# Equilibrium Pricing with Positive Externalities (Extended Abstract)

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**Abstract.** We study the problem of selling an item to strategic buyers in the presence of positive *historical externalities*, where the value of a product increases as more people buy and use it. This increase in the value of the product is the result of resolving bugs or security holes after more usage. We consider a continuum of buyers that are partitioned into *types* where each type has a valuation function based on the actions of other buyers. Given a fixed sequence of prices, or *price trajectory*, buyers choose a day on which to purchase the product, i.e., they have to decide whether to purchase the product early in the game or later after more people already own it. We model this strategic setting as a game, study existence and uniqueness of the equilibria, and design an FPTAS to compute an approximately revenue-maximizing pricing trajectory for the seller in two special cases: the *symmetric* settings in which there is just a single buyer type, and the *linear* settings that are characterized by an initial type-independent bias and a linear type-dependent influenceability coefficient.

## 1 Introduction

Many products like software, electronics, or automobiles evolve over time. When a consumer considers buying such a product, he faces a tradeoff between buying a possibly sub-par early version versus waiting for a fully functional later version. Consider, for example, the dilemma faced by a consumer who wishes to purchase the latest Windows operating system. By buying early, the consumer takes full advantage of all the new features. However, operating systems may have more bugs and security holes at the beginning, and hence a consumer may prefer to wait with the rationale that, if more people already own the operating system, then more bugs will have already been uncovered and corrected. The key observation is, the more people that have already used the operating system, or any product for that matter, the more inherent value it accrues. In other words, the product exhibits a particular type of externality, a so-called *historical externality*[5].

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[5] Note that this is different from the more well-studied notion of externalities in the computer science literature where a product (e.g., a cell phone) accrues value as more consumers buy it simply because the product is used in conjunction with other consumers.

How should a company price a product in the presence of historical externalities? A low introductory price may attract early adopters and hence help the company extract greater revenue from future customers. On the other hand, too low a price will result in significant revenue loss from the initial sales. Often, when faced with such a dilemma, a company will offer an initial promotional price at the product's release in a limited-time offer, and then raise the price after some time. For example, when releasing Windows 7, Microsoft announced a two-week pre-order option for the Home Premium Upgrade version at a discounted price of $50; thereafter the price rose to $120, where it has remained since the pre-sale ended on July 11th, 2009. Additionally, beta testers, who can be interpreted as consumers who "bought" the product even prior to release, received the release version of Windows 7 for free (as is often the case with software beta-testers).

We study this phenomenon in the following stylized model: a monopolistic seller wishes to derive a pricing and marketing plan for a product with historical externalities. To this end, she commits to a price trajectory. Potential consumers observe the price trajectory of the seller and make simultaneous decisions regarding the day on which they will buy the product (and whether to buy at all). The payoff of a consumer is a function of the day on which he bought the product, the price on that day, and the set of consumers who bought before him. We compute the equilibria of the resulting sequential game and observe that the revenue-maximizing price trajectories for the seller are increasing, as in the Windows 7 example above.

A few words are in order about our model. First, we focus on settings in which the seller has the ability to commit to a price trajectory. Such commitments are observed in many settings especially at the outset of a new product (see the Windows 7 example described above) and have been assumed in prior models in the economics literature on pricing as well as in other games in the form of Stackelberg strategies [10]. Further, commitment increases revenue: clearly a seller who commits to price trajectories can extract at least as much revenue as a seller who does not (or can not). We further observe via example in the full version of the paper that in fact commitment enables a seller to extract unboundedly higher revenue than in settings without commitment. Second, we assume a consumer's payoff is only a function of past purchases; i.e., consumers have no utility for future purchases. We motivate this in the Windows 7 example by arguing that bugs are resolved in proportion to usage rates. Of course, strictly speaking, consumers of Windows 7 benefit from future purchases as well via software updates and the like. However, this forward-looking benefit is substantially dampened in comparison to past benefits by safety and security risks, and time commitments involved in updates. Another justification for our payoff model comes from consumers' uncertainties regarding products. In many settings, consumers have signals regarding the value of a product (say an electronic gadget like the iPad for example), but do not observe its precise value until the time of purchase. Past purchases and the ensuing online reviews may help consumers improve their estimates of their values prior to purchase, an especially important factor for risk-adverse buyers.

We focus on the non-atomic setting in which we have a continuum of consumers so that each consumer is infinitesimally small and therefore his own action has a negligible effect on the actions of others. Consumers are drawn from a (possibly infinite) set of types. These types capture varying behavior among consumer groups. We study a sequential game in which the seller first commits to a price trajectory and then the consumers simultaneously choose when and whether to buy in the induced normal-form game among them. We study subgame perfect equilibria. We first observe that equilibria exist due to a slight generalization of a paper of Mas-Colell [8] (see the full version of the paper). We then turn to the question of uniqueness. We focus on *well-behaved equilibria* in which consumers with non-negative utility always purchase the product (thus indifferent consumers purchase the product). In general multiple such equilibria may exist. However, in an aggregate model in which the value function of each consumer type depends only on the *aggregate* behavior of the population (i.e., the total fraction of potential consumers that have bought the product and not the total fraction of various types), then we are able to show that when they exist the well-behaved equilibria of this game are unique in the sense that the fraction of purchases per-type-per-day is fixed among all equilibria. This enables us to search for the revenue-maximizing price trajectory. We address this question in settings in which we either have just one type or there are multiple types whose valuation functions are linear in the aggregate, both of which are special cases of the aggregate model discussed above. For each price trajectory, we define its revenue to be the amount of money consumers spend on the product. We then design an FPTAS to find the revenue-maximizing price trajectory for a monopolistic seller in these settings. We do this via a reduction to a novel rectangle covering problem in which we must find the discounted area-maximizing set of rectangles that fit underneath a given curve.

As an interesting consequence of our result, we find that the revenue maximizing price trajectory is an increasing and convex function, matching the intuition that the seller should attract a few early adopters with a low introductory price and then exploit the value they add by offering high prices to remaining consumers. We also note that the distribution of sales in the revenue maximizing equilibrium matches this intuition as well – it is also increasing and convex.

### 1.1 Related Work

Our work falls in the long line of literature investigating pricing and marketing of products that exhibit externalities [1,2,3,4,5,6,7,9]. Among these, the paper of Bensaid and Lesne [2] is most closely related to our own work. Bensaid and Lesne [2] analyzed the two and infinite period pricing problems in the presence of linear historical externalities and they study equilibria of the induced games both with and without commitment. They observe, as we do, that optimal price trajectories are increasing. The historical externalities that we study generalize the externalities of Bensaid and Lesne [2], and in this more general model, we solve for the optimal price sequence for any fixed number of price periods. Most of the remaining externalities literature studies externalities in which consumers care about the total population of users of a product and hence their utility is affected by future sales as well as past sales. Although the phenomenon studied is different from ours, some of the modeling assumptions in these papers are similar to ours. For example, in the economics literature, Cabral, Salant, and Woroch [3] also consider a seller that commits to a price trajectory and then observe that the revenue-maximizing price sequence with fully rational consumers (playing a Bayesian equilibrium) is increasing. Similar to our model, they study the pricing problem in the presence of a continuum of consumers.

In the computer science literature Akhlaghpour et al. [1] and Hartline et al. [5] study algorithmic questions regarding revenue maximization over social networks for products with externalities. However, their models assume naive behavior for consumers. Namely, they assume consumers act myopically, buying the product on the first day in which it offers them positive utility without reasoning about future prices and sales that could affect optimal buying behavior and long-term utility. Furthermore, Hartline et al. [5] allow the seller to use adaptive price discrimination. In contrast, we model consumers as fully rational agents that strategically choose the day on which to buy based on full information regarding all future states of the world and a sequence of public posted prices. While the correct model of pricing and consumer behavior probably lies somewhere between these two extremes, we believe studying fully rational consumers is an important first step in relaxing myopic assumptions.

## 2 Model

We wish to study the sale of a good by a monopolistic seller over $k$ days to a set of potential consumers or buyers. We model our setting as a sequential game whose players consist of the monopolistic seller and a continuum of potential consumers or buyers $b \in [0, 1]$. In our game, the seller moves first, selecting a *price trajectory* $p = (p_1, \ldots, p_k)$ where $p_i \in \Re$ assigning a (possibly negative) price $p_i$ to each day $i$. The buyers move next, selecting a day on which to buy the product given the complete price trajectory, as described below.

The buyers are partitioned into $n$ types $T_1, \ldots, T_n$ where each $T_t$ is a subinterval of $[0, 1]$.[6] The strategy set $A = \{1, \ldots, k\} \cup \{\emptyset\}$ indicates the day on which the product is bought ($\emptyset$ is used to indicate that the product was not purchased). Hence the strategy profile of the buyer population can be represented by a $(k + 1) \times n$ matrix $X = \{X_{i,t}\}_{i=1,\ldots,k+1;t=1,\ldots,n}$ where entry $X_{i,t}$ indicates the fraction of buyers that are of type $t$ and buy the product *before* day $i$, and we define $X_{1,t} = 0$ for all $t$. Note that by normalization $\sum_t X_{k+1,t} \leq 1$ and $1 - \sum_t X_{k+1,t}$ is the fraction of buyers that don't buy the product at any time. Corresponding to this matrix $X$ we also define the *marginal strategy profile* matrix $x = \{x_{i,t}\}_{i=1,\ldots,k;t=1,\ldots,n}$ where $x_{i,t} = X_{i+1,t} - X_{i,t}$ is the fraction of buyers who are of type $t$ and buy on day $i$. In the special case when there is only 1 type, we use $X_i$ as a scalar to denote the fraction of buyers who bought before day $i$ and $x_i$ as a scalar to denote the fraction of buyers who buy on day $i$.

Given a strategy profile $X$, we define the value of buyers of type $t$ buying on day $i$ by a value function $F_i^t(X_i)$ where $X_i$ is the $i$'th row of $X$ (hence buyers are indifferent to future buying decisions). Note the explicit dependence of $F$ on time, which allows $F_i^t(X_i)$ to be different than $F_j^t(X_j)$, for $i \neq j$. The revenue-maximization results in Section 4 further assume that the dependence of $F_i^t(X_i)$ on $i$ is of the form $F_i^t(X_i) = \beta^i F^t(X)$ for $\beta \in [0, 1]$. This special case is of particular interest as the $\beta$ factor models settings in which the value degrades over time due to, for example, a reduction in the novelty of the product.

[6] Later, we will generalize this to infinitely many types.

Given a strategy profile $X$, the payoff of buyers of type $t$ who buy on day $i$ is defined to be $F_i^t(X_i) - p_i$. We additionally allow buyers to have a discount factor $\alpha$ such that their payoff is $(1-\alpha)^i(F_i^t(X_i) - p_i)$. Thus $\alpha$ represents the way in which agents discount future payoffs with respect to present payoffs. We say that a strategy profile $X$ is a Nash equilibrium of the induced subgame given by price trajectory $p$, or equivalently $X \in NE(p)$, if for any buyer of type $t$ who buys on day $i$ we have $i \in \arg\max_j (F_j^t(X_j) - p_j)(1-\alpha)^j$, and the strategy is $\emptyset$ whenever the maximum is negative (in which case the buyer's payoff is zero). We call an equilibrium *well-behaved* if all indifferent buyers buy, i.e., a buyer does not buy if and only if his payoff $(1-\alpha)^i(F_i^t(X_i) - p_i)$ is negative on all days $1 \leq i \leq k$. We say that $(p, X)$ is a (well-behaved) equilibrium if the profile $X$ is a (well-behaved) Nash equilibrium for the subgame of price trajectory $p$. Equivalently, a marginal strategy profile $x$ is a (well-behaved) Nash equilibrium for the subgame of price trajectory $p$ if for any type $t$ and day $i$ we have $x_{i,t} > 0$ only if $i \in \arg\max_j (F_j^t(X_j) - p_j)(1-\alpha)^j$ and the value of this maximum is non-negative.

Given a price trajectory $p$ and a marginal strategy profile $x$ that arises in the subgame induced by $p$, we define the payoff of the seller to be the *revenue* of $x$ for $p$, which is $R(p, x) = \sum_{i=1}^{k} \sum_{t=1}^{n} x_{i,t} p_i (1-\alpha)^i$. A *subgame perfect equilibrium* of the sequential game is then a price trajectory $p^*$ and a set of marginal strategy profiles $x_p$ for each possible price trajectory $p$ such that: (1) $x_p$ is a Nash equilibrium of the subgame induced by $p$, and (2) $p^*$ maximizes $R(p, x_p)$. The *outcome* of this subgame perfect equilibrium is $(p^*, x_{p^*})$ and its revenue is $R(p^*, x_{p^*})$.

We are interested in computing the outcome in a revenue-maximizing subgame perfect equilibrium. To do so, we must compute a price trajectory which maximizes the revenue of the seller in equilibrium. Note that this is equal to finding the best response of the seller given the strategies $\{x_p\}$ of the buyers. We solve this problem for special settings in which there exist revenue-maximizing well-behaved equilibria in $NE(p)$ for any price trajectory $p$, allowing us to maximize over them. These settings are as follows. For the purpose of these definitions, we will allow each buyer to have a unique type and hence there are infinitely many types. We will use $b \in [0, 1]$ to denote type of buyer $b$.

**Definition 1.** *The Aggregate Model: The value function of each type in this model is a function of the aggregate behavior of the population and is invariant with respect to the behavior of each separate type. That is, the value function of buyer* $b$ *is a function of* $X_i$ *only, where* $X_i$ *is a scalar indicating the total fraction of all buyers who buy before day* $i$*. In this instance, we overload the notation for the value function and let* $F_i^b(X_i)$ *indicate the value of buyer* $b$ *(hence* $F_i^b(\cdot)$ *now maps the unit interval to the non-negative reals).*

**Definition 2.** *The Linear Model: This is a special case of the aggregate model which is defined by a function* $F_i$*, an initial bias* $I$*, and a function* $C$ *so that the value of buyer* $b$ *is* $F_i^b(X_i) = I + C(b) \cdot F_i(X_i)$*. We further define the commonly-known distribution* $\mathcal{C} : \mathbb{R} \rightarrow [0, 1]$ *such that* $\mathcal{C}(c^*)$ *indicates the fraction of buyers* $b$ *with* $C(b) \leq c^*$*.*

**Definition 3.** *The Symmetric Model: In this version we only have one type, that is,* $F_i^b = F_i$ *for all* $b$*.*

We note that alternatively, one could model this pricing game as a sequential game with multiple stages where in each day $i$ the seller selects a price $p_i$ and then buyers simultaneously choose whether to buy or not. Such a model is appropriate when it is not possible for a seller to commit to a price trajectory in advance. Again, in this setting, one could study the subgame perfect equilibria and analyze the resulting revenue. Clearly the revenue with commitment is at least as high as that without commitment. Also, there are examples in which the revenue without commitment can be unboundedly less.

## 3 Uniqueness of Equilibria

We prove that if there exists a *well-behaved equilibrium*, that is an equilibrium in which everyone with non-negative utility buys on some day, then it is unique. We show this for an infinite number of types in the aggregate model which generalizes both the linear and symmetric models.

Recall that we allow for each buyer $b \in [0, 1]$ to have a unique type in the aggregate model such that the valuation function of buyer $b$ is $F_i^b$. We will show that in all of the well-behaved equilibrium points the fraction of people buying on each day is the same. In turn, it implies that the revenue of all well-behaved equilibrium points is the same and hence the well-behaved equilibria are revenue-unique. In what follows, we consider the equilibria of a fixed price sequence $p$. We start with a definition: Consider two well-behaved equilibria $x$ and $y$. Partition the set of $k$ days to two sets as follows: We call a day $i$ a *level 1* day, and denote it by $i \in D_1(x, y)$, if $X_i < Y_i$. Otherwise, if $X_i \geq Y_i$, we call $i$ a *level 2* day and denote it by $i \in D_2(x, y)$.

**Lemma 1.** *Assume that there exist two distinct well-behaved equilibria $x$ and $y$. Then there exists a buyer whose strategy in $x$ is a day $i$ such that $i \in D_1(x, y)$ and whose strategy in $y$ is $j \in D_2(x, y)$.*

**Theorem 1.** *Let $F_i^b(X)$ be a strictly increasing function for each buyer b and day $i$. For a price sequence p and two well-behaved equilibrium points $x$ and $y$, we have $X_i = Y_i$, i.e. the fraction of buyers who have bought the product before day $i$ is unique.*

*Proof.* Assume for contradiction that we have two well-behaved equilibrium points $x$ and $y$ and a day $i$ for which $X_i \neq Y_i$. Again assume without loss of generality that $X_i < Y_i$. By lemma 1 we know that there exists a buyer $b$ who buys on a level 1 day in $x$ and buys on a level 2 day in $y$. Assume that $b$ buys on day $i$ in $x$ and on day $j$ in $y$. Then $F_i^b(X_i) - p_i \geq F_j^b(X_j) - p_j$ and $F_j^b(Y_j) - p_j \geq F_i^b(Y_i) - p_i$. Adding the two inequalities we get: $F_i^b(X_i) + F_j^b(Y_j) \geq F_j^b(X_j) + F_i^b(Y_i)$. On the other hand since $i$ is a level 1 day, $X_i < Y_i$; hence by monotonicity $F_i^b(X_i) < F_i^b(Y_i)$. Since $j$ is a level 2 day, $X_j \geq Y_j$; hence $F_j^b(Y_j) \leq F_j^b(X_j)$. The addition of these two inequalities contradicts the previous one.

## 4 Revenue Maximization

In this section, we solve the revenue-maximizing problem in two special cases: the discounted version of the symmetric model, and the general linear model without discount factors. In both cases, we provide an FPTAS to compute the revenue-maximizing price sequence. We do this by first showing that in both cases, the revenue maximizing equilibria are well-behaved ones, and then considering the problem of maximizing over well-behaved equilibria. We characterize the set of well-behaved equilibria in each section, and then use novel reductions of the problem into a new problem, called the *Rectangular Covering Problem* (RCP). The RCP is to maximize the discounted area covered by a certain number of rectangles that are fit under a given curve.

**Definition 4. Rectangular Covering Problem (RCP)** *Given an increasing function $F$ and an integer $k$, find a sequence $p$ of size at most $k$ that maximizes the discounted total area of the rectangles fit under the graph of $F$, that is, $p \in \arg\max_{p'} \sum_t (F^{-1}(p'_{t+1}) - F^{-1}(p'_t))p'_t\gamma^t$.*

We provide an FPTAS for the RCP in the full version of the paper. Given the reductions from the revenue maximization problem to rectangular covering problem, this directly gives us FPTASs for the two versions of the problem.

### 4.1 Symmetric Setting

We start by characterizing the equilibria. Since all players in this model have the same valuation function $F$, the marginal strategy profile matrix will reduce to the vector $x = (x_1, \ldots, x_k)$. Also, fixing $p$ and $x$, the utility of buyer $b$ for the item on day $i$ is $F_i^b(X_i) = F(X_i)\beta^i(1-\alpha)^i - p_i(1-\alpha)^i$, and the revenue $R(p, x) = \sum_i x_i p_i (1-\alpha)^i$. By renaming $q_i = p_i(1-\alpha)^i$ and $\gamma = \beta(1-\alpha)$, the utility of buyer $b$ for the item on day $i$ will be $F(X_i)\gamma^i - q_i$, and the revenue becomes $\sum_i x_i q_i$. Using this new notation, we may assume without loss of generality that the only discount factor is $\gamma$. For convenience, we use $p$ for the discounted prices $q$.

Since we only have one type in this model, we know that the utility of buying in day $i$ is equal among all players. We use the term *utility of a day* $i$, denoted by $u_i$, for $u_i = F(X_i)\gamma^i - p_i$. Define $u(p, x) = \max_i u_i$. Consider a price sequence and its equilibrium strategy profile $x$. We get the following properties immediately from the facts that players are utility maximizing: (i) players are allowed to choose inaction and have utility zero, (ii) they choose to buy if there is a day with a strictly positive utility. First, if there is an $i$ with $x_i > 0$, then $u(p, x) \geq 0$ and $u_i = u(p, x)$. Second, if there is a day $i$ with $x_i > 0$, then $\sum_{i=1}^k x_i = 1$.

**Lemma 2.** *Let $\hat{p}$ be the revenue-maximizing price vector that results in equilibrium $\hat{x}$. Then $u(\hat{p}, \hat{x}) = 0$.*

We use lemma 2 to find a closed form for the revenue of a price sequence. Assume that there is a price sequence $p$ with equilibrium $x$ and $u(p, x) = 0$ such that for some day $i$, we have $x_i = 0$ and $x_{i+1} > 0$. Then we can define a new price sequence $\tilde{p}$ which is equal to $p$ except that $\tilde{p}_j = p_{j+1}/\gamma$ for each $j \geq i$. Also define the vector $\tilde{x}$ to be equal to $x$ except that $\tilde{x}_j = x_{j+1}$ for each $j \geq i$, and $\tilde{x}_k = 0$. One can observe that the pair $(\tilde{p}, \tilde{x})$ is an equilibrium with no less revenue. So we can assume WLOG that for a revenue maximizing price sequence $\hat{p}$ associated with $\hat{x}$, there exists a $k' \leq k$ such that $x_i \neq 0$ if and only if $i \leq k'$. For such a price sequence, lemma 2 shows that $F(X_i)\gamma^i - p_i = 0$ for each $1 \leq i \leq k'$. As a result, we have $X_i = F^{-1}(p_i/\gamma^i)$, which is well-defined as $F$ is increasing. Now set $p'_t = p_t/\gamma^t$. The fraction of people buying on day $i$ and paying price $p_i$ is equal to $x_i = F^{-1}(p'_{i+1}) - F^{-1}(p'_i)$. So the revenue is $\sum_i x_i p_i = \sum_i (F^{-1}(p'_{i+1}) - F^{-1}(p'_i)) p'_i \gamma^i$. The revenue maximization problem therefore reduces to the rectangular covering problem.

### 4.2 Linear Version

Similar to the symmetric model, we reduce the linear model to rectangular covering problem by first characterizing the set of well-behaved equilibria. The sketch of this more technical proof is as follows. We show that in each equilibria all the purchases are sorted by the $C_b$ coefficient, i.e., a player with a lower $C_b$ buys earlier than one with higher such coefficient. We then argue that each equilibria is characterized by a sequence of thresholds $(s_1, s_2, \ldots, s_k)$ such that each person $b$ with $C_b \in [s_{i-1}, s_i]$ buys in day $i$. The problem is then to optimize the sequence $(s_1, s_2, \ldots, s_k)$ to maximize revenue. Using this characterization, we provide a closed form of the optimum revenue in the following lemma.

**Lemma 3.** *If $x$ and $p$ correspond to the revenue-maximizing equilibrium, the total revenue can be expressed by the following formula $R(p,x) = I + \sum_{i=2}^{k} (1 - X_i)\mathcal{C}^{-1}(X_i) \times (F(X_i) - F(X_{i-1}))$.*

We then show how this problem can be reduced to RCP in the following lemma.

**Lemma 4.** *The problem of maximizing $\sum_{i=2}^{k} (1 - X_i)\mathcal{C}^{-1}(X_i)(F(X_i) - F(X_{i-1}))$ can be reduced to the Rectangular Covering Problem.*

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