EE 364a: Convex Optimization I — S. Boyd
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# Midterm Quiz Solutions

1. *Convexity of some sets.* Determine if each set below is convex.

   (a) $\{(x,y) \in \mathbf{R}^2_{++} \mid x/y \leq 1\}$

   (b) $\{(x,y) \in \mathbf{R}^2_{++} \mid x/y \geq 1\}$

   (c) $\{(x,y) \in \mathbf{R}^2_{+} \mid xy \leq 1\}$

   (d) $\{(x,y) \in \mathbf{R}^2_{+} \mid xy \geq 1\}$

   **Solution.**

   (a) *Convex.* The given set is $\{(x,y) \in \mathbf{R}^2_{++} \mid x - y \leq 0\}$, which is the intersection of the positive orthant with a halfspace, thus convex.

   (b) *Convex.* The given set is $\{(x,y) \in \mathbf{R}^2_{++} \mid x - y \geq 0\}$, which is the intersection of the positive orthant with a halfspace, thus convex.

   (c) *Not convex.* The points $(1/2, 2)$ and $(2, 1/2)$ are in the given set, but their average, $(5/4, 5/4)$, is not.

   (d) *Convex.* The given set is $\{(x,y) \in \mathbf{R}^2_{+} \mid \sqrt{xy} \geq 1\}$, which is the 1-superlevel set of the geometric mean, a concave function.

2. *Curvature of some functions.* Determine the curvature of the functions below.

   (a) $f(x) = \min\{2, x, \sqrt{x}\}$, with $\mathbf{dom}\, f = \mathbf{R}_+$

   (b) $f(x) = x^3$, with $\mathbf{dom}\, f = \mathbf{R}$

   (c) $f(x,y) = \sqrt{x \min\{y, 2\}}$, with $\mathbf{dom}\, f = \mathbf{R}^2_+$

   (d) $f(x,y) = (\sqrt{x} + \sqrt{y})^2$, with $\mathbf{dom}\, f = \mathbf{R}^2_+$

   **Solution.**

   (a) *Concave.* The minimum of concave functions is concave.

   (b) *Neither convex nor concave.* The second derivative is $f''(x) = 6x$. Since $f''(1) > 0$ and $f''(-1) < 0$, $f$ is neither convex nor concave.

   (c) *Concave.* The geometric mean $\sqrt{uv}$ is (jointly) concave on $\mathbf{R}^2_{++}$. Since $h$ is increasing in both arguments, $x$ is linear, and $\min\{y, 2\}$ is positive and concave, $\sqrt{x \min\{y, 2\}}$ is concave by the composition rules.

   (d) *Concave.* By expanding the square, $f(x,y) = x + y + 2\sqrt{xy}$, which is the sum of concave functions (on $\mathbf{R}^2_{++}$), thus concave.

3. *Correlation matrices.* Determine if the following subsets of $\mathbf{S}^n$ are convex.

(a) the set of correlation matrices, $\mathcal{C}^n = \{C \in \mathbf{S}^n_+ \mid C_{ii} = 1,\ i = 1, \ldots, n\}$

(b) the set of nonnegative correlation matrices, $\{C \in \mathcal{C}^n \mid C_{ij} \geq 0,\ i, j = 1, \ldots, n\}$

(c) the set of volume-constrained correlation matrices, $\{C \in \mathcal{C}^n \mid \det C \geq (1/2)^n\}$

(d) the set of highly correlated correlation matrices, $\{C \in \mathcal{C}^n \mid C_{ij} \geq 0.8,\ i, j = 1, \ldots, n\}$

**Solution.**

(a) *Convex.* The constraints $C_{ii} = 1$ are linear, so the set is the intersection of $\mathbf{S}^n_+$ with $n$ hyperplanes.

(b) *Convex.* The constraints $C_{ij} \geq 0$ are linear, so the set is the intersection of $\mathcal{C}^n$ with $n^2$ halfspaces.

(c) *Convex.* The constraint $\det C \geq (1/2)^n$ is equivalent to $-\log\det C \leq n \log 2$. Also, note that $\det C \geq (1/2)^n$ implies $C \in \mathbf{S}^n_{++}$. Thus, the given set is the $(n \log 2)$-sublevel set of the convex function $-\log\det C$ (on $\mathbf{S}^n_{++}$), intersected with $\mathcal{C}^n$.

(d) *Convex.* The constraints $C_{ij} \geq 0.8$ are linear, so the given set is the intersection of $\mathcal{C}^n$ with $n^2$ halfspaces.

4. *DCP rules.* The function $f(x, y) = \sqrt{1 + x^4/y}$, with $\mathbf{dom}\, f = \mathbf{R} \times \mathbf{R}_{++}$, is convex. Express $f$ using disciplined convex programming (DCP), limited to the following atoms,

`inv_pos(u)`, which is $1/u$, with domain $\mathbf{R}_{++}$

`square(u)`, which is $u^2$, with domain $\mathbf{R}$

`sqrt(u)`, which is $\sqrt{u}$, with domain $\mathbf{R}_+$

`geo_mean(u,v)`, which is $\sqrt{uv}$, with domain $\mathbf{R}^2_+$

`quad_over_lin(u,v)`, which is $u^2/v$, with domain $\mathbf{R} \times \mathbf{R}_{++}$

`norm2(u,v)`, which is $\sqrt{u^2 + v^2}$, with domain $\mathbf{R}^2$.

You may also use addition, subtraction, scalar multiplication, and any constant functions. Assume that DCP is sign-sensitive, *e.g.*, `square(u)` increasing in $u$ when $u \geq 0$. Please only write down your composition. *No justification is required.*

**Solution.**

Since $f(x, y) = \|(1, x^2/\sqrt{y})\|_2$, we can write the function as

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norm2(1, quad_over_lin(x, sqrt(y))).
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The atom `quad_over_lin` is jointly convex on its domain, and since `sqrt(y)` is concave and positive, the composition `quad_over_lin(x, sqrt(y))` is DCP convex and positive on $\mathbf{R} \times \mathbf{R}_{++}$. Since `norm2` is convex and increasing in both arguments on $\mathbf{R}^2_+$, the full composition is DCP convex.