Chapter Seven
Output Feedback

One may separate the problem of physical realization into two stages: computation of the "best approximation" $\hat{x}(t_1)$ of the state from knowledge of $y(t)$ for $t \leq t_1$ and computation of $u(t_1)$ given $\hat{x}(t_1)$.

From R. E. Kalman "Contributions to the theory of optimal control" [111]

In the last chapter we considered the use of state feedback to modify the dynamics of a system. In many applications, it is not practical to measure all of the states directly and we can measure only a small number of outputs (corresponding to the sensors that are available). In this chapter we show how to use output feedback to modify the dynamics of the system, through the us of observers. We introduce the concept of observability and show that if a system is observable, it is possible to recover the state from measurements of the inputs and outputs to the system. It is then shown how to design a controller with feedback from the observer state. An important concept is the separation principle quoted above, which is also proved. The structure of the controllers derived in this chapter is quite general and is obtained by many other design methods.

7.1 OBSERVABILITY

In Section 6.2 of the previous chapter it was shown that it is possible to find a feedback that gives desired closed loop eigenvalues provided that the system is reachable and that all states are measured. For many situations, it is highly unrealistic to assume that all states are measured. In this section we investigate how the state can be estimated by using a mathematical model and a few measurements. It will be shown that the computation of the states can be carried out by a dynamical system called an *observer*.

Definition of Observability

Consider a system described by a set of differential equations

$$\frac{dx}{dt} = Ax + Bu, \qquad y = Cx + Du, \tag{7.1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^p$ the input, and $y \in \mathbb{R}^q$ the measured output. We wish to estimate the state of the system from its inputs and outputs, as illustrated in Figure 7.1. In some situations we will assume that there is only one measured signal, i.e. that the signal y is a scalar and that C is a (row) vector. This signal

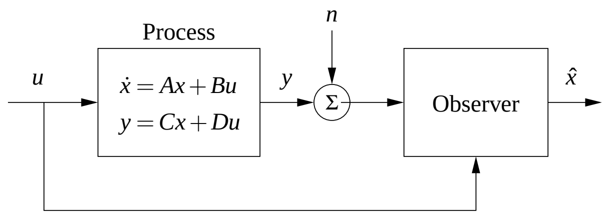


Figure 7.1: Block diagram for an observer. The observer uses the process measurement y (possibly corrupted by noise n) and the input u to estimate the current state of the process, denoted $\hat{x}$.

may be corrupted by noise, n, although we shall start by considering the noise-free case. We write $\hat{x}$ for the state estimate given by the observer.

Definition 7.1 (Observability)**.** A linear system is *observable* if for any $T > 0$ it is possible to determine the state of the system $x(T)$ through measurements of $y(t)$ and $u(t)$ on the interval $[0, T]$.

The definition above holds for nonlinear systems as well, and the results discussed here have extensions to the nonlinear case.

The problem of observability is one that has many important applications, even outside of feedback systems. If a system is observable, then there are no "hidden" dynamics inside it; we can understand everything that is going on through observation (over time) of the inputs and outputs. As we shall see, the problem of observability is of significant practical interest because it will determine if a set of sensors is sufficient for controlling a system. Sensors combined with a mathematical model can also be viewed as a "virtual sensor" that gives information about variables that are not measured directly. The process of reconciling signals from many sensors with mathematical models is also called *sensor fusion*.

Testing for Observability

When discussing reachability in the last chapter we neglected the output and focused on the state. Similarly, it is convenient here to initially neglect the input and focus on the autonomous system

$$\frac{dx}{dt} = Ax, \qquad y = Cx. \tag{7.2}$$

We wish to understand when it is possible to determine the state from observations of the output.

The output itself gives the projection of the state on vectors that are rows of the matrix C. The observability problem can immediately be solved if the matrix C is invertible. If the matrix is not invertible we can take derivatives of the output to obtain

$$\frac{dy}{dt} = C\frac{dx}{dt} = CAx.$$

From the derivative of the output we thus get the projection of the state on vectors that are rows of the matrix CA. Proceeding in this way we get

$$\begin{pmatrix} y \\ \dot{y} \\ \ddot{y} \\ \vdots \\ y^{(n-1)} \end{pmatrix} = \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix} x. \tag{7.3}$$

We thus find that the state can be determined if the matrix

$$W_o = \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix} \tag{7.4}$$

has n independent rows. It turns out that we need not consider any derivatives higher than $n-1$ (this is an application of the Cayley-Hamilton theorem (Exercise 6.11).

The calculation can easily be extended to systems with inputs. The state is then given by a linear combination of inputs and outputs and their higher derivatives. The observability criterion is unchanged. We leave this case as an exercise for the reader.

In practice, differentiation of the output can give large errors when there is measurement noise and therefore the method sketched above is not particularly practical. We will address this issue in more detail in the next section, but for now we have the following basic result:

Theorem 7.1. *A linear system of the form* (7.1) *is observable if and only if the observability matrix W_o is full rank.*

Proof. The sufficiency of the observability rank condition follows from the analysis above. To prove necessity, suppose that the system is observable but W_o is not full rank. Let $v \in \mathbb{R}^n$, $v \neq 0$ be a vector in the null space of W_o, so that $W_o v = 0$. If we let $x(0) = v$ be the initial condition for the system and choose $u = 0$, then the output is given by $y(t) = Ce^{At}v$. Since e^{At} can be written as a power series in A and since A^n and higher powers can be rewritten in terms of lower powers of A (by the Cayley-Hamilton theorem), it follows that the output will be identically zero (the reader should fill in the missing steps if this is not clear). However, if both the input and output of the system are 0, then a valid estimate of the state is $\hat{x} = 0$ for all time, which is clearly incorrect since $x(0) = v \neq 0$. Hence by contradiction we must have that W_o is full rank if the system is observable. □

Example 7.1 Compartment model

Consider the two compartment model in Figure 3.18a on page 91 but assume that

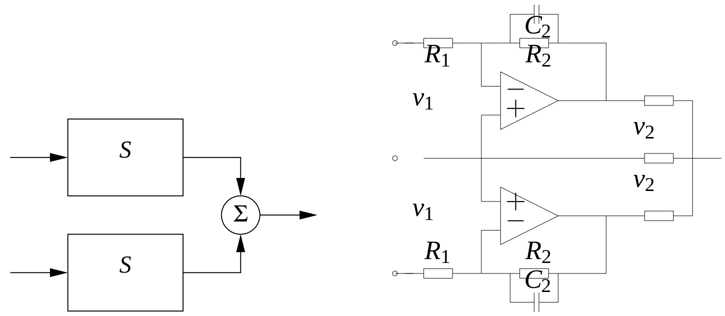


Figure 7.2: A non-observable system. Two identical subsystems have outputs that add together to form the overall system output. The individual states of the subsystem cannot be determined since the contributions of each to the output are not distinguishable. The circuit diagram on the right is an example of such a system.

the the concentration in the first compartment can be measured. The system is described by the linear system

$$\frac{dc}{dt} = \begin{pmatrix} -k_0 - k_1 & k_1 \\ k_2 & -k_2 \end{pmatrix} c + \begin{pmatrix} b_0 \\ 0 \end{pmatrix} u, \qquad y = \begin{pmatrix} 1 & 0 \end{pmatrix} x.$$

The first compartment can represent the concentration in the blood plasma and the second compartment the drug concentration in the tissue where it is active. To determine if it is possible to find the concentration in the tissue compartment from measurement of blood plasma we investigate the observability of the system by forming the observability matrix

$$W_o = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -k_0 - k_1 & k_1 \end{pmatrix}.$$

The rows are linearly independent if $k_1 \neq 0$ and under this condition it is thus possible to determine the concentration of the drug in the active compartment from measurements of the drug concentration in the blood. ∇

It is useful to have an understanding of the mechanisms that make a system unobservable. Such a system is shown in Figure 7.2. The system is composed of two identical systems whose outputs are added. It seems intuitively clear that it is not possible to deduce the states from the output since we cannot deduce the individual output contributions from the sum. This can also be seen formally (Exercise 7.2).

Observable Canonical Form

As in the case of reachability, certain canonical forms will be useful in studying observability. We define the observable canonical form to be the dual of the reachable canonical form.

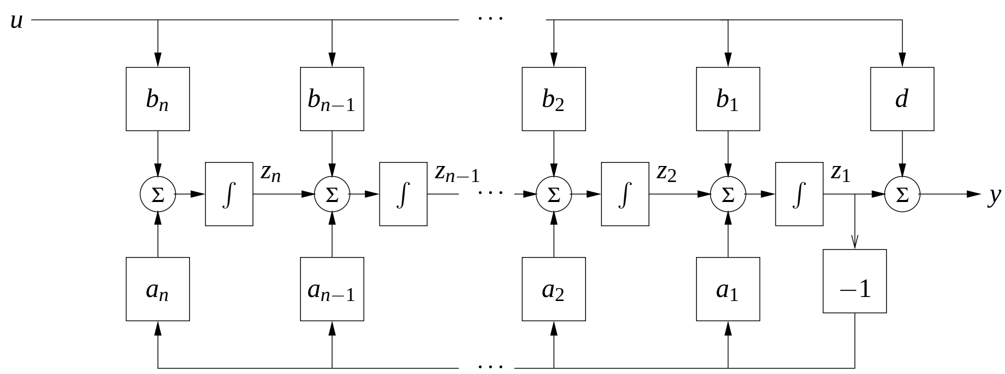


Figure 7.3: Block diagram of a system on observable canonical form. The states of the system are represented by individual integrators whose inputs are a weighted combination of the next integrator in the chain, the first state (right most integrator) and the system input. The output is a combination of the first state and the input.

Definition 7.2 (Observable canonical form)**.** A linear single input, single output (SISO) state space system is in *observable canonical form* if its dynamics are given by

$$\frac{dz}{dt} = \begin{pmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & & 0 \\ \vdots & & & \ddots & \\ -a_{n-1} & 0 & 0 & & 1 \\ -a_n & 0 & 0 & \cdots & 0 \end{pmatrix} z + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 & 0 \cdots & 0 \end{pmatrix} z + Du.$$

The definition can be extended to systems with many inputs the only difference is that the vector multiplying u is replaced by a matrix.

Figure 7.3 shows a block diagram for a system in observable canonical form. As in the case of reachable canonical form, we see that the coefficients in the system description appear directly in the block diagram. The characteristic equation for a system in observable canonical form is given by

$$\lambda(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n. \tag{7.5}$$

It is possible to reason about the observability of a system in observable canonical form by studying the block diagram. If the input u and the output y are available the state z_1 can clearly be computed. Differentiating z_1 we also obtain the input to the integrator that generates z_1 and we can now obtain $z_2 = \dot{z}_1 + a_1 z_1 - b_1 u$. Proceeding in this way we can compute all states. The computation will however require that the signals are differentiated.

To check observability more formally, we compute the observability matrix for a system in observable canonical form, which is given by

$$W_o = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ -a_1 & 1 & 0 & \dots & 0 \\ -a_1^2 - a_1 a_2 & -a_1 & 1 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ * & * & & \dots & 1 \end{pmatrix},$$

where * represents as entry whose exact value is not important. The rows of this matrix are linearly independent (since it is lower triangular) and hence W_o is full rank. A straightforward but tedious calculation shows that the inverse of the observability matrix has a simple form, given by

$$W_o^{-1} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_1 & 1 & 0 & \cdots & 0 \\ a_2 & a_1 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & 1 \end{pmatrix}.$$

As in the case of reachability, it turns out that if a system is observable then there always exists a transformation T that converts the system into reachable canonical form (Exercise 7.3). This is useful for proofs, since it lets us assume that a system is in reachable canonical form without any loss of generality. The reachable canonical form may be poorly conditioned numerically.

7.2 STATE ESTIMATION

Having defined the concept of observability, we now return to the question of how to construct an observer for a system. We will look for observers that can be represented as a linear dynamical system that takes the inputs and outputs of the system we are observing and produces an estimate of the system's state. That is, we wish to construct a dynamical system of the form

$$\frac{d\hat{x}}{dt} = F\hat{x} + Gu + Hy,$$

where u and y are the input and output of the original system and $\hat{x} \in \mathbb{R}^n$ is an estimate of the state with the property that $\hat{x}(t) \to x(t)$ as $t \to \infty$.

The Observer

We consider the system in equation (7.1) with D set to zero to simplify the exposition:

$$\frac{dx}{dt} = Ax + Bu, \qquad y = Cx, \tag{7.6}$$

We can attempt to determine the state simply by simulating the equations with the correct input. An estimate of the state is then given by

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu. \tag{7.7}$$

To find the properties of this estimate, introduce the estimation error $\tilde{x} = x - \hat{x}$. It follows from equations (7.6) and (7.7) that

$$\frac{d\tilde{x}}{dt} = A\tilde{x}.$$

If matrix A has all its eigenvalues in the left half plane, the error $\tilde{x}$ will go to zero and hence equation (7.7) is a dynamical system whose output converges to the state of the system (7.6).

The observer given by equation (7.7) uses only the process input u; the measured signal does not appear in the equation. We must also require that the system is stable, and essentially our estimator converges because the state of both the observer and the estimator are going zero. This is not very useful in a control design context since we want to have our estimate converge quickly to a nonzero state, so that we can make use of it in our controller. We will therefore attempt to modify the observer so that the output is used and its convergence properties can be designed to be fast relative to the system's dynamics. This version will also work for unstable systems.

Consider the observer

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + L(y - C\hat{x}). \tag{7.8}$$

This can be considered as a generalization of equation (7.7). Feedback from the measured output is provided by adding the term $L(y - C\hat{x})$, which is proportional to the difference between the observed output and the output that is predicted by the observer. It follows from equations (7.6) and (7.8) that

$$\frac{d\tilde{x}}{dt} = (A - LC)\tilde{x}.$$

If the matrix L can be chosen in such a way that the matrix $A - LC$ has eigenvalues with negative real parts, the error $\tilde{x}$ will go to zero. The convergence rate is determined by an appropriate selection of the eigenvalues.

Notice the similarity between the problems of finding a state feedback and finding the observer. State feedback design by eigenvalue assignment is equivalent to finding a matrix K so that $A - BK$ has given eigenvalues. Design of an observer with prescribed eigenvalues is equivalent to finding a matrix L so that $A - LC$ has given eigenvalues. Since the eigenvalues of a matrix and its transpose are the same we can established the following equivalence:

$$A \leftrightarrow A^T, \qquad B \leftrightarrow C^T, \qquad K \leftrightarrow L^T, \qquad W_r \leftrightarrow W_o^T$$

The observer design problem is the *dual* of the state feedback design problem. Using the results of Theorem 6.3, we get the following theorem on observer design:

Theorem 7.2 (Observer design by eigenvalue assignment)**.** *Consider the system given by*

$$\frac{dx}{dt} = Ax + Bu \qquad y = Cx \tag{7.9}$$

with one input and one output. Let $\lambda(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$ be the characteristic polynomial for A. If the system is observable then the dynamical system

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + L(y - C\hat{x}) \tag{7.10}$$

is an observer for the system, with L chosen as

$$L = W_o^{-1}\widetilde{W}_o \begin{pmatrix} p_1 - a_1 \\ p_2 - a_2 \\ \vdots \\ p_n - a_n \end{pmatrix}, \tag{7.11}$$

and the matrices W_o and $\widetilde{W}_o$ given by

$$W_o = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} \qquad \widetilde{W}_o = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_1 & 1 & 0 & \cdots & 0 \\ a_2 & a_1 & 1 & & 0 \\ \vdots & \vdots & & \ddots & \\ a_{n-1} & a_{n-2} & a_{n-3} & & 1 \end{pmatrix}^{-1}.$$

The resulting observer error $\tilde{x} = x - \hat{x}$ is governed by a differential equation having the characteristic polynomial

$$p(s) = s^n + p_1 s^{n-1} + \cdots + p_n.$$

The dynamical system (7.10) is called an observer for (the states of) the system (7.9) because it will generate an approximation of the states of the system from its inputs and outputs. This form of an observer is a much more useful form than the one given by pure differentiation in equation (7.3).

Example 7.2 Compartment model

Consider the compartment model in Example 7.1 which is characterized by the matrices

$$A = \begin{pmatrix} -k_0 - k_1 & k_1 \\ k_2 & -k_2 \end{pmatrix} \qquad B = \begin{pmatrix} b_0 \\ 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

The observability matrix was computed in Example 7.1 where we concluded that the system was observable if $k_1 \neq 0$. The dynamics matrix has the characteristic polynomial

$$\lambda(s) = \det \begin{pmatrix} s + k_0 + k_1 & -k_1 \\ -k_2 & s + k_2 \end{pmatrix} = s^2 + (k_0 + k_1 + k_2)s + k_0 k_2.$$

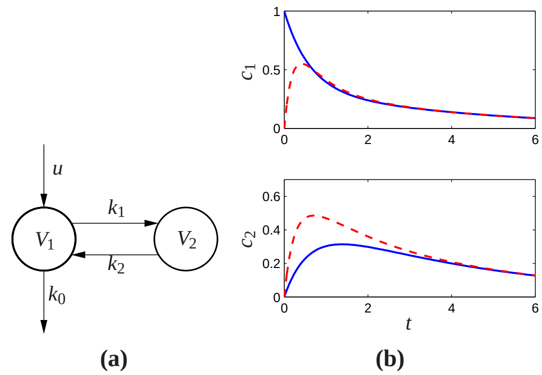


Figure 7.4: Observer for a two compartment system. The observer measures the input concentration u and output concentration y to determine the compartment concentrations, shown on the right. The true concentrations are shown in full lines and the estimates generated by the observer in dashed lines.

Let the desired characteristic polynomial of the observer be $s^2 + p_1 s + p_2$. and equation 7.1 gives the observer gain

$$\begin{aligned} L &= \begin{pmatrix} 1 & 0 \\ -k_0 - k_1 & k_1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ k_0 + k_1 + k_2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} p_1 - k_0 - k_1 - k_2 \\ p_2 - k_0 k_2 \end{pmatrix} \\ &= \begin{pmatrix} p_1 - k_0 - k_1 - k_2 \\ (p_2 - p_1 k_2 + k_1 k_2 + k_2^2)/k_1 \end{pmatrix} \end{aligned}$$

Notice that the observability condition $k_1 \neq 0$ is essential. The behavior of the observer is illustrated by the simulation in Figure 7.4b. Notice how the observed concentrations approaches the true concentrations. ∇

The observer is a dynamical system whose inputs are the process input u and process output y. The rate of change of the estimate is composed of two terms. One term, $A\hat{x} + Bu$, is the rate of change computed from the model with $\hat{x}$ substituted for x. The other term, $L(y - \hat{y})$, is proportional to the difference $e = y - \hat{y}$ between measured output y and its estimate $\hat{y} = C\hat{x}$. The estimator gain L is a matrix that tells how the error e is weighted and distributed among the states. The observer thus combines measurements with a dynamical model of the system. A block diagram of the observer is shown in Figure 7.5.

Computing the Observer Gain

For simple low order problems it is convenient to introduce the elements of the observer gain L as unknown parameters and solve for the values required to give the desired characteristic polynomial, as illustrated in the following example.

Example 7.3 Vehicle steering

The normalized, linear model for vehicle steering derived in Examples 5.12 and 6.4 gives the following state space model dynamics relating lateral path deviation y to

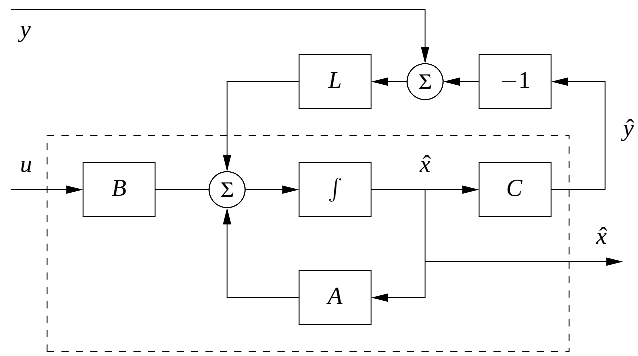


Figure 7.5: Block diagram of the observer. The observer takes the signals y and u as inputs and produces an estimate x. Notice that the observer contains a copy of the process model that is driven by $y-\hat{y}$ through the observer gain L.

steering angle u

$$\begin{aligned} \frac{dx}{dt} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} \gamma \\ 1 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 \end{pmatrix} x \end{aligned} \tag{7.12}$$

Recall that the state x_1 represents the lateral path deviation and that x_2 represents turning rate. We will now derive an observer that uses the system model to determine turning rate from the measured path deviation.

The observability matrix is

$$W_o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

i.e., the identity matrix. The system is thus observable and the eigenvalue assignment problem can be solved. We have

$$A - LC = \begin{pmatrix} -l_1 & 1 \\ -l_2 & 0 \end{pmatrix},$$

which has the characteristic polynomial

$$\det(sI - A + LC) = \det \begin{pmatrix} s+l_1 & -1 \\ l_2 & s \end{pmatrix} = s^2 + l_1 s + l_2.$$

Assuming that we want to have an observer with the characteristic polynomial

$$s^2 + p_1 s + p_2 = s^2 + 2\zeta_o \omega_o s + \omega_o^2,$$

the observer gains should be chosen as

$$l_1 = p_1 = 2\zeta_o\omega_o, \qquad l_2 = p_2 = \omega_o^2.$$

The observer is then

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + L(y - C\hat{x}) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \hat{x} + \begin{pmatrix} \gamma \\ 1 \end{pmatrix} u + \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} (y - \hat{x}_1).$$

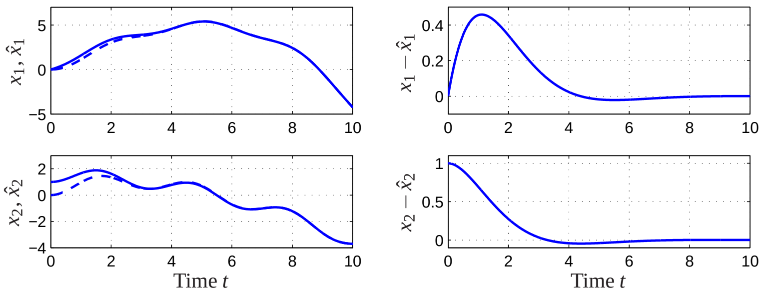


Figure 7.6: Simulation of an observer for a vehicle driving on a curvy road. The observer has an initial velocity error. The plots on the left show the lateral deviation x_1, the lateral velocity x_2 in full lines and their estimates $\hat{x}_1$ and $\hat{x}_2$ in dashed lines. The plots on the right show the estimation errors.

A simulation of the observer for a vehicle driving on a curvy road is simulated in Figure 7.6. The vehicle length is the length unit in the normalized model. The figure shows that the observer error settles in about 8 vehicle lengths. ∇

For systems of high order we have to use numerical calculations. The duality between design of a state feedback and design of an observer means that means that the computer algorithms for state feedback can be used also for the observer design; we simply use the transpose of the dynamics matrix and the output matrix. The MATLAB command `acker`, which essentially is a direct implementation of the calculations given in Theorem 7.2, can be used for systems with one output (Exercise 7.8). The MATLAB command `place` can be used for systems with many outputs. It is also better conditioned numerically.

7.3 CONTROL USING ESTIMATED STATE

In this section we will consider a state space system with no direct term (the most common case):

$$\frac{dx}{dt} = Ax + Bu, \qquad y = Cx. \tag{7.13}$$

Notice that we have assumed that there is no direct term in the system ($D = 0$). This is often a realistic assumption. The presence of a direct term in combination with a controller having proportional action creates a so called algebraic loop which will be discussed in Section 8.3. The problem can be solved even if there is a direct term but the calculations are more complicated.

We wish to design a feedback controller for the system where only the output is measured. As before, we will be assume that u and y are scalars. We also assume that the system is reachable and observable. In Chapter 6 we found a feedback of the form

$$u = -Kx + k_r r$$

for the case that all states could be measured and in Section 7.2 we developed an observer that can generate estimates of the state $\hat{x}$ based on inputs and outputs. In this section we will combine the ideas of these sections to find a feedback that gives desired closed loop eigenvalues for systems where only outputs are available for feedback.

If all states are not measurable, it seems reasonable to try the feedback

$$u = -K\hat{x} + k_r r, \tag{7.14}$$

where $\hat{x}$ is the output of an observer of the state, i.e.

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + L(y - C\hat{x}). \tag{7.15}$$

Since the system (7.13) and the observer (7.15) are both of state dimension n, the closed loop system has state dimension $2n$ with state $(x,\, \hat{x})$. The evolution of the states is described by equations (7.13), (7.14) and (7.15). To analyze the closed loop system, the state variable $\hat{x}$ is replaced by

$$\tilde{x} = x - \hat{x}. \tag{7.16}$$

Subtraction of equation (7.15) from equation (7.13) gives

$$\frac{d\tilde{x}}{dt} = Ax - A\hat{x} - L(Cx - C\hat{x}) = A\tilde{x} - LC\tilde{x} = (A - LC)\tilde{x}.$$

Returning to the process dynamics, introducing u from equation (7.14) into equation (7.13) and using equation (7.16) to eliminate $\hat{x}$ gives

$$\begin{aligned}\frac{dx}{dt} &= Ax + Bu = Ax - BK\hat{x} + Bk_r r = Ax - BK(x - \tilde{x}) + Bk_r r\\ &= (A - BK)x + BK\tilde{x} + Bk_r r.\end{aligned}$$

The closed loop system is thus governed by

$$\frac{d}{dt}\begin{pmatrix} x \\ \tilde{x} \end{pmatrix} = \begin{pmatrix} A - BK & BK \\ 0 & A - LC \end{pmatrix}\begin{pmatrix} x \\ \tilde{x} \end{pmatrix} + \begin{pmatrix} Bk_r \\ 0 \end{pmatrix} r. \tag{7.17}$$

Notice that the state $\tilde{x}$, representing the observer error, is not affected by the command signal r. This is desirable since we do not want the reference signal to generate observer errors.

Since the dynamics matrix is block diagonal, we find that the characteristic polynomial of the closed loop system is

$$\lambda(s) = \det(sI - A + BK)\det(sI - A + LC).$$

This polynomial is a product of two terms: the characteristic polynomial of the closed loop system obtained with state feedback and the characteristic polynomial of the observer error. The feedback (7.14) that was motivated heuristically thus provides a neat solution to the eigenvalue assignment problem. The result is summarized as follows.

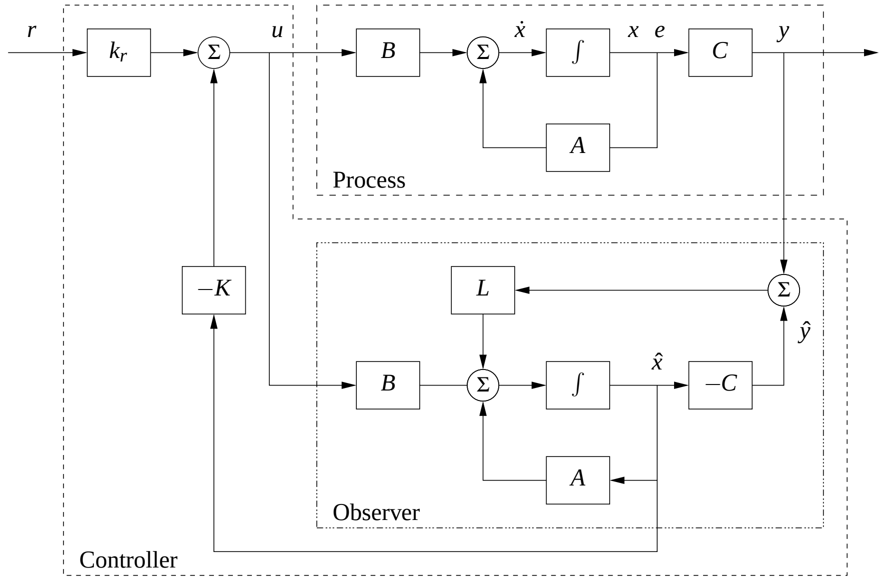


Figure 7.7: Block diagram of an observer-based control system. The observer uses the measured output y and the input u to construct an estimate of the state. This estimate is used by a state feedback controller to generate the corrective input. The controller consists of the observer and the state feedback; the observer is identical to Figure 7.5.

Theorem 7.3 (Eigenvalue assignment by output feedback)**.** *Consider the system*

$$\frac{dx}{dt} = Ax + Bu, \qquad y = Cx.$$

The controller described by

$$\begin{aligned} u &= -K\hat{x} + k_r r \\ \frac{d\hat{x}}{dt} &= A\hat{x} + Bu + L(y - C\hat{x}) = (A - BL - KC)\hat{x} + Ly \end{aligned}$$

gives a closed loop system with the characteristic polynomial

$$\lambda(s) = \det(sI - A + BK)\det(sI - A + LC).$$

This polynomial can be assigned arbitrary roots if the system is reachable and observable.

The controller has a strong intuitive appeal: it can be thought of as composed of two parts, one state feedback and one observer. The dynamics of the controller is generated by the observer. The feedback gain K can be computed as if all state variables can be measured and it only depends on A and B. The observer gain L only depends on A and C. The property that the eigenvalue assignment for output feedback can be separated into eigenvalue assignment for a state feedback and an observer is called the *separation principle*.

A block diagram of the controller is shown in Figure 7.7. Notice that the controller contains a dynamical model of the plant. This is called the *internal model principle*: the controller contains a model of the process being controlled. Indeed, the dynamics of the controller are due to the observer and the controller can thus be viewed as a dynamical system with input y and output u:

$$\frac{d\hat{x}}{dt} = (A - BK - LC)\hat{x} + Ly, \qquad u = -K\hat{x} + k_r r. \tag{7.18}$$

Example 7.4 Vehicle steering

Consider again the normalized, linear model for vehicle steering in Example 6.4. The dynamics relating steering angle u to lateral path deviation y is given by the state space model (7.12). Combining the state feedback derived in Example 6.4 with the observer determined in Example 7.3 we find that the controller is given by

$$\begin{aligned} \frac{d\hat{x}}{dt} &= A\hat{x} + Bu + L(y - C\hat{x}) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \hat{x} + \begin{pmatrix} \gamma \\ 1 \end{pmatrix} u + \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} (y - \hat{x}_1) \\ u &= -K\hat{x} + k_r r = k_1(r - x_1) - k_2 x_2. \end{aligned}$$

Elimination of the variable u gives

$$\begin{aligned} \frac{d\hat{x}}{dt} &= (A - BK - LC)\hat{x} + Ly + Bk_r r \\ &= \begin{pmatrix} -l_1 - \gamma k_1 & 1 - \gamma k_2 \\ -k_1 - l_2 & -k_2 \end{pmatrix} \hat{x} + \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} y + \begin{pmatrix} \gamma \\ 1 \end{pmatrix} k_1 r. \end{aligned}$$

The controller is a dynamical system of second order, with two inputs y and r and one output u. Figure 7.8 shows a simulation of the the system when the vehicle is driven along a curvy road. Since we are using a normalized model the length unit is the vehicle length and the time unit is the time it takes to travel one vehicle length. The estimator is initialized with all states equal to zero but the real system has an initial velocity 0.5. The figures show that the estimates converge quickly to their true values. The vehicle tracks the desired path which is in the middle of the road, but there are errors because the road is irregular. The tracking error can be improved by introducing feedforward. ∇

7.4 KALMAN FILTERING

One of the principal uses of observers in practice is to estimate the state of a system in the presence of *noisy* measurements. We have not yet treated noise in our analysis and a full treatment of stochastic dynamical systems is beyond the scope of this text. In this section, we present a brief introduction to the use of stochastic systems analysis for constructing observers. We work primarily in discrete time to avoid some of the complications associated with continuous time random processes and to keep the mathematical prerequisites to a minimum. This section assumes basic knowledge of random variables and stochastic processes; see Kumar and Varaiya [129] or Åström [15] for the required material.

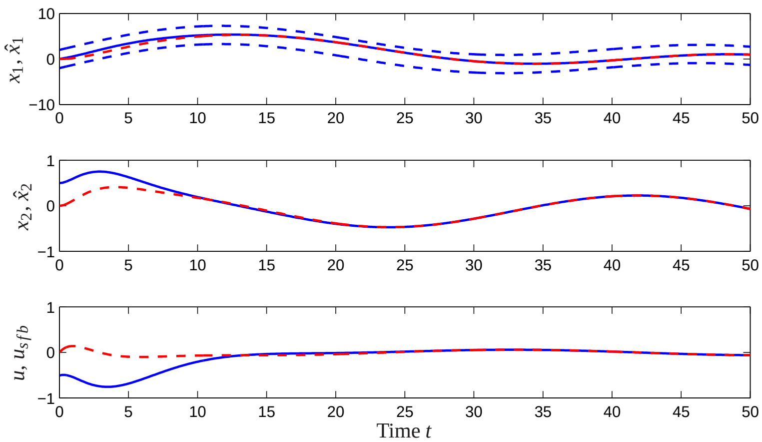


Figure 7.8: Simulation of a vehicle driving on a curvy road with a controller based on state feedback and an observer. The upper curves show the road markings (dotted), the vehicle position (full) and its estimate (dashed), the middle curve shows the velocity (full) and its estimate (dashed) and the bottom curve shows the control signal with a controller based on state feedback (full) and the control signal (dashed).

Consider a discrete time, linear system with dynamics

$$\begin{aligned} x[k+1] &= Ax[k] + Bu[k] + Fv[k] \\ y[k] &= Cx[k] + w[k], \end{aligned} \tag{7.19}$$

where $v[k]$ and $w[k]$ are Gaussian, white noise processes satisfying

$$\begin{aligned} &E\{v[k]\} = 0 && E\{w[k]\} = 0 \\ &E\{v[k]v^T[j]\} = \begin{cases} 0 & k \neq j \\ R_v & k = j \end{cases} && E\{w[k]w^T[j]\} = \begin{cases} 0 & k \neq j \\ R_w & k = j \end{cases} \\ &E\{v[k]w^T[j]\} = 0. \end{aligned} \tag{7.20}$$

$E\{v[k]\}$ represents the expected value of $v[k]$ and $E\{v[k]v^T[j]\}$ the correlation matrix. We assume that the initial condition is also modeled as a Gaussian random variable with

$$E\{x[0]\} = x_0 \qquad E\{x[0]x^T[0]\} = P_0. \tag{7.21}$$

We wish to find an estimate $\hat{x}[k]$ that minimizes the mean square error $E\{(x[k] - \hat{x}[k])(x[k] - \hat{x}[k])^T\}$ given the measurements $\{y(\tau) : 0 \leq \tau \leq t\}$. We consider an observer in the same basic form as derived previously:

$$\hat{x}[k+1] = A\hat{x}[k] + Bu[k] + L[k](y[k] - C\hat{x}[k]). \tag{7.22}$$

The following theorem summarizes the main result.

Theorem 7.4. *Consider a random process $x[k]$ with dynamics* (7.19) *and noise processes and initial conditions described by equations* (7.20) *and* (7.21). *The* *observer gain L that minimizes the mean square error is given by*

$$L[k] = AP[k]C^T(R_w + CP[k]C^T)^{-1},$$

where

$$\begin{aligned} P[k+1] &= (A-LC)P[k](A-LC)^T + FR_vF^T + LR_wL^T \\ P_0 &= E\{x[0]x^T[0]\}. \end{aligned} \tag{7.23}$$

Before we prove this result, we reflect on its form and function. First, note that the Kalman filter has the form of a *recursive* filter: given $P[k] = E\{E[k]E^T[k]\}$ at time k, can compute how the estimate and covariance *change*. Thus we do not need to keep track of old values of the output. Furthermore, the Kalman filter gives the estimate $\hat{x}[k]$ *and* the covariance $P[k]$, so we can see how reliable the estimate is. It can also be shown that the Kalman filter extracts the maximum possible information about output data. If we form the residual between the measured output and the estimated output,

$$e[k] = y[k] - C\hat{x}[k],$$

we can can show that for the Kalman filter the correlation matrix is

$$R_e(j,k) = E\{e[j]e^T[k]\} = W[k]\delta_{jk}, \qquad \delta_{jk} = \begin{cases} 0 & j \neq k \\ 1 & j = k. \end{cases}$$

In other words, the error is a white noise process, so there is no remaining dynamic information content in the error.

The Kalman filter is extremely versatile and can be used even if the process, noise or disturbances are non-stationary. When the system is stationary and *if* $P[k]$ converges, then the observer gain is constant:

$$L = APC^T(R_w + CPC^T),$$

where P satisfies

$$P = APA^T + FR_vF^T - APC^T\left(R_w + CPC^T\right)^{-1}CPA^T.$$

We see that the optimal gain depends on both the process noise and the measurement noise, but in a nontrivial way. Like the use of LQR to choose state feedback gains, the Kalman filter permits a systematic derivation of the observer gains given a description of the noise processes. The solution for the constant gain case is solved by the `dlqe` command in MATLAB.

Proof (of theorem). We wish to minimize the mean square of the error, $E\{(x[k] - \hat{x}[k])(x[k] - \hat{x}[k])^T\}$. We will define this quantity as $P[k]$ and then show that it satisfies the recursion given in equation (7.23). By definition,

$$\begin{aligned} P[k+1] &= E\{x[k+1]x^T[k+1]\} \\ &= (A-LC)P[k](A-LC)^T + FR_vF^T + LR_wL^T \\ &= AP[k]A^T - AP[k]C^TL^T - LCA^T + L(R_w + CP[k]C^T)L^T. \end{aligned}$$

Letting $R_\varepsilon = (R_w + CP[k]C^T)$, we have

$$\begin{aligned}
P[k+1] &= AP[k]A^T - AP[k]C^T L^T - LCA^T + LR_\varepsilon L^T \\
&= AP[k]A^T + \left(L - AP[k]C^T R_\varepsilon^{-1}\right) R_\varepsilon \left(L - AP[k]C^T R_\varepsilon^{-1}\right)^T \\
&\quad - AP[k]C^T R_\varepsilon^{-1} CP^T[k]A^T + R_w.
\end{aligned}$$

To minimize this expression, we choose $L = AP[k]C^T R_\varepsilon^{-1}$ and the theorem is proved. □

The Kalman filter can also be applied to continuous time stochastic processes. The mathematical derivation of this result requires more sophisticated tools, but the final form of the estimator is relatively straightforward.

Consider a continuous stochastic system

$$\begin{aligned}
\dot{x} &= Ax + Bu + Fv & E\{v(s)v^T(t)\} &= R_v(t)\delta(t-s) \\
y &= Cx + w & E\{w(s)w^T(t)\} &= R_w(t)\delta(t-s),
\end{aligned}$$

where $\delta(\tau)$ is the unit impulse function. Assume that the disturbance v and noise w are zero-mean and Gaussian (but not necessarily stationary):

$$\begin{aligned}
\mathrm{pdf}(v) &= \frac{1}{\sqrt[n]{2\pi}\sqrt{\det R_v}} e^{-\frac{1}{2}v^T R_v^{-1} v} \\
\mathrm{pdf}(w) &= \frac{1}{\sqrt[n]{2\pi}\sqrt{\det R_w}} e^{-\frac{1}{2}w^T R_w^{-1} w}
\end{aligned}$$

We wish to find the estimate $\hat{x}(t)$ that minimizes the mean square error $E\{(x(t) - \hat{x}(t))(x(t) - \hat{x}(t))^T\}$ given $\{y(\tau) : 0 \le \tau \le t\}$.

Theorem 7.5 (Kalman-Bucy, 1961)**.** *The optimal estimator has the form of a linear observer*

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

where $L(t) = P(t)C^T R_w^{-1}$ *and* $P(t) = E\{(x(t) - \hat{x}(t))(x(t) - \hat{x}(t))^T\}$ *and satisfies*

$$\begin{aligned}
\dot{P} &= AP + PA^T - PC^T R_w^{-1}(t) CP + FR_v(t)F^T \\
P[0] &= E\{x[0]x^T[0]\}
\end{aligned}$$

Example 7.5 Vectored thrust aircraft

To design a Kalman filter for the system, we must include a description of the process disturbances and the sensor noise. We thus augment the system to have the form

$$\begin{aligned}
\frac{dx}{dt} &= Ax + Bu + Gw \\
y &= Cx + v
\end{aligned}$$

where G represents the structure of the disturbances (including the effects of nonlinearities that we have ignored in the linearization), w represents the disturbance source (modeled as zero mean, Gaussian white noise) and v represents that measurement noise (also zero mean, Gaussian and white).

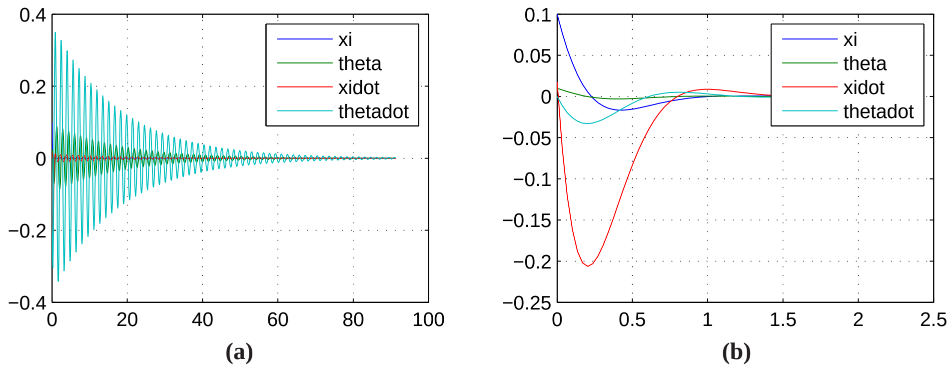


Figure 7.9: Kalman filter design for a vectored thrust aircraft. In the first design (left), only the lateral position of the aircraft is measured. Adding a direct measurement of the roll angle produces a much better observer (right).

For this example, we choose G as the identity matrix and choose disturbances w_i, $i = 1,\dots,n$ to be independent disturbances with covariance given by $R_{ii} = 0.1$, $R_{ij} = 0, i \neq j$. The sensor noise is a single random variable which we model as having covariance $R_v = 0.01$. Using the same parameters as before, the resulting Kalman gain is given by

$$L = \begin{pmatrix} 7.42 \\ -3.70 \\ 27.6 \\ 28.0 \end{pmatrix}$$

The performance of the estimator is shown in Figure 7.9a. We see that while the estimator converges to the system state, it contains significant “ringing” in the state estimate, which can lead to poor performance in a closed loop setting.

To improve the performance of the estimator, we explore the impact of adding a new output measurement. Suppose that instead of measuring just the output position ξ, we also measure the orientation of the aircraft, θ. The output becomes

$$y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

and if we assume that v_1 and v_2 are independent noise sources each with covariance $R_{v_i} = 0.0001$ then the optimal estimator gain matrix becomes

$$L = \begin{pmatrix} 7.31 & -0.019 \\ -0.019 & 6.25 \\ 26.8 & -0.368 \\ 0.110 & 19.6 \end{pmatrix}.$$

These gains provide good immunity to noise and very high performance, as illustrated in Figure 7.9b.

∇

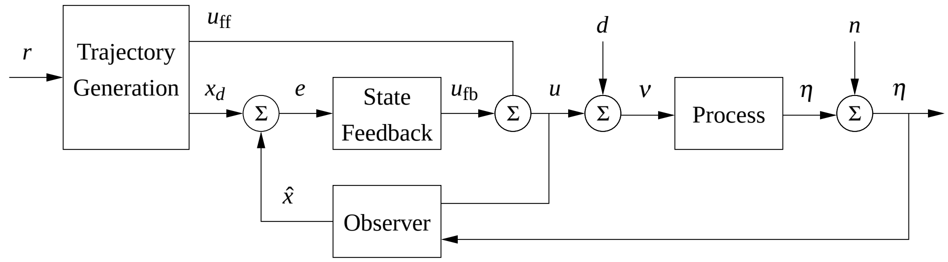


Figure 7.10: Block diagram of a controller based on a structure with two degrees of freedom. The controller consists of a trajectory generator, state feedback and an observer. The trajectory generation subsystem computes a feedforward command $u_{\rm ff}$ along with the desired state x_d. The state feedback controller uses the estimated state and desired state to compute a corrective input $u_{\rm fb}$.

7.5 FEEDFORWARD AND IMPLEMENTATION

In this section we will discuss improved ways to introduce reference values by using feedforward. This leads to a system structure is one that appears in may places in control theory and is the heart of most modern control systems. We will also briefly discuss how computers can be used to implement a controller based on output feedback.

Feedforward

In this chapter and the previous one we have emphasized feedback as a mechanism for minimizing tracking error; reference values were introduced simply by adding them to the state feedback through a gain k_r. A more sophisticated way of doing this is shown by the block diagram in Figure 7.10, where the controller consists of three parts: an observer that computes estimates of the states based on a model and measured process inputs and outputs, a state feedback, and a trajectory generator that generates the desired behavior of all states x_d and a feedforward signal $u_{\rm ff}$. Under the ideal conditions of no disturbances and no modeling errors the signal $u_{\rm ff}$ generates the desired behavior x_d when applied to the process.

To get some insight into the behavior of the system, we assume that there are no disturbances and that the system is in equilibrium with constant reference signal and with the observer state $\hat{x}$ equal to the process state x. When the command signal is changed the signals $u_{\rm ff}$ and x_d will change. The observer tracks the state perfectly because the initial state was correct. The estimated state $\hat{x}$ is thus equal to the desired state x_d and the feedback signal $u_{\rm fb} = L(x_d - \hat{x})$ will also be zero. All action is thus created by the signals from the trajectory generator. If there are some disturbances or some modeling errors the feedback signal will attempt to correct the situation.

This controller is said to have *two degrees of freedom* because the responses to command signals and disturbances are decoupled. Disturbance responses are governed by the observer and the state feedback while the response to command signals is governed by the trajectory generator (feedforward).

For an analytic description we start with the full nonlinear dynamics of the process

$$\dot{x} = f(x,u), \qquad y = h(x,u). \tag{7.24}$$

Assume that the trajectory generator is able to generate a desired trajectory (x_d, u_{ff}) that satisfies the dynamics (7.24) and satisfies $r = h(x_d, u_{\text{ff}})$. To design the controller, we construct the error system. Let $e = x - x_d$, $v = u - u_{\text{ff}}$ and compute the dynamics for the error:

$$\begin{aligned}\dot{e} = \dot{x} - \dot{x}_d &= f(x,u) - f(x_d, u_{\text{ff}})\\ &= f(e + x_d, v + u_{\text{ff}}) - f(x_d)\\ &= F(e, v, x_d(t), u_{\text{ff}}(t)).\end{aligned}$$

In general, this system is time varying.

For trajectory tracking, we can assume that e is small (if our controller is doing a good job) and so we can linearize around $e = 0$:

$$\dot{e} \approx A(t)e + B(t)v, \qquad A(t) = \left.\frac{\partial F}{\partial e}\right|_{(x_d(t), u_{\text{ff}}(t))} \qquad B(t) = \left.\frac{\partial F}{\partial v}\right|_{(x_d(t), u_{\text{ff}}(t)}.$$

It is often the case that $A(t)$ and $B(t)$ depend only on x_d, in which case it is convenient to write $A(t) = A(x_d)$ and $B(t) = B(x_d)$.

Assume now that x_d and u_{ff} are either constant or slowly varying (with respect to the performance criterion). This allows us to consider just the (constant) linear system given by $(A(x_d), B(x_d))$. If we design a state feedback controller $K(x_d)$ for each x_d, then we can regulate the system using the feedback

$$v = K(x_d)e.$$

Substituting back the definitions of e and v, our controller becomes

$$u = -K(x_d)(x - x_d) + u_{\text{ff}}$$

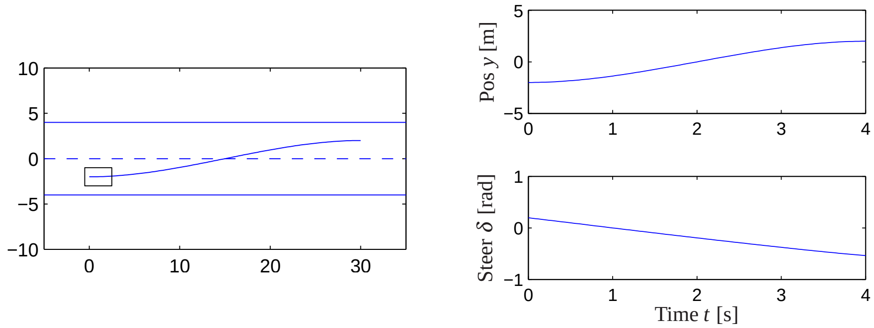
This form of controller is called a *gain scheduled* linear controller with *feedforward* u_{ff}.

Finally, we consider the observer. The full nonlinear dynamics can be used for the prediction portion of the observer and the linearized system for the correction term:

$$\dot{\hat{x}} = f(\hat{x}, u) + L(\hat{x})(y - h(\hat{x}, u))$$

where $L(\hat{x})$ is the observer gain obtained by linearizing the system around the currently estimated state. This form of the observer is known as an *extended Kalman filter* and has proved to be a very effective means of estimating the state of a nonlinear system.

There are many ways to generate the feedforward signal and there are also many different ways to compute the feedback gain K and the observer gain L. Note that once again the internal model principle applies: the controller contains a model of the system to be controlled, through the observer.



(a) **(b)**

Figure 7.11: Trajectory generation for changing lanes. We wish to change from the left lane to the right lane over a distance of 30 meters in 4 seconds. The left figure shows the planned trajectory in the xy plane and the right figure shows the lateral position y and the steering angle δ over the maneuver time interval.

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Example 7.6 Vehicle steering

To illustrate how we can use two degree of freedom design to improve the performance of the system, consider the problem of steering a car to change lanes on a road, as illustrated in Figure 7.11.

The dynamics of the system were derived in Example 2.8. Using the center of the rear wheels as the reference ($\alpha = 0$), the dynamics can be written as

$$\dot{x} = \cos\theta v, \qquad \dot{y} = \sin\theta v, \qquad \dot{\theta} = 1/b \tan\delta,$$

where v is the forward velocity of the vehicle and δ is the steering angle. To generate a trajectory for the system, we note that we can solve for the states and inputs of the system given x, y by solving the following sets of equations:

$$\begin{aligned} \dot{x} &= v\cos\theta & \ddot{x} &= \dot{v}\cos\theta - v\sin\theta\dot{\theta} \\ \dot{y} &= \sin\theta v & \ddot{y} &= \dot{v}\sin\theta + v\cos\theta\dot{\theta} \\ \dot{\theta} &= v/l\tan\delta \end{aligned} \tag{7.25}$$

This set of five equations has five unknowns (θ, $\dot{\theta}$, v, $\dot{v}$ and δ) that can be solved using trigonometry and linear algebra. It follows that we can compute a feasible trajectory for the system given any path $x(t)$, $y(t)$. (This special property of a system is something that is known as *differential flatness* [74, 75].)

To find a trajectory from an initial state (x_0, y_0, θ_0) to a final state (x_f, y_f, θ_f) at a time T, we look for a path $x(t), y(t)$ that satisfies

$$\begin{aligned} x(0) &= x_0 & x(T) &= x_f \\ y(0) &= y_0 & y(T) &= y_f \\ \dot{x}(0)\sin\theta_0 + \dot{y}(0)\cos\theta_0 &= 0 & \dot{x}(T)\sin\theta_T + \dot{y}(T)\cos\theta_T &= 0 \\ \dot{y}(0)\sin\theta_0 + \dot{y}(0)\cos\theta_0 &= 0 & \dot{y}(T)\sin\theta_T + \dot{y}(T)\cos\theta_T &= 0 \end{aligned} \tag{7.26}$$

One such trajectory can be found by choosing $x(t)$ and $y(t)$ to have the form

$$x_d(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3, \qquad y_d(t) = \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3.$$

Substituting these equations into equation (7.26), we are left with a set of linear equations that can be solved for α_i, β_i, $i = 0, 1, 2, 3$. This gives a feasible trajectory for the system by using equation (7.25) to solve for θ_d, v_d and δ_d.

Figure 7.11b shows a sample trajectory generated by solving these equations. Notice that the feedforward input is quite different from 0, allowing the controller to command a steering angle that executes the turn in the absence of errors.

∇

Kalman's Decomposition of a Linear System

In this chapter and the previous one we have seen that two fundamental properties of a linear input/output system are reachability and observability. It turns out that these two properties can be used to classify the dynamics of a system. The key result is Kalman's decomposition theorem, which says that a linear system can be divided into four subsystems: Σ_{ro} which is reachable and observable, $\Sigma_{r\bar{o}}$ which is reachable but not observable, $\Sigma_{\bar{r}o}$ which is not reachable but is observable, and $\Sigma_{\overline{ro}}$ which is neither reachable nor observable.

We will first consider this in the special case of systems where the matrix A has distinct eigenvalues. In this case we can find a set of coordinates such that the A matrix is diagonal and, with some additional reordering of the states, the system can be written as

$$\begin{aligned} \frac{dx}{dt} &= \begin{pmatrix} A_{ro} & 0 & 0 & 0 \\ 0 & A_{r\bar{o}} & 0 & 0 \\ 0 & 0 & A_{\bar{r}o} & 0 \\ 0 & 0 & 0 & A_{\overline{ro}} \end{pmatrix} x + \begin{pmatrix} B_{ro} \\ B_{r\bar{o}} \\ 0 \\ 0 \end{pmatrix} u \\ y &= \begin{pmatrix} C_{ro} & 0 & C_{\bar{r}o} & 0 \end{pmatrix} x + Du. \end{aligned} \tag{7.27}$$

All states x_k such that $B_k \neq 0$ are reachable and all states such that $C_k \neq 0$ are observable. If we set the initial state to zero (or equivalently look at the steady state response if A is stable), the states given by $x_{\bar{r}o}$ and $x_{\overline{ro}}$ will be zero and $x_{r\bar{o}}$ does not affect the output. Hence the output y can be determined from the system

$$\dot{x}_{ro} = A_{ro} x_{ro} + B_{ro} u, \qquad y = C_{ro} x_{ro} + Du$$

Thus from the input/output point of view, it is only the reachable and observable dynamics that matter. A block diagram of the system illustrating this property is given in Figure 7.12a.

The general case of the Kalman decomposition is more complicated and requires some additional linear algebra. Introduce the reachable subspace $\mathscr{X}_r$ which is the linear subspace spanned by the columns of the reachability matrix W_r. The state space is the direct sum of $\mathscr{X}_r$ and another linear subspace $\mathscr{X}_{\bar{r}}$. Notice that $\mathscr{X}_r$ is unique but that $\mathscr{X}_{\bar{r}}$ can be chosen in many different ways. Choosing coordinates

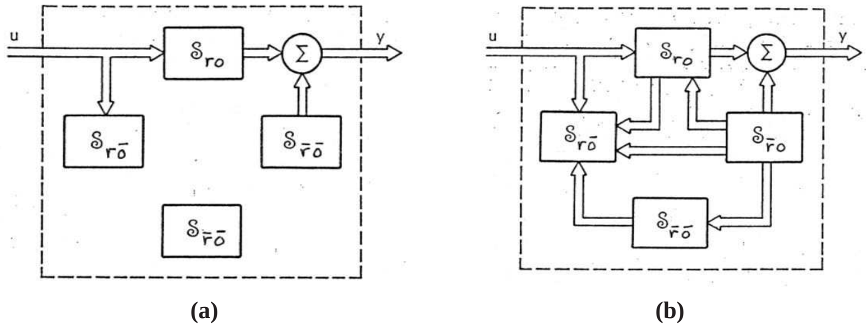

Figure 7.12: Kalman's decomposition of a linear system. The decomposition on the left is for a system with distinct eigenvalues, the one on the right is the general case. The system is broken into four subsystems, representing the various combinations of reachable and observable states. The input/output relationship only depends on the subset of states that are both reachable and observable.

with $x_r \in \mathscr{X}_r$ and $x_{\bar r} \in \mathscr{X}_{\bar r}$ the system equations can be written as

$$\frac{d}{dt}\begin{pmatrix} x_r \\ x_{\bar r}\end{pmatrix} = \begin{pmatrix} A_r^{11} & A_r^{12} \\ 0 & A_r^{22}\end{pmatrix}\begin{pmatrix} x_r \\ x_{\bar r}\end{pmatrix} + \begin{pmatrix} B_1 \\ 0\end{pmatrix} u, \tag{7.28}$$

where the states x_r are reachable and $x_{\bar r}$ are non-reachable.

Introduce the unique subspace $\mathscr{X}_{\bar o}$ of non-observable states. This is the right null space of the observability matrix W_o. The state space is the direct sum of $\mathscr{X}_{\bar o}$ and the non-unique subspace $\mathscr{X}_o$. Choosing a coordinate system with $x_o \in \mathscr{X}_o$ and $x_{\bar o} \in \mathscr{X}_{\bar o}$ the system equations can be written as

$$\begin{aligned}\frac{d}{dt}\begin{pmatrix} x_o \\ x_{\bar o}\end{pmatrix} &= \begin{pmatrix} A_o^{11} & 0 \\ A_o^{21} & A_o^{22}\end{pmatrix}\begin{pmatrix} x_o \\ x_{\bar o}\end{pmatrix} + \begin{pmatrix} B_o^1 \\ B_o^2\end{pmatrix} u \\ y &= \begin{pmatrix} C^1 & 0\end{pmatrix}\begin{pmatrix} x_o \\ x_{\bar o}\end{pmatrix},\end{aligned} \tag{7.29}$$

where the states x_o are observable and $x_{\bar o}$ are not observable.

The intersection of two linear subspaces is also a linear subspace. Introduce $\mathscr{X}_{r\bar o}$ as the intersection of $\mathscr{X}_r$ and $\mathscr{X}_{\bar o}$ and the complementary linear subspace $\mathscr{X}_{ro}$, which together with $\mathscr{X}_{r\bar o}$ spans $\mathscr{X}_r$. Finally, we introduce the linear subspace $\mathscr{X}_{\bar r o}$ which together with $\mathscr{X}_{r\bar o}$, $\mathscr{X}_{r\bar o}$ and $\mathscr{X}_{r\bar o}$ spans the full state space. Notice that the decomposition is not unique because only the subspace $\mathscr{X}_{r\bar o}$ is unique. Combining the representations (7.28) and (7.29) we find that a linear system can be transformed to the form

$$\begin{aligned}\frac{dx}{dt} &= \begin{pmatrix} A^{11} & 0 & A^{13} & 0 \\ A^{21} & A^{22} & A^{23} & A^{24} \\ 0 & 0 & A^{33} & 0 \\ 0 & 0 & A^{43} & A^{44}\end{pmatrix} x + \begin{pmatrix} B^1 \\ B^2 \\ 0 \\ 0\end{pmatrix} u \\ y &= \begin{pmatrix} C^1 & 0 & C^2 & 0\end{pmatrix} x,\end{aligned} \tag{7.30}$$

where the state vector has been partitioned as

$$x = \begin{pmatrix} x_{ro} \\ x_{r\bar{o}} \\ x_{\bar{r}o} \\ x_{\overline{ro}} \end{pmatrix}$$

A block diagram of the system is shown in Figure 7.12b. By tracing the arrows in the diagram we find that the input influences the systems Σ_{ro} and $\Sigma_{r\bar{o}}$ and that the output is influenced by Σ_{ro} and $\Sigma_{\bar{r}o}$. The system $\Sigma_{\overline{ro}}$ is neither connected to the input nor the output. The input/output response of the system is thus given by

$$\dot{x}_{ro} = A^{11}x_{ro} + B^1 u, \qquad y = C^1 x_{ro} + Du, \tag{7.31}$$

which is the dynamics of the reachable and observable subsystem Σ_{ro}.

Example 7.7 System and controller with feedback from observer states
Consider the system

$$\frac{dx}{dt} = Ax + Bu, \qquad y = Cx,$$

The following controller based on feedback from the observer state was given in Theorem 7.3

$$u = -K\hat{x} + k_r r, \qquad \frac{d\hat{x}}{dt} = A\hat{x} + Bu + L(y - C\hat{x})$$

Introducing the states x and $\tilde{x} = x - \hat{x}$ the closed loop system can be written as

$$\frac{d}{dt}\begin{pmatrix} x \\ \tilde{x} \end{pmatrix} = \begin{pmatrix} A - BK & 0 \\ BK & A - LC \end{pmatrix} \begin{pmatrix} x \\ \tilde{x} \end{pmatrix} + \begin{pmatrix} Bk_r \\ 0 \end{pmatrix} r, \qquad y = Cx.$$

The state $\tilde{x}$ is clearly not reachable from the command signal r and the relation between the reference r and the output y is the same as for a system with full state feedback. ∇

Computer Implementation

The controllers obtained so far have been described by ordinary differential equations. They can be implemented directly using analog components, whether electronic circuits, hydraulic valves or other physical devices. Since in modern engineering applications most controllers are implemented using computers we will briefly discuss how this can be done.

A computer-controlled system typically operates periodically: every cycle, signals from the sensors are sampled and converted to digital form by the A/D converter, the control signal is computed and the resulting output is converted to analog form for the actuators, as shown in Figure 7.13. To illustrate the main principles of how to implement feedback in this environment, we consider the controller

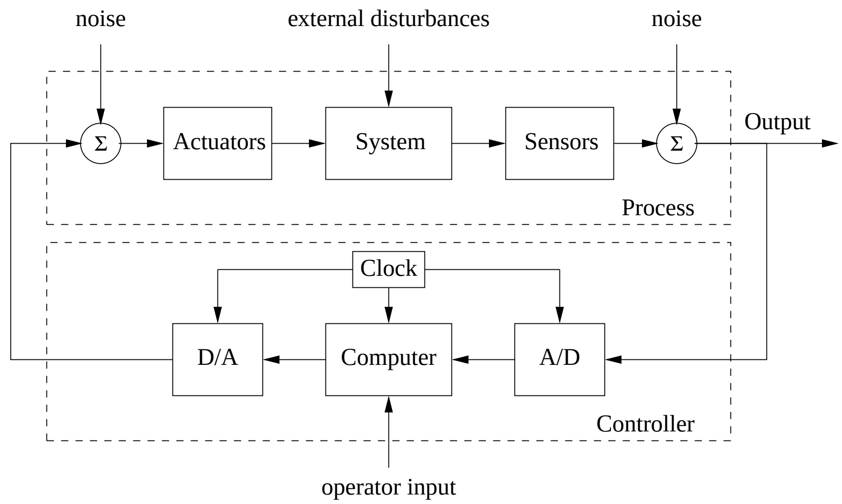

Figure 7.13: Components of a computer-controlled system. The controller consists of analog-to-digital (A/D) and digital-to-analog (D/A) converters, as well as a computer that implements the control algorithm. A system clock controls the operation of the controller, synchronizing the A/D, D/A and computing processes. The operator input is also fed to the computer as an external input.

described by equations (7.14) and (7.15), i.e.,

$$u = -K\hat{x} + k_r r, \qquad \frac{d\hat{x}}{dt} = A\hat{x} + Bu + L(y - C\hat{x}).$$

The first equation consists only of additions and multiplications and can thus be implemented directly on a computer. The second equation can be implemented by approximating the derivative by a difference

$$\frac{dx}{dt} \approx \frac{\hat{x}(t_{k+1}) - \hat{x}(t_k)}{h} = A\hat{x}(t_k) + Bu(t_k) + L\big(y(t_k) - C\hat{x}(t_k)\big),$$

where t_k are the sampling instants and $h = t_{k+1} - t_k$ is the sampling period. Rewriting the equation to isolate $\hat{x}(t_{k+1})$, we get

$$\hat{x}(t_{k+1}) = \hat{x}(t_k) + h\Big(A\hat{x}(t_k) + Bu(t_k) + L\big(y(t_k) - C\hat{x}(t_k)\big)\Big). \tag{7.32}$$

The calculation of the estimated state at time t_{k+1} only requires addition and multiplication and can easily be done by a computer. A section of pseudo code for the program that performs this calculation is

```
% Control algorithm - main loop
r = adin(ch1)                       % read reference
y = adin(ch2)                       % get process output
u = -K*xhat + Kr*r                  % compute control variable
daout(ch1, u)                       % set analog output
xhat = xhat + h*(A*x+B*u+L*(y-C*x)) % update state estimate
```

The program runs periodically at a fixed rate h. Notice that the number of computations between reading the analog input and setting the analog output has been minimized by updating the state after the analog output has been set. The program has an array of states, `xhat`, that represents the state estimate. The choice of sampling period requires some care.

There are more sophisticated ways of approximating a differential equation by a difference equation. If the control signal is constant between the sampling instants it is possible to obtain exact equations; see [20].

There are several practical issues that also must be dealt with. For example, it is necessary to filter a signal before it is sampled so that the filtered signal has little frequency content above $f_s/2$ where f_s is the sampling frequency. If controllers with integral action are used, it is also necessary to provide protection so that the integral does not become too large when the actuator saturates. This issue, called *integrator windup*, is studied in more detail in Chapter 10. Care must also be taken so that parameter changes do not cause disturbances.

7.6 FURTHER READING

The notion of observability is due to Kalman [113] and, combined with the dual notion of reachability, it was a major stepping stone toward establishing state space control theory beginning in the 1960s. The observer first appeared as the Kalman filter, in the paper by Kalman [112] for the discrete time case and Kalman and Bucy [114] for the the continuous time case. Kalman also conjectured that the controller for output feedback could be obtained by combining a state feedback with an observer; see the quote in the beginning of this chapter. This result was formally proved by Josep and Tou [109] and Gunckel and Franklin [93]. The combined result is known as the linear quadratic Gaussian control theory; a compact treatment is given in the books by Anderson and Moore [7] and Åström [15]. Much later it was shown that solutions to robust control problems also had a similar structure but with different ways of computing observer and state feedback gains [64].

EXERCISES

7.1 (Coordinate transformations) Consider a system under a coordinate transformation $z = Tx$, where $T \in \mathbb{R}^{n\times n}$ is an invertible matrix. Show that the observability matrix for the transformed system is given by $\widetilde{W}_o = W_o T^{-1}$ and hence observability is independent of the choice of coordinates.

7.2 Show that the system depicted in Figure 7.2 is not observable.

7.3 (Coordinate transformations) Show that if a system is observable, then there exists a change of coordinates $z = Tx$ that puts the transformed system into observable canonical form.

7.4 (Bicycle dynamics) The linearized model for a bicycle are given in equation (3.5), which has the form

$$J\frac{d^2\varphi}{dt^2} - \frac{Dv_0}{b}\frac{d\delta}{dt} = mgh\varphi + \frac{mv_0^2 h}{b}\delta,$$

where φ is the tilt of the bicycle and δ is the steering angle. Given conditions under which the system is observable and explain any special situations were it loses observability.

7.5 (Pendulum on cart) Consider the linearized model of a pendulum on a cart given in Example **??**. Is the system is observable from the cart position? What happens if the ratio m/M goes to zero? Discuss qualitatively the effect of friction on the cart.

7.6 (Pendulum on cart) Design an observer for the pendulum on the cart. Combine the observer with the state feedback developed in Example **??** to obtain an output feedback. Simulate the system and investigate the effect of a bias error in the angle sensor.

7.7 (Pendulum on cart) A normalized model of the pendulum on a cart is described by the equations

$$\ddot{x} = u, \qquad \ddot{\theta} = \theta + u,$$

where it has been assumed that the cart is very heavy, see Example **??**. Assume that cart position x and the pendulum angle θ are measured, but that there is a bias in the measurement of the angle, which is modeled by $y_2 = \theta + \theta_0$, where θ_0 is a constant bias, hence $\dot{\theta}_0 = 0$. Introduce state variables $x_1 = x$, $x_2 = \theta$, $x_3 = \dot{x}$, $x_4 = \dot{\theta}$ and $x_5 = \theta_0$. Show that the system is observable. What is the engineering implication of the result?

7.8 (Duality) Show that the the following MATLAB function computes the gain L of a an observer for the system $\dot{x} = Ax$, $y = Cx$ which gives and observer whose eigenvalues are the elements of the vector p.

```
function L=observer(A,C,p)
L=place(A',C',p);L=L';
```

Test the program on some examples where you have calculated the result by hand.

7.9 (Selection of Eigenvalues) Pick up the program for simulating Figure 7.4 from the `wiki`. Read the program and make sure that you understand it. Explore the behavior of the estimates for different choices of the eigenvalues.

7.10 (Uniqueness) Show that design of an observer by eigenvalue placement is unique for single output systems. Construct examples that show that the problem is not necessarily unique for systems with many outputs. Suggest how the lack of uniqueness can be exploited.