{-1}B$ has no zeroes at the origin.

(c) *The dimension of z is less than that of u:* In this case there are too many degrees of freedom and the dimension of z can be increased by adding components to the controlled variable.

On the basis of these considerations, we from now on restrict ourselves to the case where dim (z) = dim (u). Then the present analysis shows that a *necessary* condition for the successful operation of the integral scheme under consideration is that the open-loop transfer matrix $H(s) = D(sI - A)^{-1}B$ have no zeroes at the origin. In fact, it can be shown, by a slight extension of the argument of Power and Porter (1970) involving the controllability canonical form of the system **3-445**, that necessary and sufficient conditions for the existence of an asymptotically stable control law of the form **3-449** are that

(i) the system **3-445** is stabilizable; and
(ii) the open-loop transfer matrix $H(s) = D(sI - A)^{-1}B$ has no zeroes at the origin.

Power and Porter (1970) and Davison and Smith (1971) prove that necessary and sufficient conditions for arbitrary placement of the closed-loop system poles are that the system **3-445** be completely controllable and that the open-loop transfer matrix have no zeroes at the origin. Davison and Smith (1971) state the latter condition in an alternative form.

In the literature alternative approaches to determining integral control schemes can be found (see, e.g., Anderson and Moore, 1971, Chapter 10; Johnson, 1971b).

Example 3.17. *Integral control of the positioning system*

Let us consider the positioning system of previous examples and assume that a constant disturbance can enter into the system in the form of a constant torque τ_0 on the shaft of the motor. We thus modify the state differential equation 3-59 to

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ 0 & -\alpha \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ \kappa \end{pmatrix} \mu(t) + \begin{pmatrix} 0 \\ \gamma \end{pmatrix} \tau_0, \qquad \text{3-462}$$

where $\gamma = 1/J$, with J the moment of inertia of all the rotating parts. As before, the controlled variable is given by

$$\zeta(t) = (1, 0)x(t). \qquad \text{3-463}$$

We add to the system the scalar integral state $q(t)$, defined by

$$\dot{q}(t) = \zeta(t). \qquad \text{3-464}$$

From Example 3.15 we know that the open-loop transfer function has no zeroes at the origin; moreover, the system is completely controllable so that we expect no difficulties in finding an integral control system. Let us consider the optimization criterion

$$\int_{t_0}^{\infty} [\zeta^2(t) + \lambda q^2(t) + \rho\mu^2(t)]\, dt. \qquad \text{3-465}$$

As in previous examples, we choose

$$\rho = 0.00002 \text{ rad}^2/\text{V}^2. \qquad \text{3-466}$$

Inspection of Fig. 3.9 shows that in the absence of integral control $q(t)$ will reach a steady-state value of roughly 0.01 rad s for the given initial condition. Choosing

$$\lambda = 10 \text{ s}^{-2} \qquad \text{3-467}$$

can therefore be expected to affect the control scheme significantly.

Numerical solution of the corresponding regulator problem with the numerical values of Example 3.4 (Section 3.3.1) and $\gamma = 0.1 \text{ kg}^{-1} \text{ m}^{-2}$ yields the steady-state control law

$$\mu(t) = -F_1 x(t) - F_2 q(t), \qquad \text{3-468}$$

with

$$F_1 = (299.8, 22.37), \quad F_2 = 707.1. \qquad \text{3-469}$$

The corresponding closed-loop characteristic values are $-9.519 \pm j9.222\ s^{-1}$ and $-3.168\ s^{-1}$. Upon comparison with the purely proportional scheme of Example 3.8 (Section 3.4.1), we note that the proportional part of the feedback, represented by F_1, has hardly changed (compare **3-169**), and that the corresponding closed-loop poles, which are $-9.658 \pm j9.094\ s^{-1}$ in Example 3.8 also have moved very little. Figure 3.18 gives the response of the integral

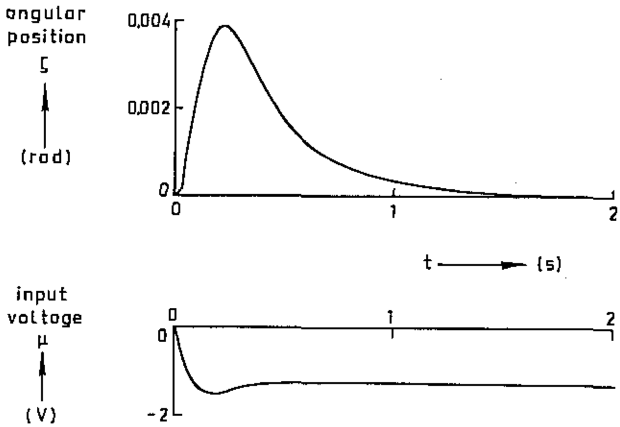


Fig. 3.18. Response of the integral position control system to a constant torque of 10 N m on the shaft of the motor.

control system from zero initial conditions to a constant torque τ_0 of 10 N m on the shaft of the motor. The maximum deviation of the angular displacement caused by this constant torque is about 0.004 rad.

3.8* ASYMPTOTIC PROPERTIES OF TIME-INVARIANT OPTIMAL CONTROL LAWS

3.8.1* Asymptotic Behavior of the Optimal Closed-Loop Poles

In Section 3.2 we saw that the stability of time-invariant linear state feedback control systems can be achieved or improved by assigning the closed-loop poles to suitable locations in the left-half complex plane. We were not able to determine which pole patterns are most desirable, however. In Sections 3.3 and 3.4, the theory of linear optimal state feedback control systems was developed. For time-invariant optimal systems, a question of obvious interest concerns the closed-loop pole patterns that result. This section is devoted to a study of these patterns. This will supply valuable information about the response that can be expected from optimal regulators.

Suppose that in the time-invariant regulator problem we let

$$R_2 = \rho N, \tag{3-470}$$

where N is a positive-definite symmetric matrix and ρ a positive scalar. With this choice of R_2, the optimization criterion is given by

$$\int_{t_0}^{\infty} [z^T(t)R_3 z(t) + \rho u^T(t)Nu(t)]\, dt. \tag{3-471}$$

The parameter ρ determines how much weight is attributed to the input; a large value of ρ results in small input amplitudes, while a small value of ρ permits large input amplitudes. We study in this subsection how the locations of the optimal closed-loop regulator poles vary as a function of ρ. For this investigation we employ root locus methods.

In Section 3.4.4 we saw that the optimal closed-loop poles are the left-half plane characteristic values of the matrix Z, where

$$Z = \begin{pmatrix} A & -BR_2^{-1}B^T \\ -R_1 & -A^T \end{pmatrix} = \begin{pmatrix} A & -\dfrac{1}{\rho} BN^{-1}B^T \\ -D^TR_3D & -A^T \end{pmatrix}. \tag{3-472}$$

Using Lemma 1.2 (Section 1.5.4) and Lemma 1.1 (Section 1.5.3), we expand $\det(sI - Z)$ as follows:

$$\begin{aligned}
\det(sI - Z) &= \det \begin{pmatrix} sI - A & \dfrac{1}{\rho} BN^{-1}B^T \\ D^TR_3D & sI + A^T \end{pmatrix} \\
&= \det(sI - A) \\
&\quad \cdot \det\left[(sI + A^T) - D^TR_3D(sI - A)^{-1}\frac{1}{\rho} BN^{-1}B^T\right] \\
&= \det(sI - A)\det(sI + A^T) \\
&\quad \cdot \det\left[I - D^TR_3D(sI - A)^{-1}\frac{1}{\rho} BN^{-1}B^T(sI + A^T)^{-1}\right] \\
&= \det(sI - A)(-1)^n \det(-sI - A) \\
&\quad \cdot \det\left[I + \frac{1}{\rho} N^{-1}B^T(-sI - A^T)^{-1}D^TR_3D(sI - A)^{-1}B\right] \\
&= (-1)^n\phi(s)\phi(-s)\det\left[I + \frac{1}{\rho} N^{-1}H^T(-s)R_3H(s)\right],
\end{aligned} \tag{3-473}$$

where n is the dimension of the state x, and

$$\begin{aligned}\phi(s) &= \det(sI - A),\\ H(s) &= D(sI - A)^{-1}B.\end{aligned} \qquad \textbf{3-474}$$

For simplicity, we first study the case where both the input u and the controlled variable z are scalars, while

$$R_3 = 1, \qquad N = 1. \qquad \textbf{3-475}$$

We return to the multiinput multioutput case at the end of this section. It follows from **3-473** that in the single-input single-output case the closed-loop poles are the left-half plane zeroes of

$$(-1)^n\phi(s)\phi(-s)\left[1 + \frac{1}{\rho}H(-s)H(s)\right], \qquad \textbf{3-476}$$

where $H(s)$ is now a scalar transfer function. Let us represent $H(s)$ in the form

$$H(s) = \frac{\psi(s)}{\phi(s)}, \qquad \textbf{3-477}$$

where $\psi(s)$ is the numerator polynomial of $H(s)$. It follows that the closed-loop poles are the left-half plane roots of

$$\phi(s)\phi(-s) + \frac{1}{\rho}\psi(s)\psi(-s) = 0. \qquad \textbf{3-478}$$

We can apply two techniques in determining the loci of the closed-loop poles. The first method is to recognize that **3-478** is a function of s^2, to substitute $s^2 = s'$, and to find the root loci in the s'-plane. The closed-loop poles are then obtained as the left-half plane square roots of the roots in the s'-plane. This is the *root-square locus* method (Chang, 1961).

For our purposes it is more convenient to trace the loci in the s-plane. Let us write

$$\begin{aligned}\psi(s) &= \alpha\prod_{i=1}^{p}(s - \nu_i),\\ \phi(s) &= \prod_{i=1}^{n}(s - \pi_i),\end{aligned} \qquad \textbf{3-479}$$

where the ν_i, $i = 1, 2, \cdots, p$, are the zeroes of $H(s)$, and the π_i, $i = 1, 2, \cdots, n$, the poles of $H(s)$. To bring **3-478** in standard form, we rewrite it with **3-479** as

$$\prod_{i=1}^{n}(s - \pi_i)(s + \pi_i) + (-1)^{n-p}\frac{\alpha^2}{\rho}\prod_{i=1}^{p}(s - \nu_i)(s + \nu_i) = 0. \qquad \textbf{3-480}$$

Applying the rules of Section 1.5.5, we conclude the following.

(a) As $\rho \to 0$, of the $2n$ roots of **3-480** a total number of $2p$ asymptotically approach the p zeroes ν_i, $i = 1, 2, \cdots, p$, and their negatives $-\nu_i$, $i = 1, 2, \cdots, p$.

(b) As $\rho \to 0$, the other $2(n - p)$ roots of **3-480** asymptotically approach straight lines which intersect in the origin and make angles with the positive real axis of

$$\begin{aligned} &\frac{k\pi}{n-p}, \qquad k = 0, 1, 2, \cdots, 2n - 2p - 1, \qquad n - p \text{ odd}, \\ &\frac{(k + \frac{1}{2})\pi}{n-p}, \qquad k = 0, 1, 2, \cdots, 2n - 2p - 1, \qquad n - p \text{ even}. \end{aligned} \qquad \textbf{3-481}$$

(c) As $\rho \to 0$, the $2(n - p)$ faraway roots of **3-480** are asymptotically at a distance

$$\left(\frac{\alpha^2}{\rho}\right)^{1/[2(n-p)]} \qquad \textbf{3-482}$$

from the origin.

(d) As $\rho \to \infty$, the $2n$ roots of **3-480** approach the n poles π_i, $i = 1, 2, \cdots, n$, and their negatives $-\pi_i$, $i = 1, 2, \cdots, n$.

Since the optimal closed-loop poles are the left-half plane roots of **3-480** we easily conclude the following (Kalman, 1964).

Theorem 3.11. *Consider the steady-state solution of the single-input single-output regulator problem with $R_3 = 1$ and $R_2 = \rho$. Assume that the open-loop system is stabilizable and detectable and let its transfer function be given by*

$$H(s) = \frac{\alpha \prod_{i=1}^{p} (s - \nu_i)}{\prod_{i=1}^{n} (s - \pi_i)}, \qquad \alpha \neq 0, \qquad \textbf{3-483}$$

where the π_i, $i = 1, 2, \cdots, n$, are the characteristic values of the system. Then we have the following.

(a) *As $\rho \downarrow 0$, p of the n optimal closed-loop characteristic values asymptotically approach the numbers $\hat{\nu}_i$, $i = 1, 2, \cdots, p$, where*

$$\hat{\nu}_i = \begin{cases} \nu_i & \text{if } \operatorname{Re}(\nu_i) \leq 0, \\ -\nu_i & \text{if } \operatorname{Re}(\nu_i) > 0. \end{cases} \qquad \textbf{3-484}$$

(b) *As $\rho \downarrow 0$, the remaining $n - p$ optimal closed-loop characteristic values asymptotically approach straight lines which intersect in the origin and make* *angles with the negative real axis of*

$$\pm l \frac{\pi}{n-p}, \quad l = 0, 1, \cdots, \frac{n-p-1}{2}, \quad n-p \text{ odd},$$
$$\pm \frac{(l+\frac{1}{2})\pi}{n-p}, \quad l = 0, 1, \cdots, \frac{n-p}{2} - 1, \quad n-p \text{ even}. \qquad \text{3-485}$$

These faraway closed-loop characteristic values are asymptotically at a distance

$$\omega_0 = \left(\frac{\alpha^2}{\rho}\right)^{1/[2(n-p)]} \qquad \text{3-486}$$

from the origin.

(c) *As $\rho \rightarrow \infty$, the n closed-loop characteristic values approach the numbers $\hat{\pi}_i$, $i = 1, 2, \cdots, n$, where*

$$\hat{\pi}_i = \begin{cases} \pi_i & \text{if } \operatorname{Re}(\pi_i) \leq 0, \\ -\pi_i & \text{if } \operatorname{Re}(\pi_i) > 0. \end{cases} \qquad \text{3-487}$$

The configuration of poles indicated by (b) is known as a *Butterworth configuration* of order $n - p$ with radius ω_0 (Weinberg, 1962). In Fig. 3.19

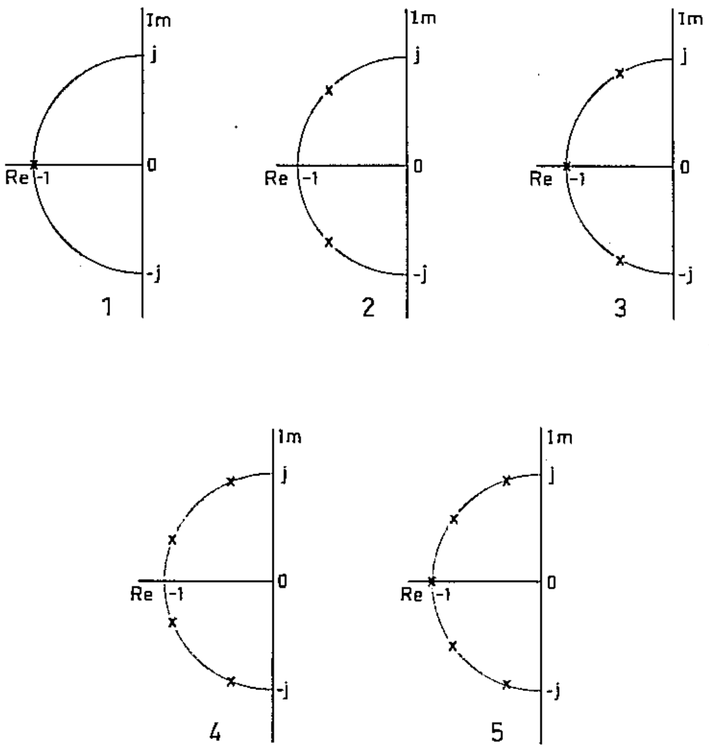


Fig. 3.19. Butterworth pole configurations of orders one through five and unit radii.

some low-order Butterworth configurations are indicated. In the next section we investigate what responses correspond to such configurations.

Figure 3.20 gives an example of the behavior of the closed-loop poles for a fictitious open-loop pole-zero configuration. Crosses mark the open-loop poles, circles the open-loop zeroes. Since the excess of poles over zeroes is two, a second-order Butterworth configuration results as $\rho \downarrow 0$. The remaining

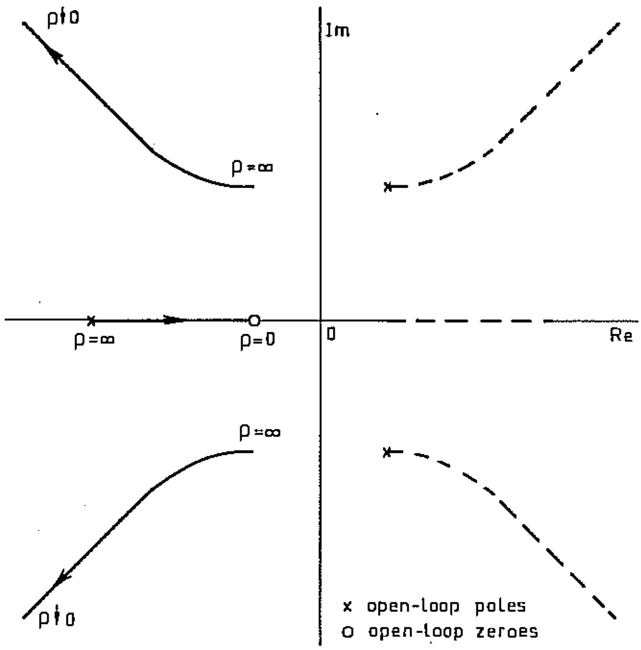


Fig. 3.20. Root loci of the characteristic values of the matrix Z (dashed and solid lines) and of the closed-loop poles (solid lines only) for a single-input single-output system with a fictitious open-loop pole-zero configuration.

closed-loop pole approaches the open-loop zero as $\rho \downarrow 0$. For $\rho \to \infty$ the closed-loop poles approach the single left-half plane open-loop pole and the mirror images of the two right-half plane open-loop poles.

We now return to the multiinput case. Here we must investigate the roots of

$$\phi(s)\phi(-s) \det \left[I + \frac{1}{\rho} N^{-1} H^T(-s) R_3 H(s) \right] = 0. \qquad \textbf{3-488}$$

The problem of determining the root loci for this expression is not as simple as in the single-input case. Evaluation of the determinant leads to an expression of the form

$$\sum_{i=0}^{n} \alpha_i(1/\rho)s^{2i} = 0, \qquad \textbf{3-489}$$

where the functions $\alpha_i(1/\rho)$, $i = 0, 1, 2, \cdots, n$ are polynomials in $1/\rho$. Rosenau (1968) has given rules that are helpful in obtaining root loci for such an expression. We are only interested in the asymptotic behavior of the roots as $\rho \to 0$ and $\rho \to \infty$. The roots of **3-488** are also the roots of

$$\phi(s)\phi(-s) \det [\rho I + N^{-1}H^T(-s)R_3H(s)] = 0. \qquad \textbf{3-490}$$

As $\rho \to 0$ some of the roots go to infinity; those that stay finite approach the zeroes of

$$\phi(s)\phi(-s) \det [N^{-1}H^T(-s)R_3H(s)], \qquad \textbf{3-491}$$

provided this expression is not identically zero. Let us suppose that $H(s)$ is a square transfer matrix (in Section 3.7 we saw that this is a natural assumption). Then we know from Section 1.5.3 that

$$\det [H(s)] = \frac{\psi(s)}{\phi(s)}, \qquad \textbf{3-492}$$

where $\psi(s)$ is a polynomial at most of degree $n - k$, with n the dimension of the system and k the dimension of u and z. As a result, we can write for **3-491**

$$\frac{\det (R_3)}{\det (N)} \psi(-s)\psi(s). \qquad \textbf{3-493}$$

Thus it follows that as $\rho \downarrow 0$ those roots of **3-490** that stay finite approach the zeroes of the transfer matrix $H(s)$ and their negatives. This means that those optimal closed-loop poles of the regulator that stay finite approach those zeroes of $H(s)$ that have negative real parts and the negatives of the zeroes that have nonnegative real parts.

It turns out (Rosenau, 1968) that as $\rho \downarrow 0$ the far-off closed-loop regulator poles, that is, those poles that go to infinity, generally do not form a single Butterworth configuration, such as in the single-input case, but that they group into *several* Butterworth configurations of different orders and different radii (see Examples 3.19 and 3.21). A rough estimate of the distance of the faraway poles from the origin can be obtained as follows. Let $\phi_c(s)$ denote the closed-loop characteristic polynomial. Then we have

$$\phi_c(s)\phi_c(-s) = \phi(s)\phi(-s) \det \left[I + \frac{1}{\rho} N^{-1}H^T(-s)R_3H(s)\right]. \qquad \textbf{3-494}$$

For small ρ we can approximate the right-hand side of this expression by

$$\phi(s)\phi(-s)\det\left[\frac{1}{\rho}N^{-1}H^T(-s)R_3H(s)\right] = \frac{\det(R_3)}{\rho^k\det(N)}\psi(s)\psi(-s) \quad \text{3-495}$$

where k is the dimension of the input u. Let us write

$$\psi(s) = \alpha\prod_{i=1}^{p}(s-\nu_i). \quad \text{3-496}$$

Then the leading term in **3-491** is given by

$$\alpha^2\frac{\det(R_3)}{\rho^k\det(N)}(-1)^p s^{2p}. \quad \text{3-497}$$

This shows that the polynomial $\phi_c(s)\phi_c(-s)$ contains the following terms

$$\phi_c(s)\phi_c(-s) = (-1)^n s^{2n} + \cdots + \alpha^2\frac{\det(R_3)}{\rho^k\det(N)}(-1)^p s^{2p} + \cdots. \quad \text{3-498}$$

The terms given are the term with the highest power of s and the term with the highest power of $1/\rho$. An approximation of the faraway roots of this polynomial (for small ρ) is obtained from

$$(-1)^n s^{2n} + \alpha^2\frac{\det(R_3)}{\rho^k\det(N)}(-1)^p s^{2p} = 0. \quad \text{3-499}$$

It follows that the closed-loop poles are approximated by the left-half plane solutions of

$$(-1)^{(n-p-1)/[2(n-p)]}\left(\alpha^2\frac{\det(R_3)}{\rho^k\det(N)}\right)^{1/[2(n-p)]}. \quad \text{3-500}$$

This first approximation indicates a Butterworth configuration of order $n-p$. We use this expression to estimate the distance of the faraway poles to the origin; this (crude) estimate is given by

$$\left(\alpha^2\frac{\det(R_3)}{\rho^k\det(N)}\right)^{1/[2(n-p)]}. \quad \text{3-501}$$

We consider finally the behavior of the closed-loop poles for $\rho \rightarrow \infty$. In this case we see from **3-494** that the characteristic values of the matrix Z approach the roots of $\phi(s)\phi(-s)$. This means that the closed-loop poles approach the numbers $\hat{\pi}_i$, $i = 1, 2, \cdots, n$, as given by **3-487**.

We summarize our results for the multiinput case as follows.

Theorem 3.12. *Consider the steady-state solution of the multiinput time-invariant regulator problem. Assume that the open-loop system is stabilizable and detectable, that the input u and the controlled variable z have the same* *dimension k, and that the state x has dimension n. Let $H(s)$ be the $k \times k$ open-loop transfer matrix*

$$H(s) = D(sI - A)^{-1}B. \tag{3-502}$$

Suppose that $\phi(s)$ is the open-loop characteristic polynomial and write

$$\det [H(s)] = \frac{\psi(s)}{\phi(s)} = \frac{\alpha \prod_{i=1}^{p} (s - \nu_i)}{\prod_{i=1}^{n} (s - \pi_i)}. \tag{3-503}$$

Assume that $\alpha \neq 0$ and take $R_2 = \rho N$ with $N > 0$, $\rho > 0$.

(a) *Then as $\rho \to 0$, p of the optimal closed-loop regulator poles approach the values $\hat{\nu}_i$, $i = 1, 2, \cdots, p$, where*

$$\hat{\nu}_i = \begin{cases} \nu_i & \text{if } \operatorname{Re}(\nu_i) \leq 0 \\ -\nu_i & \text{if } \operatorname{Re}(\nu_i) > 0. \end{cases} \tag{3-504}$$

The remaining closed-loop poles go to infinity and group into several Butterworth configurations of different orders and different radii. A rough estimate of the distance of the faraway closed-loop poles to the origin is

$$\left(\alpha^2 \frac{\det (R_3)}{\rho^k \det (N)}\right)^{1/[2(n-p)]}. \tag{3-505}$$

(b) *As $\rho \to \infty$, the n closed-loop regulator poles approach the numbers $\hat{\pi}_i$, $i = 1, 2, \cdots, n$, where*

$$\hat{\pi}_i = \begin{cases} \pi_i & \text{if } \operatorname{Re}(\pi_i) \leq 0 \\ -\pi_i & \text{if } \operatorname{Re}(\pi_i) > 0. \end{cases} \tag{3-506}$$

We conclude this section with the following comments. When ρ is very small, large input amplitudes are permitted. As a result, the system can move fast, which is reflected in a great distance of the faraway poles from the origin. Apparently, Butterworth pole patterns give good responses. Some of the closed-loop poles, however, do not move away but shift to the locations of open-loop zeroes. As is confirmed later in this section, in systems with left-half plane zeroes only these nearby poles are "canceled" by the open-loop zeroes, which means that their effect in the controlled variable response is not noticeable.

The case $\rho = \infty$ corresponds to a very heavy constraint on the input amplitudes. It is interesting to note that the "cheapest" stabilizing control law ("cheap" in terms of input amplitude) is a control law that relocates the unstable system poles to their mirror images in the left-half plane.

Problem 3.14 gives some information concerning the asymptotic behavior of the closed-loop poles for systems for which dim $(u) \neq$ dim (z).

Example 3.18. *Position control system*

In Example 3.8 (Section 3.4.1), we studied the locations of the closed-loop poles of the optimal position control system as a function of the parameter ρ. As we have seen, the closed-loop poles approach a Butterworth configuration of order two. This is in agreement with the results of this section. Since the open-loop transfer function

$$H(s) = \frac{\kappa}{s(s + \alpha)} \qquad \textbf{3-507}$$

has no zeroes, both closed-loop poles go to infinity as $\rho \downarrow 0$.

Example 3.19. *Stirred tank*

As an example of a multiinput multioutput system consider the stirred tank regulator problem of Example 3.9 (Section 3.4.1). From Example 1.15 (Section 1.5.3), we know that the open-loop transfer matrix is given by

$$H(s) = \begin{pmatrix} \dfrac{0.01}{s + 0.01} & \dfrac{0.01}{s + 0.01} \\ \dfrac{-0.25}{s + 0.02} & \dfrac{0.75}{s + 0.02} \end{pmatrix}. \qquad \textbf{3-508}$$

For this transfer matrix we have

$$\det [H(s)] = \frac{0.01}{(s + 0.01)(s + 0.02)}. \qquad \textbf{3-509}$$

Apparently, the transfer matrix has no zeroes; all closed-loop poles are therefore expected to go to ∞ as $\rho \downarrow 0$. With the numerical values of Example 3.9 for R_3 and N, we find for the characteristic polynomial of the matrix Z

$$s^4 + s^2\left(-0.5 \times 10^{-3} - \frac{0.02416}{\rho}\right) + \left(0.4 \times 10^{-7} + \frac{0.7416 \times 10^{-5}}{\rho} + \frac{10^{-4}}{\rho^2}\right). \qquad \textbf{3-510}$$

Figure 3.21 gives the behavior of the two closed-loop poles as ρ varies. Apparently, each pole traces a first-order Butterworth pattern. The asymptotic behavior of the roots for $\rho \downarrow 0$ can be found by solving the equation

$$s^4 - \frac{0.02416}{\rho} s^2 + \frac{10^{-4}}{\rho^2} = 0, \qquad \textbf{3-511}$$

which yields for the asymptotic closed-loop pole locations

$$-\frac{0.1373}{\sqrt{\rho}} \quad \text{and} \quad -\frac{0.07280}{\sqrt{\rho}}. \qquad \textbf{3-512}$$

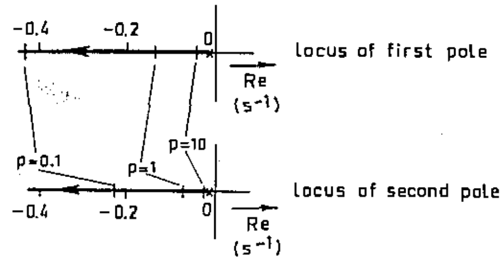


Fig. 3.21. Loci of the closed-loop roots for the stirred tank regulator. The locus on top originates from −0.02, the one below from −0.01.

The estimate **3-505** yields for the distance of the faraway poles to the origin

$$\frac{0.1}{\sqrt{\rho}}. \qquad \textbf{3-513}$$

We see that this is precisely the geometric average of the values **3-512**.

Example 3.20. *Pitch control of an airplane*

As an example of a more complicated system, we consider the longitudinal motions of an airplane (see Fig. 3.22). These motions are characterized by the velocity u along the x-axis of the airplane, the velocity w along the z-axis of the airplane, the pitch θ, and the pitch rate $q = \dot{\theta}$. The x- and z-axes are rigidly connected to the airplane. The x-axis is chosen to coincide with the horizontal axis when the airplane performs a horizontal stationary flight.

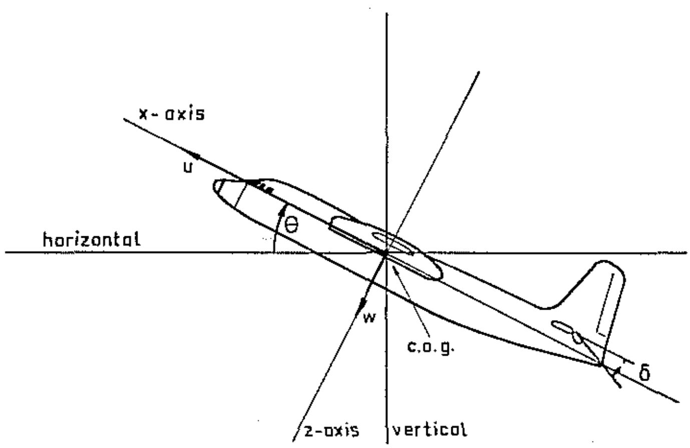


Fig. 3.22. The longitudinal motions of an airplane.

The control variables for these motions are the engine thrust T and the elevator deflection δ. The equations of motion can be linearized around a nominal solution which consists of horizontal flight with constant speed. It can be shown (Blakelock, 1965) that the linearized longitudinal equations of motion are independent of the lateral motions of the plane.

We choose the components of the state as follows:

$$\begin{aligned} \xi_1(t) &= u(t), && \text{incremental speed along } x\text{-axis}, \\ \xi_2(t) &= w(t), && \text{speed along } z\text{-axis}, \\ \xi_3(t) &= \theta(t), && \text{pitch}, \\ \xi_4(t) &= q(t), && \text{pitch rate}. \end{aligned} \qquad \text{3-514}$$

The input variable, this time denoted by c, we define as

$$c(t) = \begin{pmatrix} T(t) \\ \delta(t) \end{pmatrix} \quad \begin{matrix} \text{incremental engine thrust,} \\ \text{elevator deflection.} \end{matrix} \qquad \text{3-515}$$

With these definitions the state differential equations can be found from the inertial and aerodynamical laws governing the motion of the airplane (Blakelock, 1965). For a particular medium-weight transport aircraft under cruising conditions, the following linearized state differential equation results:

$$\dot{x}(t) = \begin{pmatrix} -0.01580 & 0.02633 & -9.810 & 0 \\ -0.1571 & -1.030 & 0 & 120.5 \\ 0 & 0 & 0 & 1 \\ 0.0005274 & -0.01652 & 0 & -1.466 \end{pmatrix} x(t) + \begin{pmatrix} 0.0006056 & 0 \\ 0 & -9.496 \\ 0 & 0 \\ 0 & -5.565 \end{pmatrix} c(t). \qquad \text{3-516}$$

Here the following physical units are employed: u and w in m/s, θ in rad, q in rad/s, T in N, and δ in rad.

In this example we assume that the thrust is constant, so that the elevator deflection $\delta(t)$ is the only control variable. With this the system is described by the state differential equation

$$\dot{x}(t) = \begin{pmatrix} -0.01580 & 0.02633 & -9.810 & 0 \\ -0.1571 & -1.030 & 0 & 120.5 \\ 0 & 0 & 0 & 1 \\ 0.0005274 & -0.01652 & 0 & -1.466 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ -9.496 \\ 0 \\ -5.565 \end{pmatrix} \delta(t). \qquad \textbf{3-517}$$

As the controlled variable we choose the pitch $\theta(t)$:

$$\theta(t) = (0, 0, 1, 0)x(t). \qquad \textbf{3-518}$$

It can be found that the transfer function from the elevator deflection $\delta(t)$ to the pitch $\theta(t)$ is given by

$$\frac{-5.565s^2 - 5.663s - 0.1112}{s^4 + 2.512s^3 - 3.544s^2 + 0.06487s + 0.03079}. \qquad \textbf{3-519}$$

The poles of the transfer function are

$$\begin{aligned} -0.006123 \pm j0.09353, \\ -1.250 \pm j1.394, \end{aligned} \qquad \textbf{3-520}$$

while the zeroes are given by

$$-0.02004 \quad \text{and} \quad -0.9976. \qquad \textbf{3-521}$$

The loci of the closed-loop poles can be found by machine computation. They are given in Fig. 3.23. As expected, the faraway poles group into a Butterworth pattern of order two and the nearby closed-loop poles approach the open-loop zeroes. The system is further discussed in Example 3.22.

Example 3.21. *The control of the longitudinal motions of an airplane*

In Example 3.20 we considered the control of the pitch of an airplane through the elevator deflection. In the present example we extend the system by controlling, in addition to the pitch, the speed along the x-axis. As an additional control variable, we use the incremental engine thrust $T(t)$. Thus we choose for the input variable

$$c(t) = \begin{pmatrix} T(t) \\ \delta(t) \end{pmatrix} \quad \begin{matrix} \text{incremental engine thrust,} \\ \text{elevator deflection,} \end{matrix} \qquad \textbf{3-522}$$

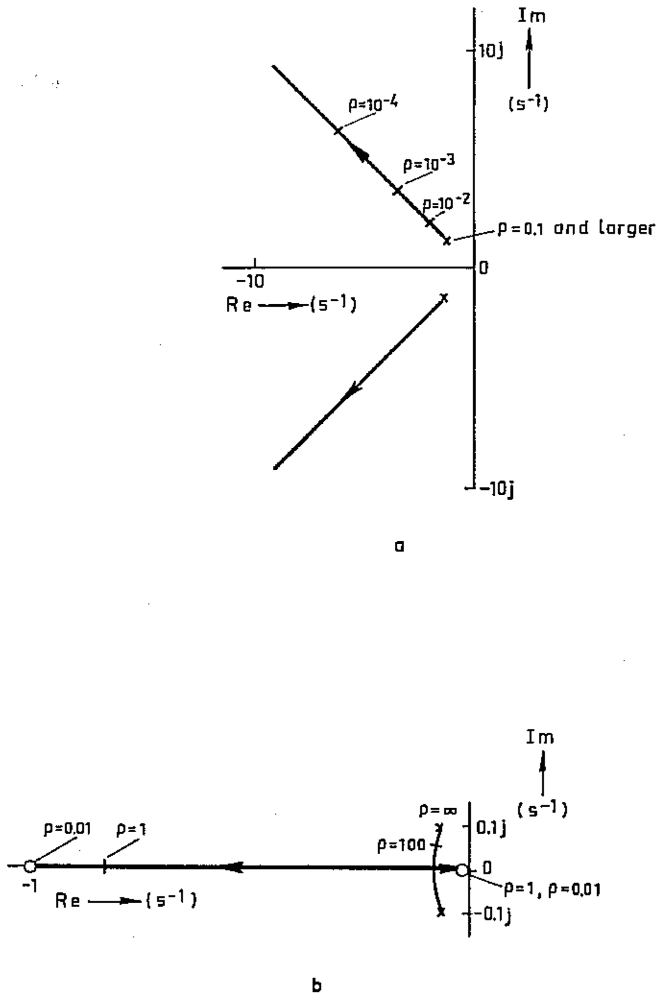


Fig. 3.23. Loci of the closed-loop poles of the pitch stabilization system. (*a*) Faraway poles; (*b*) nearby poles.

and for the controlled variable

$$z(t) = \begin{pmatrix} u(t) \\ \theta(t) \end{pmatrix} \quad \begin{matrix} \text{incremental speed along the } x\text{-axis,} \\ \text{pitch.} \end{matrix} \qquad \textbf{3-523}$$

From the system state differential equation **3-516**, it can be computed that the system transfer matrix has the numerator polynomial

$$\psi(s) = -0.003370(s + 1.002), \qquad \textbf{3-524}$$

which results in a single open-loop zero at -1.002. The open-loop poles are at $-0.006123 \pm j0.09353$ and $-1.250 \pm j1.394$.

Before analyzing the problem any further, we must establish the weighting matrices R_3 and N. For both we adopt a diagonal form and to determine their values we proceed in essentially the same manner as in Example 3.9 (Section 3.4.1) for the stirred tank. Suppose that $R_3 = \text{diag}(\sigma_1, \sigma_2)$. Then

$$z^T(t)R_3z(t) = \sigma_1 u^2(t) + \sigma_2\theta^2(t). \qquad \textbf{3-525}$$

Now let us assume that a deviation of 10 m/s in the speed along the x-axis is considered to be about as bad as a deviation of 0.2 rad (12°) in the pitch. We therefore select σ_1 and σ_2 such that

$$\sigma_1(10)^2 = \sigma_2(0.2)^2, \qquad \textbf{3-526}$$

or

$$\frac{\sigma_1}{\sigma_2} = 0.0004. \qquad \textbf{3-527}$$

Thus we choose

$$R_3 = \begin{pmatrix} 0.02 & 0 \\ 0 & 50 \end{pmatrix}, \qquad \textbf{3-528}$$

where for convenience we have let $\det(R_3) = 1$. Similarly, suppose that $N = \text{diag}(\rho_1, \rho_2)$ so that

$$c^T(t)Nc(t) = \rho_1 T^2(t) + \rho_2\,\delta^2(t). \qquad \textbf{3-529}$$

To determine ρ_1 and ρ_2, we assume that a deviation of 500 N in the engine thrust is about as acceptable as a deviation of 0.2 rad (12°) in the elevator deflection. This leads us to select

$$\rho_1(500)^2 = \rho_2(0.2)^2, \qquad \textbf{3-530}$$

which results in the following choice of N:

$$N = \begin{pmatrix} 0.0004 & 0 \\ 0 & 2500 \end{pmatrix}. \qquad \textbf{3-531}$$

With these values of R_3 and N, the relation **3-505** gives us the following estimate for the distance of the far-off poles:

$$\omega_0 = \left(\alpha^2 \frac{\det(R_3)}{\rho^k \det(N)}\right)^{1/[2(n-p)]} = \frac{0.15}{\rho^{1/3}}. \qquad \textbf{3-532}$$

The closed-loop pole locations must be found by machine computation. Table 3.4 lists the closed-loop poles for various values of ρ and also gives the estimated radius ω_0. We note first that one of the closed-loop poles approaches the open-loop zero at -1.002. Furthermore, we see that ω_0 is only a very crude estimate for the distance of the faraway poles from the origin.

The complete closed-loop loci are sketched in Fig. 3.24. It is noted that the appearance of these loci is quite different from those for single-input systems. Two of the faraway poles assume a second-order Butterworth configuration, while the third traces a first-order Butterworth pattern. The system is further discussed in Example 3.24.

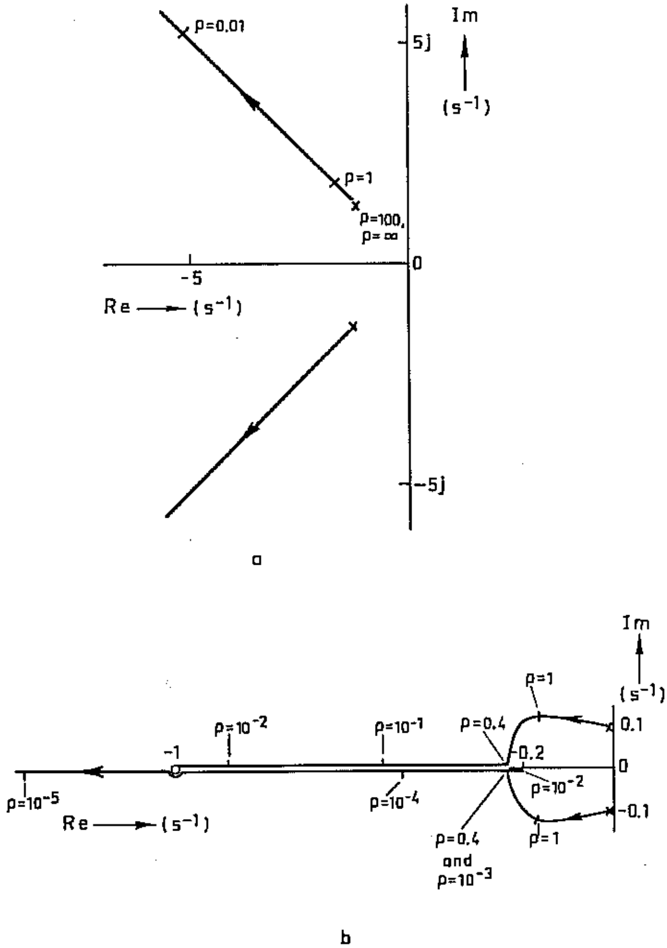


Fig. 3.24. Loci of the closed-loop poles for the longitudinal motion control system. (*a*) Faraway poles; (*b*) nearby pole and one faraway pole. For clarity the coinciding portions of the loci on the real axis are represented as distinct lines; in reality they coincide with the real axis.

Table 3.4 Closed-Loop Poles for the Longitudinal Motion Stability Augmentation System

ρ	Closed-loop poles (s^{-1})			ω_0 (s^{-1})
∞	$-0.006123 \pm j0.09353$		$-1.250 \pm j1.394$	0
1	$-0.1734 \pm j0.1184$		$-1.263 \pm j1.415$	0.15
10^{-1}	-0.5252	-0.2166	$-1.376 \pm j1.564$	0.32
10^{-2}	-0.8877	-0.2062	$-1.986 \pm j2.179$	0.70
10^{-3}	-0.9745	-0.2431	$-3.484 \pm j3.609$	1.5
10^{-4}	-0.9814	-0.4806	$-6.241 \pm j6.312$	3.2
10^{-5}	-1.020	-1.344	$-11.14 \pm j11.18$	7.0
10^{-6}	-1.003	-4.283	$-19.83 \pm j19.83$	15
10^{-8}	-1.002	-42.82	$-62.73 \pm j62.73$	70

3.8.2* Asymptotic Properties of the Single-Input Single-Output Nonzero Set Point Regulator

In this section we discuss the single-input single-output nonzero set point optimal regulator in the light of the results of Section 3.8.1. Consider the single-input system

$$\dot{x}(t) = Ax(t) + b\mu(t) \tag{3-533}$$

with the scalar controlled variable

$$\zeta(t) = dx(t). \tag{3-534}$$

Here b is a column vector and d a row vector. From Section 3.7 we know that the nonzero set point optimal control law is given by

$$\mu(t) = -\bar{f}x(t) + \frac{1}{H_c(0)}\zeta_0, \tag{3-535}$$

where $\bar{f}$ is the row vector

$$\bar{f} = \frac{1}{\rho} b^T \bar{P}, \tag{3-536}$$

with $\bar{P}$ the solution of the appropriate Riccati equation. Furthermore, $H_c(s)$ is the closed-loop transfer function

$$H_c(s) = d(sI - A + b\bar{f})^{-1}b, \tag{3-537}$$

and ζ_0 is the set point for the controlled variable.

In order to study the response of the regulator to a step change in the set point, let us replace ζ_0 with a time-dependent variable $\zeta_0(t)$. The interconnection of the open-loop system and the nonzero set point optimal control law is then described by

$$\dot{x}(t) = (A - b\bar{f})x(t) + b\,\frac{1}{H_c(0)}\,\zeta_0(t), \qquad \text{3-538}$$
$$\zeta(t) = dx(t).$$

Laplace transformation yields for the transfer function $T(s)$ from the variable set point $\zeta_0(t)$ to the controlled variable $\zeta(t)$:

$$T(s) = d(sI - A + b\bar{f})^{-1}b\,\frac{1}{H_c(0)}. \qquad \text{3-539}$$

Let us consider the closed-loop transfer function $d(sI - A + b\bar{f})^{-1}b$. Obviously,

$$d(sI - A + b\bar{f})^{-1}b = \frac{\psi_c(s)}{\phi_c(s)}, \qquad \text{3-540}$$

where $\phi_c(s) = \det(sI - A + b\bar{f})$ is the closed-loop characteristic polynomial and $\psi_c(s)$ is another polynomial. Now we saw in Section 3.7 (Eq. **3-428**) that the numerator of the determinant of a square transfer matrix $D(sI - A + BF)^{-1}B$ *is independent of the feedback gain matrix* F and is equal to the numerator polynomial of the open-loop transfer matrix $D(sI - A)^{-1}B$. Since in the single-input single-output case the determinant of the transfer function reduces to the transfer function itself, we can immediately conclude that $\psi_c(s)$ equals $\psi(s)$, which is defined from

$$H(s) = \frac{\psi(s)}{\phi(s)}. \qquad \text{3-541}$$

Here $H(s) = d(sI - A)^{-1}b$ is the open-loop transfer function and $\phi(s) = \det(sI - A)$ the open-loop characteristic polynomial.

As a result of these considerations, we conclude that

$$T(s) = \frac{\psi(s)}{\phi_c(s)}\,\frac{\phi_c(0)}{\psi(0)}. \qquad \text{3-542}$$

Let us write

$$\psi(s) = \alpha \prod_{i=1}^{p} (s - \nu_i), \qquad \text{3-543}$$

where the ν_i, $i = 1, 2, \cdots, p$, are the zeroes of $H(s)$. Then it follows from Theorem 3.11 that as $\rho \downarrow 0$ we can write for the closed-loop characteristic polynomial

$$\phi_c(s) \simeq \prod_{i=1}^{p} (s - \hat{\nu}_i) \prod_{i=1}^{n-p} (s - \eta_i\omega_0), \qquad \text{3-544}$$

where the $\hat{\nu}_i$, $i = 1, 2, \cdots, p$, are defined by **3-484**, the η_i, $i = 1, 2, \cdots,$

$n - p$, form a Butterworth configuration of order $n - p$ and radius 1, and where

$$\omega_0 = \left(\frac{\alpha^2}{\rho}\right)^{1/[2(n-p)]} \qquad \textbf{3-545}$$

Substitution of **3-544** into **3-542** yields the following approximation for $T(s)$:

$$T(s) \simeq \frac{1}{\prod_{i=1}^{n-p}\left(-\dfrac{s}{\eta_i\omega_0}+1\right)} \prod_{i=1}^{p}\left(\frac{-\dfrac{s}{\nu_i}+1}{-\dfrac{s}{\hat{\nu}_i}+1}\right). \qquad \textbf{3-546}$$

This we rewrite as

$$T(s) \simeq \frac{1}{\chi_{n-p}(s/\omega_0)} \prod_{i=1}^{p}\left(\frac{-\dfrac{s}{\nu_i}+1}{-\dfrac{s}{\hat{\nu}_i}+1}\right), \qquad \textbf{3-547}$$

where $\chi_{n-p}(s)$ is a *Butterworth polynomial* of order $n - p$, that is, $\chi_{n-p}(s)$ is defined by

$$\chi_{n-p}(s) = \prod_{i=1}^{n-p}\left(-\frac{s}{\eta_i}+1\right). \qquad \textbf{3-548}$$

Table 3.5 lists some low-order Butterworth polynomials (Weinberg, 1962).

Table 3.5 Butterworth Polynomials of Orders One through Five

$\chi_1(s) = s + 1$
$\chi_2(s) = s^2 + 1.414s + 1$
$\chi_3(s) = s^3 + 2s^2 + 2s + 1$
$\chi_4(s) = s^4 + 2.613s^3 + 3.414s^2 + 2.613s + 1$
$\chi_5(s) = s^5 + 3.236s^4 + 5.236s^3 + 5.236s^2 + 3.236s + 1$

The expression **3-547** shows that, if the open-loop transfer function has *zeroes in the left-half plane only*, the control system transfer function $T(s)$ approaches

$$\frac{1}{\chi_{n-p}(s/\omega_0)} \qquad \textbf{3-549}$$

as $\rho \downarrow 0$. We call this a *Butterworth transfer function* of order $n - p$ and break frequency ω_0. In Figs. 3.25 and 3.26, plots are given of the step responses and Bode diagrams of systems with Butterworth transfer functions

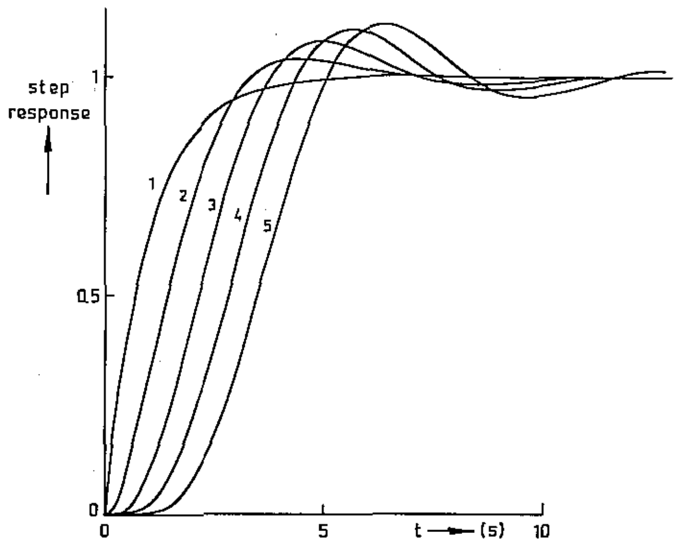


Fig. 3.25. Step responses of systems with Butterworth transfer functions of orders one through five with break frequencies 1 rad/s.

of various orders. The plots of Fig. 3.25 give an indication of the type of response obtained to steps in the set point. This response is asymptotically independent of the open-loop system poles and zeroes (provided the latter are in the left-half complex plane). We also see that by choosing ρ small enough the break frequency ω_0 can be made arbitrarily high, and correspondingly the settling time of the step response can be made arbitrarily small. An extremely fast response is of course obtained at the expense of large input amplitudes.

This analysis shows that the response of the controlled variable to changes in the set point is dominated by the far-off poles $\eta_i\omega_0$, $i = 1, 2, \cdots, n - p$. The nearby poles, which nearly coincide with the open-loop zeroes, have little effect on the response of the controlled variable because they nearly cancel against the zeroes. As we see in the next section, the far-off poles dominate not only the response of the controlled variable to changes in the set point but also the response to arbitrary initial conditions. As can easily be seen, and as illustrated in the examples, the nearby poles *do* show up in the *input*. The settling time of the tracking error is therefore determined by the faraway poles, but that of the input by the nearby poles.

The situation is less favorable for systems with *right-half plane zeroes.* Here the transmission $T(s)$ contains extra factors of the form

$$\frac{s + \vartheta_i}{s - \vartheta_i} \qquad \textbf{3-550}$$

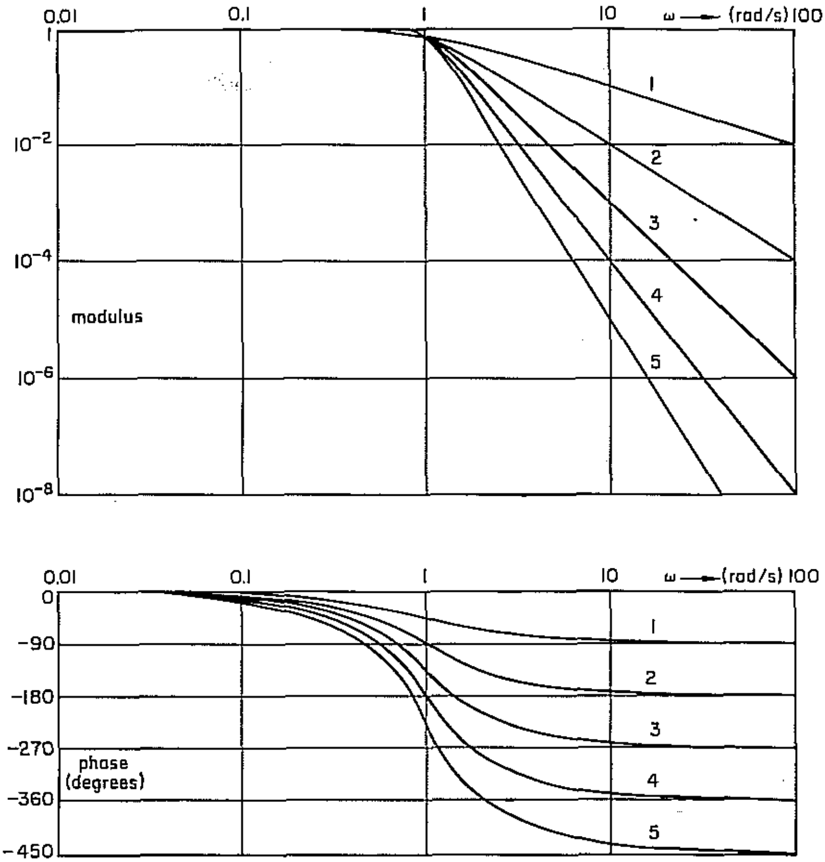


Fig. 3.26. Modulus and phase of Butterworth transfer functions of orders one through five with break frequencies 1 rad/s.

and the tracking error response is dominated by the nearby pole at $\hat{\nu}_i$. This points to an inherent limitation in the speed of response of systems with right-half plane zeroes. In the next subsection we further pursue this topic. First, however, we summarize the results of this section:

Theorem 3.13. *Consider the nonzero set point optimal control law* 3-535 *for the time-invariant, single-input single-output, stabilizable and detectable system*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + b\mu(t), \\ \zeta(t) &= dx(t), \end{aligned} \qquad \text{3-551}$$

where $R_3 = 1$ *and* $R_2 = \rho$. *Then as* $\rho \downarrow 0$ *the control system transmission* $T(s)$ *(i.e., the closed-loop transfer function from the variable set point* $\zeta_0(t)$

to the controlled variable $\zeta(t)$) approaches

$$T(s) \rightarrow \frac{1}{\chi_{n-p}(s/\omega_0)} \prod_{i=1}^{p} \left(\frac{-\dfrac{s}{\nu_i} + 1}{-\dfrac{s}{\hat{\nu}_i} + 1} \right), \qquad \text{3-552}$$

where $\chi_{n-p}(s)$ is a Butterworth polynomial of order $n - p$ and radius 1, n is the order of the system, p is the number of zeroes of the open-loop transfer function of the system, ω_0 is the asymptotic radius of the Butterworth configuration of the faraway closed-loop poles as given by **3-486**, *ν_i, $i = 1, 2, \cdots, p$, are the zeroes of the open-loop transfer function, and $\hat{\nu}_i$, $i = 1, 2, \cdots, p$, are the open-loop transfer function zeroes mirrored into the left-half complex plane.*

Example 3.22. *Pitch control*

Consider the pitch control problem of Example 3.20. For $\rho = 0.01$ the steady-state feedback gain matrix can be computed to be

$$\bar{F} = (-0.0001174, 0.002813, -10.00, -1.619). \qquad \text{3-556}$$

The corresponding closed-loop characteristic polynomial is given by

$$\phi_c(s) = s^4 + 11.49s^3 + 66.43s^2 + 56.84s + 1.112. \qquad \text{3-557}$$

The closed-loop poles are

$$-0.02004, \ -0.9953, \quad \text{and} \quad -0.5239 \pm j5.323. \qquad \text{3-558}$$

We see that the first two poles are very close to the open-loop zeroes at -0.2004 and -0.9976. The closed-loop transfer function is given by

$$H_c(s) = \frac{\psi(s)}{\phi_c(s)} = \frac{-5.565s^2 - 5.663s - 0.1112}{s^4 + 11.49s^3 + 66.43s^2 + 56.84s + 1.112}, \qquad \text{3-559}$$

so that $H_c(0) = -0.1000$. As a result, the nonzero set point control law is given by

$$\delta(t) = -\bar{F}x(t) - 10.00\theta_0(t), \qquad \text{3-560}$$

where $\theta_0(t)$ is the set point of the pitch.

Figure 3.27 depicts the response of the system to a step of 0.1 rad in the set point $\theta_0(t)$. It is seen that the pitch θ quickly settles at the desired value; its response is completely determined by the second-order Butterworth configuration at $-5.239 \pm j5.323$. The pole at -0.9953 (corresponding to a time constant of about 1 s) shows up most clearly in the response of the speed along the z-axis w and can also be identified in the behavior of the elevator deflection δ. The very slow motion with a time constant of 50 s, which

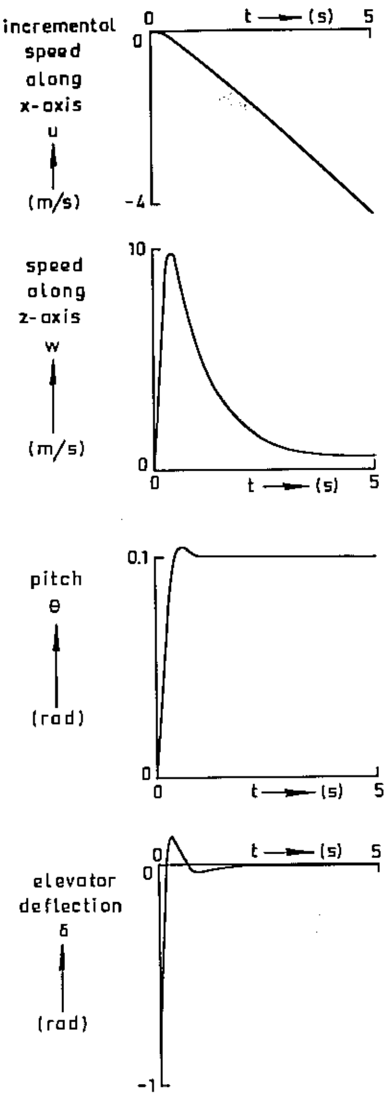


Fig. 3.27. Response of the pitch control system to a step of 0.1 rad in the pitch angle set point.

corresponds to the pole at -0.02004, is represented in the response of the speed along the x-axis u, the speed along the z-axis w, and also in the elevator deflection δ, although this is not visible in the plot. It takes about 2 min for u and w to settle at the steady-state values -49.16 and 7.754 m/s.

Note that this control law yields an initial elevator deflection of -1 rad which, practically speaking, is far too large.

Example 3.23. *System with a right-half plane zero*

As a second example consider the single-input system with state differential equation

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mu(t). \qquad \textbf{3-561}$$

Let us choose for the controlled variable

$$\zeta(t) = (1, \quad -1)x(t). \qquad \textbf{3-562}$$

This system has the open-loop transfer function

$$H(s) = \frac{-s+1}{s(s+2)}, \qquad \textbf{3-563}$$

and therefore has a zero in the right-half plane. Consider for this system the criterion

$$\int_{t_0}^{\infty} [\zeta^2(t) + \rho\mu^2(t)]\, dt. \qquad \textbf{3-564}$$

It can be found that the corresponding Riccati equation has the steady-state solution

$$\bar{P} = \begin{pmatrix} 1 + \sqrt{1 + 4\rho + 2\sqrt{\rho}} & \sqrt{\rho} \\ \sqrt{\rho} & \rho\left(-2 + \sqrt{4 + \dfrac{1}{\rho} + \dfrac{2}{\sqrt{\rho}}}\right) \end{pmatrix}. \qquad \textbf{3-565}$$

The corresponding steady-state feedback gain vector is

$$\bar{f} = \left(\frac{1}{\sqrt{\rho}}, \quad -2 + \sqrt{4 + \frac{1}{\rho} + \frac{2}{\sqrt{\rho}}}\right). \qquad \textbf{3-566}$$

The closed-loop poles can be found to be

$$\frac{1}{2}\left(-\sqrt{4 + \frac{1}{\rho} + \frac{2}{\sqrt{\rho}}} \pm \sqrt{4 + \frac{1}{\rho} - \frac{2}{\sqrt{\rho}}}\right). \qquad \textbf{3-567}$$

Figure 3.28 gives a sketch of the loci of the closed-loop poles. As expected, one of the closed-loop poles approaches the mirror image of the right-half plane zero, while the other pole goes to $-\infty$ along the real axis.

For $\rho = 0.04$ the closed-loop characteristic polynomial is given by

$$s^2 + 6.245s + 5, \qquad \textbf{3-568}$$

and the closed-loop poles are located at -0.943 and -5.302. The closed-loop

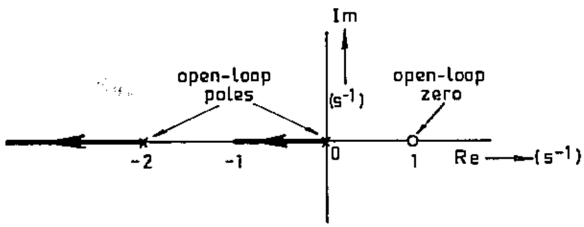


Fig. 3.28. Loci of the closed-loop poles for a system with a right-half plane zero.

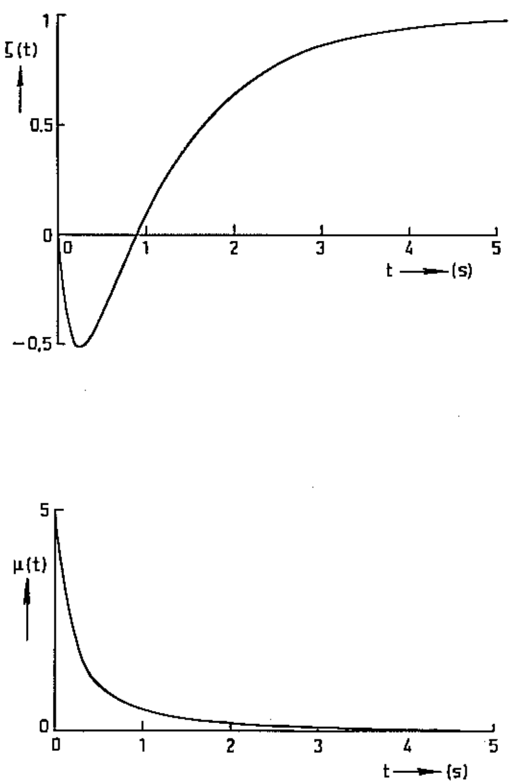


Fig. 3.29. Response of a closed-loop system with a right-half plane zero to a unit step in the set point.

transfer function is

$$H_c(s) = \frac{\psi(s)}{\phi_c(s)} = \frac{-s+1}{s^2+6.245s+5}, \qquad \textbf{3-569}$$

so that $H_c(0) = 0.2$. The steady-state feedback gain vector is

$$\bar{F} = (5, \quad 4.245). \qquad \textbf{3-570}$$

As a result, the nonzero set point control law is

$$\mu(t) = -(5, \quad 4.245)x(t) + 5\zeta_0(t). \qquad \textbf{3-571}$$

Figure 3.29 gives the response of the closed-loop system to a step in the set point $\zeta_0(t)$. We see that in this case the response is dominated by the closed-loop pole at -0.943. It is impossible to obtain a response that is faster and at the same time has a smaller integrated square tracking error.

3.8.3* The Maximally Achievable Accuracy of Regulators and Tracking Systems

In this section we study the steady-state solution of the Riccati equation as ρ approaches zero in

$$R_2 = \rho N. \qquad \textbf{3-572}$$

The reason for our interest in this asymptotic solution is that it will give us insight into the maximally achievable accuracy of regulator and tracking systems when no limitations are imposed upon the input amplitudes.

This section is organized as follows. First, the main results are stated in the form of a theorem. The proof of this theorem (Kwakernaak and Sivan, 1972), which is long and technical, is omitted. The remainder of the section is devoted to a discussion of the results and to examples.

We first state the main results:

Theorem 3.14. *Consider the time-invariant stabilizable and detectable linear system*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ z(t) &= Dx(t), \end{aligned} \qquad \textbf{3-573}$$

where B and D are assumed to have full rank. Consider also the criterion

$$\int_{t_0}^{\infty} [z^T(t)R_3z(t) + u^T(t)R_2u(t)]\, dt \qquad \textbf{3-574}$$

where $R_3 > 0$, $R_2 > 0$. Let

$$R_2 = \rho N, \qquad \textbf{3-575}$$

with $N > 0$ and ρ a positive scalar, and let $\bar{P}_\rho$ be the steady-state solution of *the Riccati equation*

$$-\dot{P}_\rho(t) = D^T R_3 D - P_\rho(t) B R_2^{-1} B^T P_\rho(t) + A^T P_\rho(t) + P_\rho(t) A, \qquad P_\rho(t_1) = 0. \tag{3-576}$$

Then the following facts hold.

(a) *The limit*

$$\lim_{\rho \downarrow 0} \bar{P}_\rho = P_0 \tag{3-577}$$

exists.

(b) *Let $z_\rho(t)$, $t \geq t_0$, denote the response of the controlled variable for the regulator that is steady-state optimal for $R_2 = \rho N$. Then*

$$\lim_{\rho \downarrow 0} \int_{t_0}^{\infty} z_\rho^T(t) R_3 z_\rho(t)\, dt = x^T(t_0) P_0 x(t_0). \tag{3-578}$$

(c) *If* dim $(z) >$ dim (u), *then* $P_0 \neq 0$.

(d) *If* dim $(z) =$ dim (u) *and the numerator polynomial $\psi(s)$ of the open-loop transfer matrix $H(s) = D(sI - A)^{-1}B$ is nonzero, $P_0 = 0$ if and only if $\psi(s)$ has zeroes with nonpositive real parts only.*

(e) *If* dim $(z) <$ dim (u), *then a sufficient condition for P_0 to be* 0 *is that there exists a rectangular matrix M such that the numerator polynomial $\psi(s)$ of the square transfer matrix $D(sI - A)^{-1}BM$ is nonzero and has zeroes with nonpositive real parts only.*

A discussion of the significance of the various parts of the theorem now follows. Item (a) states that, as we let the weighting coefficient of the input ρ decrease, the criterion

$$\int_{t_0}^{\infty} [z_\rho^T(t) R_3 z_\rho(t) + \rho u_\rho^T(t) N u_\rho(t)]\, dt = x^T(t_0) \bar{P}_\rho x(t_0) \tag{3-579}$$

approaches a limit $x^T(t_0) P_0 x(t_0)$. If we identify R_3 with W_e and N with W_u, the expression **3-579** can be rewritten as

$$\int_{t_0}^{\infty} C_{e,\rho}(t)\, dt + \rho \int_{t_0}^{\infty} C_{u,\rho}(t)\, dt, \tag{3-580}$$

where $C_{e,\rho}(t) = z_\rho^T(t) W_e z_\rho(t)$ is the weighted square regulating error and $C_{u,\rho}(t) = u_\rho^T(t) W_u u_\rho(t)$ the weighted square input. It follows from item (b) of the theorem that as $\rho \downarrow 0$, of the two terms in **3-580** the first term, that is, the integrated square regulating error, fully accounts for the two terms together so that in the limit the integrated square regulating error is given by

$$\lim_{\rho \downarrow 0} \int_{t_0}^{\infty} C_{e,\rho}(t)\, dt = x^T(t_0) P_0 x(t_0). \tag{3-581}$$

If the weighting coefficient ρ is zero, no costs are spared in the sense that no limitations are imposed upon the input amplitudes. Clearly, under this condition the greatest accuracy in regulation is achieved in the sense that the integrated square regulation error is the least that can ever be obtained.

Parts (c), (d), and (e) of the theorem are concerned with the conditions under which $P_0 = 0$, which means that ultimately perfect regulation is approached since

$$\lim_{\rho \downarrow 0} \int_{t_0}^{\infty} C_{e,\rho}(t)\, dt = 0. \tag{3-582}$$

Part (c) of the theorem states that, if the dimension of the controlled variable is greater than that of the input, perfect regulation is impossible. This is very reasonable, since in this case the number of degrees of freedom to control the system is too small. In order to determine the maximal accuracy that can be achieved, P_0 must be computed. Some remarks on how this can be done are given in Section 4.4.4.

In part (d) the case is considered where the number of degrees of freedom is sufficient, that is, the input and the controlled variable have the same dimensions. Here the maximally achievable accuracy is dependent upon the properties of the open-loop system transfer matrix $H(s)$. Perfect regulation is possible only when the numerator polynomial $\psi(s)$ of the transfer matrix has no right-half plane zeroes (assuming that $\psi(s)$ is not identical to zero). This can be made intuitively plausible as follows. Suppose that at time 0 the system is in the initial state x_0. Then in terms of Laplace transforms the response of the controlled variable can be expressed as

$$\mathbf{Z}(s) = H(s)\mathbf{U}(s) + D(sI - A)^{-1}x_0, \tag{3-583}$$

where $\mathbf{Z}(s)$ and $\mathbf{U}(s)$ are the Laplace transforms of z and u, respectively. $\mathbf{Z}(s)$ can be made identical to zero by choosing

$$\mathbf{U}(s) = -H^{-1}(s)D(sI - A)^{-1}x_0. \tag{3-584}$$

The input $u(t)$ in general contains delta functions and derivatives of delta functions at time 0. These delta functions instantaneously transfer the system from the state x_0 at time 0 to a state $x(0^+)$ that has the property that $z(0^+) = Dx(0^+) = 0$ *and* that $z(t)$ can be maintained at 0 for $t > 0$ (Sivan, 1965). Note that in general the state $x(t)$ undergoes a delta function and derivative of delta function type of trajectory at time 0 but that $z(t)$ moves from $z(0) = Dx_0$ to 0 directly, without infinite excursions, as can be seen by inserting **3-584** into **3-583**.

The expression **3-584** leads to a stable behavior of the input only if the inverse transfer matrix $H^{-1}(s)$ is stable, that is if the numerator polynomial $\psi(s)$ of $H(s)$ has no right-half plane zeroes. The reason that the input **3-584** cannot be used in the case that $H^{-1}(s)$ has unstable poles is that although the input **3-584** drives the controlled variable $z(t)$ to zero and maintains $z(t)$ at zero, the input itself grows indefinitely (Levy and Sivan, 1966). By our problem formulation such inputs are ruled out, so that in this case **3-584** is not the limiting input as $\rho \downarrow 0$ and, in fact, costless regulation cannot be achieved.

Finally, in part (e) of the theorem, we see that if dim $(z) <$ dim (u), then $P_0 = 0$ if the situation can be reduced to that of part (d) by replacing the input u with an input u' of the form

$$u'(t) = Mu(t). \qquad \textbf{3-585}$$

The existence of such a matrix M is not a necessary condition for P_0 to be zero, however.

Theorem 3.14 extends some of the results of Section 3.8.2. There we found that for single-input single-output systems without zeroes in the right-half complex plane the response of the controlled variable to steps in the set point is asymptotically completely determined by the faraway closed-loop poles and not by the nearby poles. The reason is that the nearby poles are canceled by the zeroes of the system. Theorem 3.14 leads to more general conclusions. It states that for multiinput multioutput systems without zeroes in the right-half complex plane the integrated square regulating error goes to zero asymptotically. This means that for small values of ρ the closed-loop response of the controlled variable to any initial condition of the system is very fast, which means that this response is determined by the faraway closed-loop poles only. Consequently, also in this case the effect of the nearby poles is canceled by the zeroes. The slow motion corresponding to the nearby poles of course shows up in the response of the input variable, so that in general the input can be expected to have a much longer settling time than the controlled variable. For illustrations we refer to the examples.

It follows from the theory that optimal regulator systems can have "hidden modes" which do not appear in the controlled variable but which do appear in the state and the input. These modes may impair the operation of the control system. Often this phenomenon can be remedied by redefining or extending the controlled variable so that the requirements upon the system are more faithfully reflected.

It also follows from the theory that systems with right-half plane zeroes are fundamentally deficient in their capability to regulate since the mirror images of the right-half plane zeroes appear as nearby closed-loop poles which are not canceled by zeroes. If these right-half plane zeroes are far away from the origin, however, their detrimental effect may be limited.

It should be mentioned that ultimate accuracy can of course never be achieved since this would involve infinite feedback gains and infinite input amplitudes. The results of this section, however, give an idea of the ideal performance of which the system is capable. In practice, this limit may not nearly be approximated because of the constraints on the input amplitudes.

So far the discussion has been confined to the deterministic regulator problem. Let us now briefly consider the stochastic regulator problem, which includes tracking problems. As we saw in Section 3.6, we have for the stochastic regulator problem

$$C_{e\infty,\rho} + \rho C_{u\infty,\rho} = \mathrm{tr}\,(\bar{P}V), \qquad \textbf{3-586}$$

where $C_{e\infty}$ and $C_{u\infty}$ indicate the steady-state mean square regulation error and the steady-state mean square input, respectively. It immediately follows that

$$\lim_{\rho \downarrow 0} (C_{e\infty,\rho} + \rho C_{u\infty,\rho}) = \mathrm{tr}\,(P_0 V). \qquad \textbf{3-587}$$

It is not difficult to argue [analogously to the proof of part (b) of Theorem 3.14] that of the two terms in **3-587** the first term fully accounts for the left-hand side so that

$$\lim_{\rho \downarrow 0} C_{e\infty,\rho} = \mathrm{tr}\,(P_0 V). \qquad \textbf{3-588}$$

This means that perfect stochastic regulation ($P_0 = 0$) can be achieved under the same conditions for which perfect deterministic regulation is possible. It furthermore is easily verified that, for the regulator with nonwhite disturbances (Section 3.6.1) and for the stochastic tracking problem (Section 3.6.2), perfect regulation or tracking, respectively, is achieved if and only if in both cases the *plant* transfer matrix $H(s) = D(sI - A)^{-1}B$ satisfies the conditions outlined in Theorem 3.14. This shows that it is the plant alone that determines the maximally achievable accuracy and not the properties of the disturbances or the reference variable.

In conclusion, we note that Theorem 3.14 gives no results for the case in which the numerator polynomial $\psi(s)$ is identical to zero. This case rarely seems to occur, however.

Example 3.24. *Control of the longitudinal motions of an airplane*

As an example of a multiinput system, we consider the regulation of the longitudinal motions of an airplane as described in Example 3.21. For $\rho = 10^{-6}$ we found in Example 3.21 that the closed-loop poles are -1.003, -4.283, and $-19.83 \pm j19.83$. The first of these closed-loop poles practically coincides with the open-loop zero at -1.002.

Figure 3.30 shows the response of the closed-loop system to an initial deviation in the speed along the x-axis u, and to an initial deviation in the pitch θ. It is seen that the response of the speed along the x-axis is determined

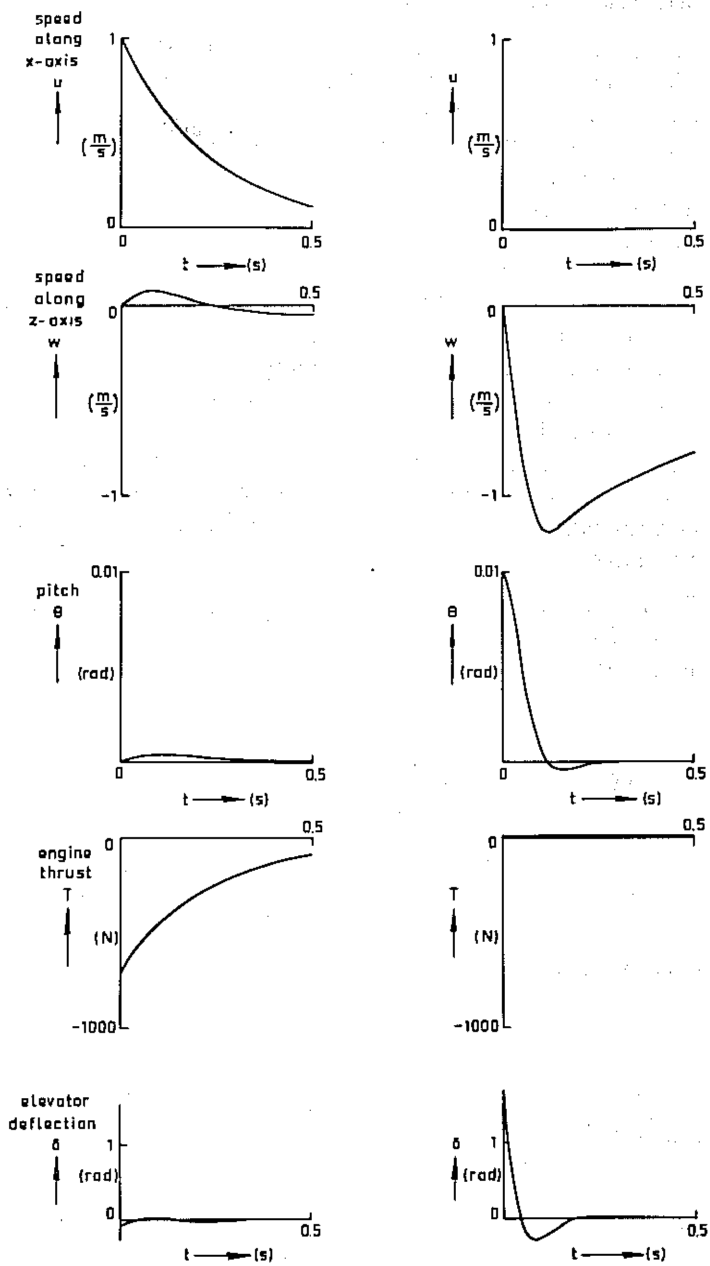


Fig. 3.30. Closed-loop responses of a longitudinal stability augmentation system for an airplane. Left column: Responses to the initial state $u(0)=1$ m/s, while all other components of the initial state are zero. Right column: Response to the initial state $\theta(0) = 0.01$ rad, while all other components of the initial state are zero.

mainly by a time constant of about 0.24 s which corresponds to the pole at -4.283. The response of the pitch is determined by the Butterworth configuration at $-19.83 \pm j19.83$. The slow motion with a time constant of about 1 s that corresponds to the pole at -1.003 only affects the response of the speed along the z-axis w.

We note that the controlled system exhibits very little *interaction* in the sense that the restoration of the speed along the x-axis does not result in an appreciable deviation of the pitch, and conversely.

Finally, it should be remarked that the value $\rho = 10^{-6}$ is not suitable from a practical point of view. It causes far too large a change in the engine thrust and the elevator angle. In addition, the engine is unable to follow the fast thrust changes that this control law requires. Further investigation should take into account the dynamics of the engine.

The example confirms, however, that since the plant has no right-half plane zeroes an arbitrarily fast response can be obtained, and that the nearby pole that corresponds to the open-loop zero does not affect the response of the controlled variable.

Example 3.25. *A system with a right-half plane zero*

In Example 3.23 we saw that the system described by **3-561** and **3-562** with the open-loop transfer function

$$H(s) = \frac{-s+1}{s(s+2)} \qquad \textbf{3-589}$$

has the following steady-state solution of the Riccati equation

$$\bar{P} = \begin{pmatrix} 1+\sqrt{1+4\rho+2\sqrt{\rho}} & \sqrt{\rho} \\ \sqrt{\rho} & \rho\left(-2+\sqrt{4+\dfrac{1}{\rho}+\dfrac{2}{\sqrt{\rho}}}\right) \end{pmatrix}. \qquad \textbf{3-590}$$

As ρ approaches zero, $\bar{P}$ approaches P_0, where

$$P_0 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}. \qquad \textbf{3-591}$$

As we saw in Example 3.23, in the limit $\rho \downarrow 0$ the response is dominated by the closed-loop pole at -1.

3.9* SENSITIVITY OF LINEAR STATE FEEDBACK CONTROL SYSTEMS

In Chapter 2 we saw that a very important property of a feedback system is its ability to suppress disturbances and to compensate for parameter changes.

In this section we investigate to what extent optimal regulators and tracking systems possess these properties. When we limit ourselves to time-invariant problems and consider only the steady-state case, where the terminal time is at infinity, the optimal regulator and tracking systems we have derived have the structure of Fig. 3.31. The optimal control law can generally be represented

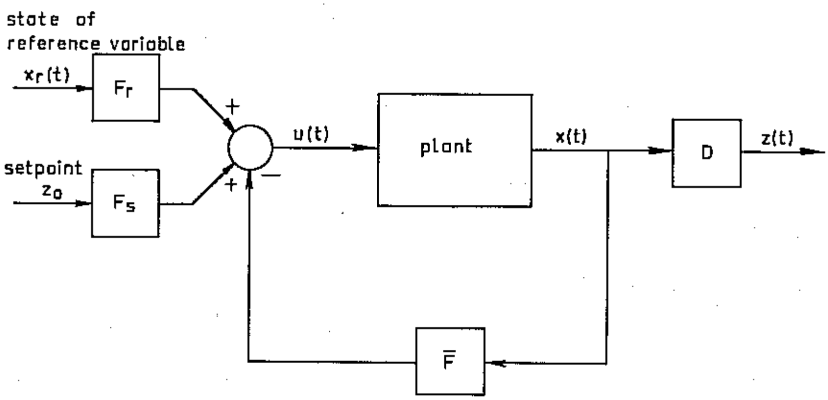


Fig. 3.31. The structure of a time-invariant linear state feedback control system.

in the form

$$u(t) = -\bar{F}x(t) + F_r x_r(t) + F_s z_0, \qquad \text{3-592}$$

where $x_r(t)$ is the state of the reference variable, z_0 the set point, and $\bar{F}$, F_r, and F_s are constant matrices. The matrix $\bar{F}$ is given by

$$\bar{F} = R_2^{-1}B^T\bar{P}, \qquad \text{3-593}$$

where $\bar{P}$ is the nonnegative-definite solution of the algebraic Riccati equation

$$0 = D^T R_3 D - \bar{P}BR_2^{-1}B^T\bar{P} + A^T\bar{P} + \bar{P}A. \qquad \text{3-594}$$

In Chapter 2 (Section 2.10) we saw that the ability of the closed-loop system to suppress disturbances or to compensate for parameter changes as compared to an equivalent open-loop configuration is determined by the behavior of the return difference matrix $J(s)$. Let us derive $J(s)$ in the present case. The transfer matrix of the plant is given by $(sI - A)^{-1}B$, while that of the feedback link is simply $\bar{F}$. Thus the return difference matrix is

$$J(s) = I + (sI - A)^{-1}B\bar{F}. \qquad \text{3-595}$$

Note that we consider the complete state $x(t)$ as the controlled variable (see Section 2.10).

We now derive an expression for $J(s)$ starting from the algebraic Riccati equation **3-594**. Addition and substraction of an extra term $s\bar{P}$ yields after rearrangement

$$0 = D^T R_3 D - \bar{P}BR_2^{-1}B^T\bar{P} - (-sI - A^T)\bar{P} - \bar{P}(sI - A). \quad \textbf{3-596}$$

Premultiplication by $B^T(-sI - A^T)^{-1}$ and postmultiplication by $(sI - A)^{-1}B$ gives

$$0 = B^T(-sI - A^T)^{-1}(-\bar{P}BR_2^{-1}B^T\bar{P} + D^TR_3D^T)(sI - A)^{-1}B \\ - B^T\bar{P}(sI - A)^{-1}B - B^T(-sI - A^T)^{-1}\bar{P}B. \quad \textbf{3-597}$$

This can be rearranged as follows:

$$[I + B^T(-sI - A^T)^{-1}\bar{P}BR_2^{-1}]R_2[I + R_2^{-1}B^T\bar{P}(sI - A)^{-1}B] \\ = R_2 + B^T(-sI - A^T)^{-1}D^TR_3D(sI - A)^{-1}B. \quad \textbf{3-598}$$

After substitution of $R_2^{-1}B^T\bar{P} = \bar{F}$, this can be rewritten as

$$[I + B^T(-sI - A^T)^{-1}\bar{F}^T]R_2[I + \bar{F}(sI - A)^{-1}B] \\ = R_2 + H^T(-s)R_3H(s), \quad \textbf{3-599}$$

where $H(s) = D(sI - A)^{-1}B$. Premultiplication of both sides of **3-599** by $\bar{F}^T$ and postmultiplication by $\bar{F}$ yields after a simple manipulation

$$[I + \bar{F}^TB^T(-sI - A^T)^{-1}]\bar{F}^TR_2\bar{F}[I + (sI - A)^{-1}B\bar{F}] \\ = \bar{F}^TR_2\bar{F} + \bar{F}^TH^T(-s)R_3H(s)\bar{F}, \quad \textbf{3-600}$$

or

$$J^T(-s)\bar{F}^TR_2\bar{F}J(s) = \bar{F}^TR_2\bar{F} + \bar{F}^TH^T(-s)R_3H(s)\bar{F}. \quad \textbf{3-601}$$

If we now substitute $s = j\omega$, we see that the second term on the right-hand side of this expression is nonnegative-definite Hermitian; this means that we can write

$$J^T(-j\omega)WJ(j\omega) \geq W \qquad \text{for all real } \omega, \quad \textbf{3-602}$$

where

$$W = \bar{F}^TR_2\bar{F}. \quad \textbf{3-603}$$

We know from Section 2.10 that a condition of the form **3-602** guarantees disturbance suppression and compensation of parameter changes as compared to the equivalent open-loop system *for all frequencies*. This is a useful result. We know already from Section 3.6 that the optimal regulator gives *optimal* protection against *white* noise disturbances entering at the input side of the plant. The present result shows, however, that protection against disturbances is not restricted to this special type of disturbances only. By the same token, compensation of parameter changes is achieved.

Thus we have obtained the following result (Kreindler, 1968b; Anderson and Moore, 1971).

Theorem 3.15. *Consider the system configuration of Fig. 3.31, where the "plant" is the detectable and stabilizable time-invariant system*

$$\dot{x}(t) = Ax(t) + Bu(t). \tag{3-604}$$

Let the feedback gain matrix be given by

$$\bar{F} = R_2^{-1}B^T\bar{P}, \tag{3-605}$$

where $\bar{P}$ is the nonnegative-definite solution of the algebraic Riccati equation

$$0 = D^T R_3 D - \bar{P}BR_2^{-1}B^T\bar{P} + A^T\bar{P} + \bar{P}A. \tag{3-606}$$

Then the return difference

$$J(s) = I + (sI - A)^{-1}B\bar{F} \tag{3-607}$$

satisfies the inequality

$$J^T(-j\omega)WJ(j\omega) \geq W \quad \text{for all real } \omega, \tag{3-608}$$

where

$$W = \bar{F}^T R_2 \bar{F}. \tag{3-609}$$

For an extension of this result to time-varying systems, we refer the reader to Kreindler (1969).

It is clear that with the configuration of Fig. 3.31 improved protection is achieved only against disturbances and parameter variations *inside* the feedback loop. In particular, variations in D fully affect the controlled variable $z(t)$. It frequently happens, however, that D does not exhibit variations. This is especially the case if the controlled variable is composed of components of the state vector, which means that $z(t)$ is actually inside the loop (see Fig. 3.32).

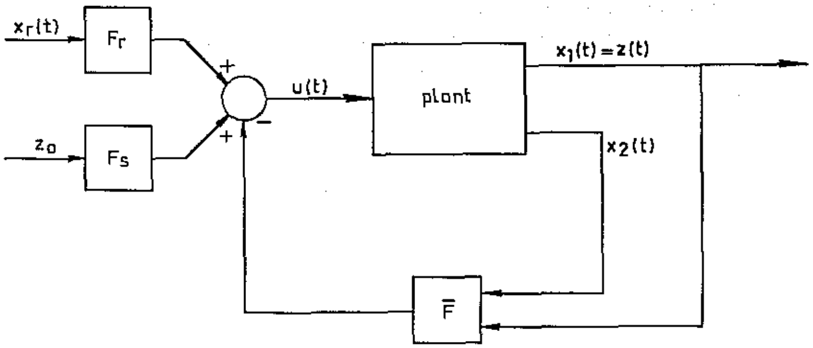


Fig. 3.32. Example of a situation in which the controlled variable is inside the feedback loop.

Theorem 3.15 has the shortcoming that the weighting matrix $\bar{F}^T R_2 \bar{F}$ is known only *after* the control law has been computed; this makes it difficult to choose the design parameters R_3 and R_2 such as to achieve a given weighting matrix. We shall now see that under certain conditions it is possible to determine an asymptotic expression for W. In Section 3.8.3 it was found that if $\dim(z) = \dim(u)$, and the open-loop transfer matrix $H(s) = D(sI - A)^{-1}B$ does not have any right-half plane zeroes, the solution $\bar{P}$ of the algebraic Riccati equation approaches the zero matrix as the weighting matrix R_2 approaches the zero matrix. A glance at the algebraic Riccati equation **3-594** shows that this implies that

$$\bar{P}BR_2^{-1}B^T\bar{P} \rightarrow D^T R_3 D \tag{3-610}$$

as $R_2 \rightarrow 0$, or, since $R_2^{-1}B^T\bar{P} = \bar{F}$, that

$$\bar{F}^T R_2 \bar{F} \rightarrow D^T R_3 D \tag{3-611}$$

as $R_2 \rightarrow 0$. This proves that the weighting matrix W in the sensitivity criterion **3-608** approaches $D^T R_3 D$ as $R_2 \rightarrow 0$.

We have considered the entire state $x(t)$ as the feedback variable. This means that the weighted square tracking error is

$$x^T(t) W x(t). \tag{3-612}$$

From the results we have just obtained, it follows that as $R_2 \rightarrow 0$ this can be replaced with

$$x^T(t) D^T R_3 D x(t) = z^T(t) R_3 z(t). \tag{3-613}$$

This means (see Section 2.10) that in the limit $R_2 \rightarrow 0$ the controlled variable receives all the protection against disturbances and parameter variations, and that the components of the controlled variable are weighted by R_3. This is a useful result because it is the controlled variable we are most interested in.

The property derived does *not* hold, however, for plants with zeroes in the right-half plane, or with too few inputs, because here $\bar{P}$ does not approach the zero matrix.

We summarize our conclusions:

Theorem 3.16. *Consider the weighting matrix*

$$W = \bar{F}^T R_2 \bar{F}, \tag{3-614}$$

where

$$\bar{F} = R_2^{-1} B^T \bar{P}, \tag{3-615}$$

with $\bar{P}$ the nonnegative-definite symmetric solution of

$$0 = D^T R_3 D - \bar{P}BR_2^{-1}B^T\bar{P} + A^T\bar{P} + \bar{P}A. \tag{3-616}$$

If the conditions are satisfied (Theorem 3.14) under which $\bar{P} \rightarrow 0$ *as* $R_2 \rightarrow 0$, *then*

$$W \rightarrow D^T R_3 D \tag{3-617}$$

as $R_2 \rightarrow 0$.

The results of this section indicate in a general way that state feedback systems offer protection against disturbances and parameter variations. Since sensitivity matrices are not very convenient to work with, indications as to what to do for specific parameter variations are not easily found. The following general conclusions are valid, however.

1. As the weighting matrix R_2 is decreased the protection against disturbances and parameter variations improves, since the feedback gains increase. For plants with zeroes in the left-half complex plane only, the break frequency up to which protection is obtained is determined by the faraway closed-loop poles, which move away from the origin as R_2 decreases.

2. For plants with zeroes in the left-half plane only, most of tbe protection extends to the controlled variable. The weight attributed to the various components of the controlled variable is determined by the weighting matrix R_3.

3. For plants with zeroes in the right-half plane, the break frequency up to which protection is obtained is limited by those nearby closed-loop poles that are not canceled by zeroes.

Example 3.26. *Position control system*

As an illustration of the theory of this section, let us perform a brief sensitivity analysis of the position control system of Example 3.8 (Section 3.4.1). With the numerical values given, it is easily found that the weighting matrix in the sensitivity criterion is given by

$$W = \bar{F}^T R_2 \bar{F} = \begin{pmatrix} 1 & 0.08364 \\ 0.08364 & 0.006994 \end{pmatrix}. \tag{3-618}$$

This is quite close to the limiting value

$$D^T R_3 D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{3-619}$$

To study the sensitivity of the closed-loop system to parameter variations, in Fig. 3.33 the response of the closed-loop system is depicted for nominal and off-nominal conditions. Here the off-nominal conditions are caused by a change in the inertia of the load driven by the position control system. The curves a correspond to the nominal case, while in the case of curves b and c the combined inertia of load and armature of the motor is $\frac{2}{3}$ of nominal

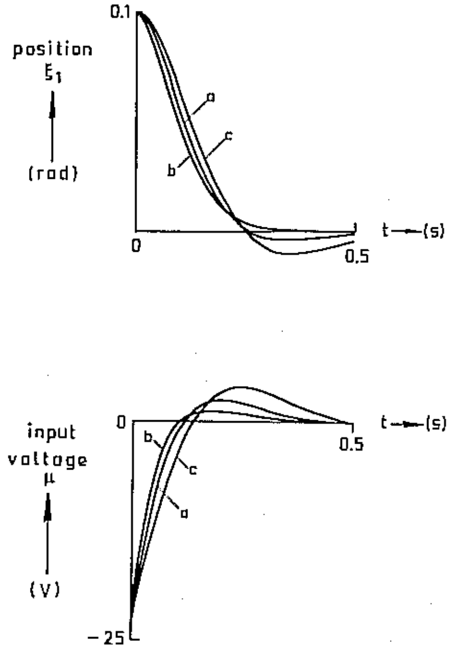


Fig. 3.33. The effect of parameter variations on the response of the position control system: (*a*) Nominal load; (*b*) inertial load $\frac{2}{3}$ of nominal; (*c*) inertial load $\frac{3}{2}$ of nominal.

and $\frac{3}{2}$ of nominal, respectively. A change in the total moment of inertia by a certain factor corresponds to division of the constants α and κ by the same factor. Thus $\frac{2}{3}$ of the nominal moment of inertia yields 6.9 and 1.18 for α and κ, respectively, while $\frac{3}{2}$ of the nominal moment of inertia results in the values 3.07 and 0.525 for α and κ, respectively. Figure 3.33 vividly illustrates the limited effect of relatively large parameter variations.

3.10 CONCLUSIONS

This chapter has dealt with state feedback control systems where all the components of the state can be accurately measured at all times. We have discussed quite extensively how linear state feedback control systems can be designed that are optimal in the sense of a quadratic integral criterion. Such systems possess many useful properties. They can be made to exhibit a satisfactory transient response to nonzero initial conditions, to an external reference variable, and to a change in the set point. Moreover, they have excellent stability characteristics and are insensitive to disturbances and parameter variations.

All these properties can be achieved in the desired measure by appropriately choosing the controlled variable of the system and properly adjusting the weighting matrices R_3 and R_2. The results of Sections 3.8 and 3.9, which concern the asymptotic properties and the sensitivity properties of steady-state control laws, give considerable insight into the influence of the weighting matrices.

A major objection to the theory of this section, however, is that very often it is either too costly or impossible to measure all components of the state. To overcome this difficulty, we study in Chapter 4 the problem of reconstructing the state of the system from incomplete and inaccurate measurements. Following this in Chapter 5 it is shown how the theory of linear state feedback control can be integrated with the theory of state reconstruction to provide a general theory of optimal linear feedback control.

3.11 PROBLEMS

3.1. *Stabilization of the position control system*

Consider the position control system of Example 3.4 (Section 3.3.1). Determine the set of all linear control laws that stabilize the position control system.

3.2. *Position control of a frictionless dc motor*

A simplification of the regulator problem of Example 3.4 (Section 3.3.1) occurs when we neglect the friction in the motor; the state differential equation then takes the form

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ \kappa \end{pmatrix} \mu(t), \qquad \textbf{3-620}$$

where $x(t) = \text{col}\,[\xi_1(t), \xi_2(t)]$. Take as the controlled variable

$$\zeta(t) = (1, \quad 0)x(t), \qquad \textbf{3-621}$$

and consider the criterion

$$\int_{t_0}^{t_1} [\zeta^2(t) + \rho\mu^2(t)]\, dt. \qquad \textbf{3-622}$$

(a) Determine the steady-state solution $\bar{P}$ of the Riccati equation.

(b) Determine the steady-state control law.

(c) Compute the closed-loop poles. Sketch the loci of the closed-loop poles as ρ varies.

(d) Use the numerical values $\kappa = 150$ rad/(V s^2) and $\rho = 2.25$ rad^2/V^2 and determine by computation or simulation the response of the closed-loop system to the initial condition $\xi_1(0) = 0.1$ rad, $\xi_2(0) = 0$ rad/s.

3.3. *Regulation of an amplidyne*

Consider the amplidyne of Problem 1.2.

(a) Suppose that the output voltage is to be kept at a constant value e_{20} Denote the nominal input voltage as e_{00} and represent the system in terms of a shifted state variable with zero as nominal value.

(b) Choose as the controlled variable

$$\zeta(t) = e_2(t) - e_{20}, \tag{3-623}$$

and consider the criterion

$$\int_{t_0}^{t_1} [\zeta^2(t) + \rho\mu'^2(t)]\, dt \tag{3-624}$$

where

$$\mu'(t) = e_0(t) - e_{00}. \tag{3-625}$$

Find the steady-state solution of the resulting regulator problem for the following numerical values:

$$\begin{aligned} \frac{R_1}{L_1} &= 10\ \text{s}^{-1}, & \frac{R_2}{L_2} &= 1\ \text{s}^{-1}, \\ R_1 &= 5\ \Omega, & R_2 &= 10\ \Omega, \\ k_1 &= 20\ \text{V/A}, & k_2 &= 50\ \text{V/A}, \\ \rho &= 0.025. \end{aligned} \tag{3-626}$$

(c) Compute the closed-loop poles.

(d) Compute or simulate the response of the closed-loop system to the initial conditions $x(0) = \text{col}\,(1, 0)$ and $x(0) = \text{col}\,(0, 1)$.

3.4. *Stochastic position control system*

Consider the position control problem of Example 3.4 (Section 3.3.1) but assume that in addition to the input a stochastically varying torque operates upon the system so that the state differential equation **3-59** must be extended as follows:

$$\dot{x}'(t) = \begin{pmatrix} 0 & 1 \\ 0 & -\alpha \end{pmatrix} x'(t) + \begin{pmatrix} 0 \\ \kappa \end{pmatrix} \mu(t) + \begin{pmatrix} 0 \\ \nu(t) \end{pmatrix}. \tag{3-627}$$

Here $\nu(t)$ represents the effect of the disturbing torque. We model $\nu(t)$ as exponentially correlated noise:

$$\dot{\nu}(t) = -\frac{1}{\theta}\nu(t) + \omega(t), \tag{3-628}$$

where $\omega(t)$ is white noise with intensity $2\sigma^2/\theta$.

(a) Consider the controlled variable

$$\zeta(t) = (1, \quad 0)x'(t) \tag{3-629}$$

and the criterion

$$E\left\{\int_{t_0}^{t_1} [\zeta^2(t) + \rho\mu^2(t)]\, dt\right\}. \tag{3-630}$$

Find the steady-state solution of the corresponding stochastic regulator problem.

(b) Use the numerical values

$$\begin{aligned} \kappa &= 0.787 \text{ rad/(V s}^2), \\ \alpha &= 4.6 \text{ s}^{-1}, \\ \sigma &= 5 \text{ rad/s}^2, \\ \theta &= 1 \text{ s}. \end{aligned} \tag{3-631}$$

Compute the steady-state rms values of the controlled variable $\zeta(t)$ and the input $\mu(t)$ for $\rho = 0.2 \times 10^{-4}$ rad^2/V^2.

3.5. *Angular velocity tracking system*

Consider the angular velocity tracking problem of Examples 3.12 (Section 3.6.2) and 3.14 (Section 3.6.3). In Example 3.14 we found that the value of ρ that was chosen ($\rho = 1000$) leaves considerable room for improvement.

(a) Vary ρ and select that value of ρ that results in a steady-state rms input voltage of 3 V.

(b) Compute the corresponding steady-state rms tracking error.

(c) Compute the corresponding break frequency of the closed-loop system and compare this to the break frequency of the reference variable.

3.6. *Nonzero set point regulator for an amplidyne*

Consider Problem 3.3 where a regulator has been derived for an amplidyne.

(a) Using the results of this problem, find the nonzero set point regulator.

(b) Simulate or calculate the response of the regulator to a step in the output voltage set point of 10 V.

3.7. *Extension of the regulator problem*

Consider the linear time-varying system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{3-632}$$

with the generalized quadratic criterion

$$\int_{t_0}^{t_1} [x^T(t)R_1(t)x(t) + 2x^T(t)R_{12}(t)u(t) + u^T(t)R_2(t)u(t)]\, dt + x^T(t_1)P_1x(t_1), \tag{3-633}$$

where $R_1(t)$, $R_{12}(t)$, and $R_2(t)$ are matrices of appropriate dimensions.

(a) Show that the problem of minimizing **3-633** for the system **3-632** can be reformulated as minimizing the criterion

$$\int_{t_0}^{t_1} [x^T(t)R_1'(t)x(t) + u'^T(t)R_2(t)u'(t)]\, dt + x^T(t_1)P_1x(t_1) \qquad \textbf{3-634}$$

for the system

$$\dot{x}(t) = A'(t)x(t) + B(t)u'(t), \qquad \textbf{3-635}$$

where

$$\begin{aligned} R_1'(t) &= R_1(t) - R_{12}(t)R_2^{-1}(t)R_{12}^T(t),\\ u'(t) &= u(t) + R_2^{-1}(t)R_{12}^T(t)x(t),\\ A'(t) &= A(t) - B(t)R_2^{-1}(t)R_{12}^T(t) \end{aligned} \qquad \textbf{3-636}$$

(Kalman, 1964; Anderson, 1966a; Anderson and Moore, 1971).

(b) Show that **3-633** is minimized for the system **3-632** by letting

$$u(t) = -F^0(t)x(t), \qquad \textbf{3-637}$$

where

$$F^0(t) = R_2^{-1}(t)[B^T(t)P(t) + R_{12}^T(t)], \qquad \textbf{3-638}$$

with $P(t)$ the solution of the matrix Riccati equation

$$\begin{aligned} -\dot{P}(t) = {} & [A(t) - B(t)R_2^{-1}(t)R_{12}^T(t)]^T P(t)\\ & + P(t)[A(t) - B(t)R_2^{-1}(t)R_{12}^T(t)]\\ & + R_1(t) - R_{12}(t)R_2^{-1}(t)R_{12}^T(t)\\ & - P(t)B(t)R_2^{-1}(t)B^T(t)P(t), \qquad t \le t_1,\\ P(t_1) = {} & P_1. \end{aligned} \qquad \textbf{3-639}$$

(c) For arbitrary $F(t)$, $t \le t_1$, let $\bar{P}(t)$ be the solution of the matrix differential equation

$$\begin{aligned} -\dot{\bar{P}}(t) = {} & [A(t) - B(t)F(t)]^T\bar{P}(t) + \bar{P}(t)[A(t) - B(t)F(t)]\\ & + R_1(t) - R_{12}(t)F(t) - F^T(t)R_{12}^T(t)\\ & + F^T(t)R_2(t)F(t), \qquad t \le t_1,\\ P(t_1) = {} & P_1. \end{aligned} \qquad \textbf{3-640}$$

Show that by choosing $F(t)$ equal to $F^0(t)$, $\bar{P}(t)$ is minimized in the sense that $\bar{P}(t) \ge P(t)$, $t \le t_1$, where $P(t)$ is the solution of **3-639**. *Remark:* The proof of (c) follows from (b). One can also prove that **3-637** is the best *linear* control law by rearranging **3-640** and applying Lemma 3.1 (Section 3.3.3) to it.

3.8*. *Solutions of the algebraic Riccati equation* (O'Donnell, 1966; Anderson, 1966b; Potter, 1964)

Consider the algebraic Riccati equation

$$0 = R_1 - \bar{P}BR_2^{-1}B^T\bar{P} + \bar{P}A + A^T\bar{P}. \qquad \textbf{3-641}$$

Let Z be the matrix

$$Z = \begin{pmatrix} A & -BR_2^{-1}B^T \\ -R_1 & -A^T \end{pmatrix}. \qquad \textbf{3-642}$$

Z can always be represented as

$$Z = WJW^{-1}, \qquad \textbf{3-643}$$

where J is the Jordan canonical form of Z. It is always possible to arrange the columns of W such that J can be partitioned as

$$J = \begin{pmatrix} J_{11} & 0 \\ J_{21} & J_{22} \end{pmatrix}. \qquad \textbf{3-644}$$

Here J_{11}, J_{21} and J_{22} are $n \times n$ blocks. Partition W accordingly as

$$W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}. \qquad \textbf{3-645}$$

(a) Consider the equality

$$ZW = WJ, \qquad \textbf{3-646}$$

and show by considering the 12- and 22-blocks of this equality that if W_{12} is nonsingular $\bar{P} = W_{22}W_{12}^{-1}$ is a solution of the algebraic Riccati equation. Note that in this manner many solutions can be obtained by permuting the order of the characteristic values in J.

(b) Show also that the characteristic values of the matrix $A - BR_2^{-1}B^TW_{22}W_{12}^{-1}$ are precisely the characteristic values of J_{22} and that the (generalized) characteristic vectors of this matrix are the columns of W_{12}. *Hint:* Evaluate the 12-block of the identity **3-646**.

3.9*. *Steady-state solution of the Riccati equation by diagonalization*

Consider the $2n \times 2n$ matrix Z as given by **3-247** and suppose that it cannot be diagonalized. Then Z can be represented as

$$Z = WJW^{-1}, \qquad \textbf{3-647}$$

where J is the Jordan canonical form of Z, and W is composed of the characteristic vectors and generalized characteristic vectors of Z. It is always possible to arrange the columns of W such that J can be partitioned as follows

$$J = \begin{pmatrix} J_{11} & 0 \\ J_{21} & J_{22} \end{pmatrix}, \qquad \textbf{3-648}$$

where the $n \times n$ matrix J_{11} has as diagonal elements those characteristic values of Z that have positive real parts and half of those that have zero real parts. Partition W and $V = W^{-1}$ accordingly as

$$W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}, \qquad V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}. \qquad \textbf{3-649}$$

Assume that $\{A, B\}$ is stabilizable and $\{A, D\}$ detectable. Follow the argument of Section 3.4.4 closely and show that for the present case the following conclusions hold.

(a) The steady-state solution $\bar{P}$ of the Riccati equation

$$-\dot{P}(t) = R_1 - P(t)BR_2^{-1}B^TP(t) + A^TP(t) + P(t)A \qquad \textbf{3-650}$$

satisfies

$$V_{11} + V_{12}\bar{P} = 0. \qquad \textbf{3-651}$$

(b) W_{12} is nonsingular and

$$\bar{P} = W_{22}W_{12}^{-1}. \qquad \textbf{3-652}$$

(c) The steady-state optimal behavior of the state is given by

$$x(t) = W_{12}e^{J_{22}(t-t_0)}W_{12}^{-1}x(t_0). \qquad \textbf{3-653}$$

Hence Z has no characteristic values with zero real parts, and the steady-state closed-loop poles consist of those cbaracterstic values of Z that have negative real parts. *Hint:* Show that

$$e^{Jt} = \begin{pmatrix} e^{J_{11}t} & 0 \\ X(t) & e^{J_{22}t} \end{pmatrix}, \qquad \textbf{3-654}$$

where the precise form of $X(t)$ is unimportant.

3.10*. *Bass' relation for $\bar{P}$* (Bass, 1967)

Consider the algebraic Riccati equation

$$0 = R_1 - \bar{P}BR_2^{-1}B^T\bar{P} + A^T\bar{P} + \bar{P}A \qquad \textbf{3-655}$$

and suppose that the conditions are satisfied under which it has a unique nonnegative-definite symmetric solution. Let the matrix Z be given by

$$Z = \begin{pmatrix} A & -BR_2^{-1}B^T \\ -R_1 & -A^T \end{pmatrix}. \qquad \textbf{3-656}$$

It follows from Theorem 3.8 (Section 3.4.4) that Z has no characteristic values with zero real parts. Factor the characteristic polynomial of Z as follows

$$\det(sI - Z) = \phi(s)\phi(-s) \qquad \textbf{3-657}$$

such that the roots of $\phi(s)$ have strictly negative real parts. Show that $\bar{P}$

satisfies the relation:

$$\phi(Z)\begin{pmatrix} I \\ \bar{P} \end{pmatrix} = 0. \qquad \textbf{3-658}$$

Hint: Write $\phi(Z) = \phi(WJW^{-1}) = W\phi(J)W^{-1} = W\phi(J)V$ where $V = W^{-1}$ and $J = \text{diag}(\Lambda, -\Lambda)$ in the notation of Section 3.4.4.

3.11*. *Negative exponential solution of the Riccati equation* (Vaughan, 1969)

Using the notation of Section 3.4.4, show that the solution of the time-invariant Riccati equation

$$\begin{aligned} -\dot{P}(t) &= R_1 - P(t)BR_2^{-1}B^TP(t) + A^TP(t) + P(t)A, \\ P(t_1) &= P_1, \end{aligned} \qquad \textbf{3-659}$$

can be expressed as follows:

$$P(t) = [W_{22} + W_{21}G(t_1 - t)][W_{12} + W_{11}G(t_1 - t)]^{-1}, \qquad \textbf{3-660}$$

where

$$G(t) = e^{-\Lambda t}Se^{-\Lambda t}, \qquad \textbf{3-661}$$

with

$$S = (V_{11} + V_{12}P_1)(V_{21} + V_{22}P_1)^{-1}. \qquad \textbf{3-662}$$

Show with the aid of Problem 3.12 that S can also be written in terms of W as

$$S = -(W_{22} - P_1W_{12})^{-1}(W_{21} - P_1W_{11}). \qquad \textbf{3-663}$$

3.12*. *The relation between W and V*

Consider the matrix Z as defined in Section 3.4.4.

(a) Show that if $e = \text{col}(e', e'')$, where e' and e'' both are n-dimensional vectors, is a right characteristic vector of Z corresponding to the characteristic value λ, that is, $Ze = \lambda e$, then $(e''^T, -e'^T)$ is a left characteristic vector of Z corresponding to the characteristic value $-\lambda$, that is,

$$(e''^T, -e'^T)Z = -\lambda(e''^T, -e'^T). \qquad \textbf{3-664}$$

(b) Assume for simplicity that all characteristic values λ_i, $i = 1, 2, \cdots, 2n$, of Z are distinct and let the corresponding characteristic vectors be given by e_i, $i = 1, 2, \cdots, 2n$. Scale the e_i such that if the characteristic vector $e = \text{col}(e', e'')$ corresponds to a characteristic value λ, and $f = \text{col}(f', f'')$ corresponds to $-\lambda$, then

$$f''^Te' - f'^Te'' = 1. \qquad \textbf{3-665}$$

Show that if W is a matrix of which the columns are e_i, $i = 1, 2, \cdots, 2n$, and we partition

$$W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}, \qquad \textbf{3-666}$$

then (O'Donnell, 1966; Walter, 1970)

$$W^{-1} = V = \begin{pmatrix} W_{22}^T & -W_{12}^T \\ -W_{21}^T & W_{11}^T \end{pmatrix}. \qquad \textbf{3-667}$$

Hint: Remember that left and right characteristic vectors for different characteristic values are orthogonal.

3.13*. *Frequency domain solution of regulator problems*

For single-input time-invariant systems in phase-variable canonical form, the regulator problem can be conveniently solved in the frequency domain. Let

$$\dot{x}(t) = Ax(t) + b\mu(t) \qquad \textbf{3-668}$$

be given in phase-variable canonical form and consider the problem of minimizing

$$\int_{t_0}^{\infty} [\zeta^2(t) + \rho\mu^2(t)]\, dt, \qquad \textbf{3-669}$$

where

$$\zeta(t) = dx(t). \qquad \textbf{3-670}$$

(a) Show that the closed-loop characteristic polynomial can be found by factorization of the polynomial

$$1 + \frac{1}{\rho} H(s)H(-s), \qquad \textbf{3-671}$$

where $H(s)$ is the open-loop transfer function $H(s) = d(sI - A)^{-1}b$.

(b) For a given closed-loop characteristic polynomial, show how the corresponding control law

$$\mu(t) = -fx(t) \qquad \textbf{3-672}$$

can be found. *Hint:* Compare Section 3.2.

3.14*. *The minimum number of faraway closed-loop poles*

Consider the problem of minimizing

$$\int_{t_0}^{\infty} [x^T(t)R_1x(t) + \rho u^T(t)Nu(t)]\, dt, \qquad \textbf{3-673}$$

where $R_1 \geq 0$, $N > 0$, and $\rho > 0$, for the system

$$\dot{x}(t) = Ax(t) + Bu(t). \qquad \textbf{3-674}$$

(a) Show that as $\rho \downarrow 0$ some of the closed-loop poles go to infinity while the others stay finite. Show that those poles that remain finite approach the left-half plane zeroes of

$$\det\,[B^T(-sI - A^T)^{-1}R_1(sI - A)^{-1}B]. \qquad \textbf{3-675}$$

(b) Prove that at least k closed-loop poles approach infinity, where k is the dimension of the input u. *Hint:* Let $|s| \rightarrow \infty$ to determine the maximum number of zeroes of 3-675. Compare the proof of Theorem 1.19 (Section 1.5.3).

(c) Prove that as $\rho \rightarrow \infty$ the closed-loop poles approach the numbers $\hat{\pi}_i$, $i = 1, 2, \cdots, n$, which are the characteristic values of the matrix A mirrored into the left-half complex plane.

3.15*. *Estimation of the radius of the faraway closed-loop poles from the Bode plot* (Leake, 1965; Schultz and Melsa, 1967, Section 8.4)

Consider the problem of minimizing

$$\int_{t_0}^{\infty} [\zeta^2(t) + \rho\mu^2(t)]\, dt \qquad \text{3-676}$$

for the single-input single-output system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + b\mu(t), \\ \zeta(t) &= dx(t). \end{aligned} \qquad \text{3-677}$$

Suppose that a Bode plot is available of the open-loop frequency response function $H(j\omega) = d(j\omega I - A)^{-1}b$. Show that for small ρ the radius of the faraway poles of the steady-state optimal closed-loop system can be estimated: as the frequency ω_0 for which $|H(j\omega_0)| = \sqrt{\rho}$.