

# On the Non-progressive Spread of Influence through Social Networks

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**Abstract.** The spread of influence in social networks is studied in two main categories: the progressive model and the non-progressive model (see e.g. the seminal work of Kempe, Kleinberg, and Tardos in KDD 2003). While the progressive models are suitable for modeling the spread of influence in monopolistic settings, non-progressive are more appropriate for modeling non-monopolistic settings, e.g., modeling diffusion of two competing technologies over a social network. Despite the extensive work on the progressive model, non-progressive models have not been studied well. In this paper, we study the spread of influence in the non-progressive model under the strict majority threshold: given a graph  $G$  with a set of initially infected nodes, each node gets infected at time  $\tau$  iff a majority of its neighbors are infected at time  $\tau - 1$ . Our goal in the *MinPTS* problem is to find a minimum-cardinality initial set of infected nodes that would eventually converge to a steady state where all nodes of  $G$  are infected.

We prove that while the MinPTS is NP-hard for a restricted family of graphs, it admits an improved constant-factor approximation algorithm for power-law graphs. We do so by proving lower and upper bounds in terms of the minimum and maximum degree of nodes in the graph. The upper bound is achieved in turn by applying a natural greedy algorithm. Our experimental evaluation of the greedy algorithm also shows its superior performance compared to other algorithms for a set of real-world graphs as well as the random power-law graphs. Finally, we study the convergence properties of these algorithms and show that the non-progressive model converges in at most  $O(|E(G)|)$  steps.

## 1 Introduction

Studying the *spread of social influence* over in networks under various *propagation models* is a central issue in social network analysis[18, 12, 27, 32]. This

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issue plays an important role in several real-world applications including viral marketing [5, 13, 30, 23]. As categorized by Kempe et al. [23], there are two main types of influence propagation models: the progressive and the non-progressive models. In *progressive models*, infected (or influenced) vertices will remain infected forever, but in the *non-progressive model*, under some conditions, infected vertices may become uninfected again. In the context of viral marketing and diffusion of technologies over social networks, the progressive model captures the monopolistic settings where one new service is propagated among nodes of the social network. On the other hand, in non-monopolistic settings, multiple service providers might be competing to get people adopting their services, and thus users may switch among two or more services back and forth. As a result, in these non-monopolistic settings, the more relevant model to capture the spread of influence is the non-progressive model [20, 4, 14, 22].

While the progressive model has been studied extensively in the literature [23, 31, 19, 3, 8–10], the non-progressive model has not received much attention in the literature. In this paper, we study the non-progressive influence models, and report both theoretical and experimental results for our models. We focus on the *strict majority* propagation rule in which the state of each vertex at time  $\tau$  is determined by the state of the majority of its neighbors at time  $\tau - 1$ . As an application of this propagation model, consider two competing technologies (e.g. IM service) that are competing in attracting nodes of a social network to adopt their service, and nodes tend to adopt a service that the majority of their neighbors already adopted. This type of influence propagation process can be captured by applying the strict majority rule. Moreover, as an illustrative example of the linear threshold model [23], the strict majority propagation model is suitable for modeling transient faults in fault tolerant systems [16, 28, 17], and also used in verifying convergence of consensus problems on social networks [26]. Here we study the non-progressive influence models under the strict majority rule. In particular, we are mainly interested in the *minimum perfect target set* problem where the goal is to identify a target set of nodes to infect at the beginning of the process so that all nodes get infected at the end of the process. We will present approximation algorithms and hardness results for the problem as well experimental evaluation of our results. As our main contributions, we provide improved upper and lower bounds on the size of the minimum perfect target set, which in turn, result in improved constant-factor approximations for power-law graphs. Finally, we also study the convergence rate of our algorithms and report preliminary results. Before stating our results, we define the problems and models formally.

**Problem Formulations.** Consider a graph  $G(V, E)$ . Let  $N(v)$  denote the set of neighbors of vertex  $v$ , and  $d(v) = |N(v)|$ . Also, let  $\Delta(G)$  and  $\delta(G)$  denote the maximum and minimum degree of vertices in  $G$  respectively. A *0/1 initial assignment* is a function  $f_0 : V(G) \rightarrow \{0, 1\}$ . For any 0/1 initial assignment  $f_0$ , let  $f_\tau : V(G) \rightarrow \{0, 1\}$  ( $\tau \geq 1$ ) be the state of vertices at time  $\tau$  and  $t(v)$  be the threshold associated with vertex  $v$ . For the strict majority model, the threshold  $t(v) = \lceil \frac{d(v)+1}{2} \rceil$  for each vertex  $v$ .

In the *non-progressive strict majority* model:

$$f_\tau(v) = \begin{cases} 0 & \text{if } \sum_{u \in N(v)} f_{\tau-1}(u) < t(v) \\ 1 & \text{if } \sum_{u \in N(v)} f_{\tau-1}(u) \geq t(v) . \end{cases}$$

In the *progressive strict majority* model:

$$f_\tau(v) = \begin{cases} 0 & \text{if } f_{\tau-1}(v) = 0 \text{ and } \sum_{u \in N(v)} f_{\tau-1}(u) < t(v) \\ 1 & \text{if } f_{\tau-1}(v) = 1 \text{ or } \sum_{u \in N(v)} f_{\tau-1}(u) \geq t(v) . \end{cases}$$

Strict majority model is related to the *linear threshold model* in which  $t(v)$  is chosen at random and not necessarily equal to  $\lceil \frac{d(v)+1}{2} \rceil$ .

A 0/1 initial assignment  $f_0$  is called a *perfect target set* (PTS) if for a finite  $\tau$ ,  $f_\tau(v) = 1$  for all  $v \in V(G)$ , i.e., the dynamics will converge to a steady state of all 1's. The cost of a target set  $f_0$ , denoted by  $\text{cost}(f_0)$ , is the number of vertices  $v$  with  $f_0(v) = 1$ . The minimum *perfect target set* (MinPTS) problem is to find a perfect target set with minimum cost. The cost of this minimum PTS is denoted by  $PPTS(G)$  and  $NPPTS(G)$  respectively for progressive and non-progressive models. This problem is also called *target set selection* [1]. Another variant of this problem is the *maximum active set* problem [1] where the goal is to find at most  $k$  nodes to activate (or infect) at time zero such that the number of finally infected vertices is maximized.

A graph is power-law if and only if its degree distribution follows a power-law distribution asymptotically. That is, the fraction  $P(x)$  of nodes in the network having degree  $x$  goes for large number of nodes as  $P(x) = \alpha x^{-\gamma}$  where  $\alpha$  is a constant and  $\gamma > 1$  is called power-law coefficient. It is widely observed that most social networks are power-law [11].

**Our Results and Techniques.** In this paper, we study the spread of influence in the non-progressive model under the strict majority threshold. We present approximation algorithms and hardness results for the problem as well experimental evaluation of our results. As our main contributions, we provide improved upper and lower bounds on the size of the minimum perfect target set, which in turn, result in improved constant-factor approximations for power-law graphs. In addition, we prove that the MinPTS problem (or computing  $NPPTS(G)$ ) is NP-hard for a restricted family of graphs. In particular, we prove lower and upper bounds on  $NPPTS(G)$  in terms of the minimum degree ( $\delta(G)$ ) and maximum degree ( $\Delta(G)$ ) of nodes in the graph, i.e., we show that  $\frac{2n}{\Delta(G)+1} \leq NPPTS(G) \leq \frac{n\Delta(G)(\delta(G)+2)}{4\Delta(G)+(\Delta(G)+1)(\delta(G)-2)}$ . The proofs of these bounds are combinatorial and start by observing that in order to bound  $NPPTS(G)$  for general graphs, one can bound it for bipartite graphs. The upper bound is achieved in turn by applying a natural greedy algorithm which can be easily implemented. Our experimental evaluation of the greedy algorithm also shows its superior performance compared to other algorithms for a set of real-world graphs as well as the random power-law graphs<sup>4</sup>. Finally, we study the convergence properties of these algorithms. We first observe that the process will

<sup>4</sup> The report of our experimental works is given in the journal version due to space shortage

always converges to a fixed point or a cycle of size two. Then we focus on the convergence time and prove that for a given graph  $G$ , it takes at most  $O(|E(G)|)$  rounds for the process to converge. We also evaluate the convergence rate of the non-progressive influence models on some real-world social networks, and report the average convergence time for a randomly chosen set of initially infected nodes.

**More Related Work.** The non-progressive spread of influence under the strict majority rule is related to the diffusion of two or more competing technologies over a social network [20, 4, 14, 22]. As an example, an active line of research in economics and mathematical sociology is concerned with modeling these types of diffusion processes as a coordination game played on a social network [20, 4, 14, 22]. Note that none of these previous prior work provide a bound for the perfect target set problem.

It has been brought to our attention that in a relevant unpublished work by Chang [6], the MinPTS problem on power-law graphs is studied and the bound of  $NPPTS(G) = O(\lceil \frac{|V|}{2^{\gamma-1}} \rceil)$  is proved under non-progressive majority models in a power-law graph. But his results do not practically provide any bound for the strict majority model. We will show that our upper bound is better and practically applicable for different amounts of  $\gamma$  under the strict majority threshold.

Tight or nearly tight bounds on the  $PPTS(G)$  are known for special types of graphs such as torus, hypercube, butterfly and chordal rings [15, 16, 24, 28, 29]. The best bounds for progressive strict majority model in general graphs are due to Chang and Lyuu. In [8], they showed that for a directed graph  $G$ ,  $PPTS(G) \leq \frac{23}{27}|V(G)|$ . In [7], they improved their upper bound to  $\frac{2}{3}|V(G)|$  for directed graphs and  $\frac{|V(G)|}{2}$  for undirected graphs. However, to the best of our knowledge, there is no known bound for  $NPPTS(G)$  for any type of graphs. In this paper, we will combinatorially prove that  $\frac{2n}{\Delta(G)+1} \leq NPPTS(G) \leq \frac{n\Delta(G)(\delta(G)+2)}{4\Delta(G)+(\Delta(G)+1)(\delta(G)-2)}$ .

It is known that the Target Set Selection problem and Maximum Active Set problem are both NP-hard in the linear threshold model [23], and approximation algorithms have been developed for these problems. Kempe et al. [23] and Mossel and Roch [25] present a  $(1 - \frac{1}{e})$ -approximation algorithm for the maximum active set problem by showing that the set of finally influenced vertices as a function of the originally influenced nodes is submodular. On the other hand, it has been shown that the target set selection problem is not approximable for different propagation models [19, 3, 7, 9]. The inapproximability result of Chang and Lyuu in [7] on the progressive strict majority threshold model is the most relevant result to our results. They show that unless  $NP \subseteq TIME(n^{O(\ln \ln n)})$ , no polynomial time  $((1/2 - \epsilon) \ln |V|)$ -approximation algorithm exists for computing  $PPTS(G)$ . To the best of our knowledge, no complexity theoretic results have been obtained for the non-progressive models.

The problem of maximizing social influence for specific family of graphs has been studied under the name of *dynamic monopolies* in the combinatorics literature [15, 16, 24, 28, 29, 7, 1, 6]. All these results are for the progressive model. The optimization problems related to the non-progressive influence models are

not well-studied in the literature. The one result in the area is due to Kempe et al. [23] who presented a general reduction from non-progressive models to progressive models. Their reduction, however, is not applicable to the perfect target set selection problem.

## 2 Non-Progressive Spread of Influence in General Graphs

In this section, we prove lower bound and upper bound for minimum PTS in graphs, and finally show that finding the minimum PTS in general graphs is NP-Hard.

**Lower bound.** The following theorem shows that if we have some lower bound and upper bound for minimum Perfect Target Set in bipartite graphs then these bounds could be generalized to all graphs (Theorem 1).

**Lemma 1.** *If  $\alpha|V(H)| \leq NPPTS(H) \leq \beta|V(H)|$  for every bipartite graph  $H$  under strict majority threshold, then  $\alpha|V(G)| \leq NPPTS(G) \leq \beta|V(G)|$  under strict majority threshold for every graph  $G$ .*

The following lemma shows characteristics of PTSs in some special cases. These will be used in proof of our theorems.

**Lemma 2.** *Consider the non-progressive model and let  $G = (X, Y)$  be a bipartite graph and  $f_0$  be a perfect target set under strict majority threshold. For every  $S \subseteq V(G)$  if  $\sum_{v \in S \cap X} f_0(v) = 0$  or  $\sum_{v \in S \cap Y} f_0(v) = 0$ , then there exists at least one vertex  $u$  in  $S$  such that  $d_S(u) \leq d(u) - t(u)$ .*

If the conditions of previous lemma hold, we can obtain an upper bound for number of edges in the graph. Following lemma provides this upper bound. This will help us finding a lower bound for NPPTS of graphs. The function  $t : V(G) \rightarrow \mathbb{N}$  may be any arbitrary function but here it is interpreted as the threshold function.

**Lemma 3.** *Consider a graph  $G$  with  $n$  vertices. If for every  $S \subseteq V(G)$  there exists at least one vertex  $v$  for which  $d_S(v) \leq d(v) - t(v)$ , then  $|E(G)| \leq \sum_{u \in V(G)} (d(u) - t(u))$ .*

The following theorem shows that for every bipartite graph  $G$ ,  $NPPTS(G) \geq \frac{2|V(G)|}{\Delta(G)+1}$ . Lemma 1 generalizes this theorem to all graphs. Also, Lemma 4 shows that this bound is tight. In the following, the induced subgraph of  $G$  with a vertex set  $S \subseteq V(G)$  is denoted by  $G[S]$ .

**Theorem 1.** *For every bipartite graph  $G = (X, Y)$  of order  $n$ ,  $NPPTS(G) \geq \frac{2n}{\Delta(G)+1}$ .*

*Proof.* Let  $f_0$  be an arbitrary PTS for  $G$ . Partition the vertices of graph  $G$  into three subsets  $B_X$ ,  $B_Y$  and  $W$  as follows.

$$\begin{aligned} B_X &= \{v \in X \mid f_0(v) = 1\} \\ B_Y &= \{v \in Y \mid f_0(v) = 1\} \\ W &= \{v \in V(G) \mid f_0(v) = 0\} \end{aligned}$$

Consider the induced subgraph of  $G$  with vertex set  $B_X \cup W$  and assume that  $S \subseteq B_X \cup W$ . For every vertex  $v \in Y \cap S$ , we have  $f_0(v) = 0$ . So By Lemma 2, for every  $S \subseteq B_X \cup W$  there is at least one vertex  $u$  such that  $d_S(u) \leq d(u) - t(u)$ . By Lemma 3, this implies that  $G[B_X \cup W]$  has at most  $\sum_{u \in B_X \cup W} (d(u) - t(u))$  edges. Similarly we can prove that  $G[B_Y \cup W]$  has at most  $\sum_{u \in B_Y \cup W} (d(u) - t(u))$  edges. Let  $e_W$  be the number of edges in  $G[W]$ ,  $e_{WX}$  be the number of edges with one end point in  $B_X$  and the other end point in  $W$  and  $e_{WY}$  be the number of edges with one end point in  $B_Y$  and the other end point in  $W$ . we have:

$$\begin{aligned} e_{WX} + e_W &\leq \sum_{v \in B_X \cup W} (d(v) - t(v)) \\ e_{WY} + e_W &\leq \sum_{v \in B_Y \cup W} (d(v) - t(v)) \end{aligned}$$

and so,

$$e_{WX} + e_{WY} + 2e_W \leq \sum_{v \in V(G)} (d(v) - t(v)) + \sum_{v \in W} (d(v) - t(v))$$

The total degree of vertices in  $W$  is  $\sum_{v \in W} d(v) = e_{WX} + e_{WY} + 2e_W$ . Thus

$$\sum_{v \in W} d(v) \leq \sum_{v \in V(G)} (d(v) - t(v)) + \sum_{v \in W} (d(v) - t(v))$$

If we denote the set of vertices for which  $f_0$  is equal to 1 by  $B$ , we have

$$\sum_{v \in W} (2t(v) - d(v)) \leq \sum_{v \in B} (d(v) - t(v)) \quad (1)$$

For every vertex  $v$ ,  $t(v) \geq \frac{d(v)+1}{2}$ , so

$$\begin{aligned} |W| &\leq \sum_{v \in B} \frac{d(v) - 1}{2} \Rightarrow |W| \leq \frac{\Delta - 1}{2} (|B|) \\ \Rightarrow |B| &\geq \frac{2n}{\Delta + 1} \end{aligned}$$

And the proof is complete.  $\square$

We now show that the bound in Theorem 1 is tight.

**Lemma 4.** *For infinitely many  $n$ 's, there exists a  $2d + 1$ -regular graph with  $n$  vertices such that  $NPPTS(G) = \frac{n}{d+1}$  under strict majority rule.*

**Upper bound.** In this section, we present a greedy algorithm which gives an upper bound for  $NPPTS(G)$ .

**Theorem 2.** *For every graph  $G$  of order  $n$ ,  $NPPTS(G) \leq \frac{n\Delta(\delta+2)}{4\Delta+(\Delta+1)(\delta-2)}$  under strict majority threshold.*

**Algorithm 1** Greedy NPPTS

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sort the vertices in  $G$  in ascending order of their degrees as the sequence  $v_1, \dots, v_n$ .
for  $i = 1$  to  $n$  do
  whiteadj $[v_i] = 0$ 
  blocked $[v_i] = 0$ 
end for
for  $i = 1$  to  $n$  do
  for each  $u \in N(v_i)$  do
    if whiteadj $[u] = d(u) - t(u)$  then
      blocked $[v_i] = 1$ 
    end if
  end for
  if (blocked $[v_i] = 1$ ) then
     $f_0(v) = 1$ 
  else
     $f_0(v) = 0$ 
    for each  $u \in N(v_i)$  do
      whiteadj $[u] + = 1$ 
    end for
  end if
end for

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Algorithm 1 guarantees this upper bound. This algorithm gets a graph  $G$  of order  $n$  and the threshold function  $t$  as input and determines the values of  $f_0$  for each vertex.

**Lemma 5.** *The algorithm Greedy NPPTS finds a Perfect Target Set for non-progressive spread of influence.*

**Lemma 6.** *For every graph  $G$  of order  $n$ , Greedy NPPTS guarantees the upper bound of  $\frac{n\Delta(\delta+2)}{4\Delta+(\Delta+1)(\delta-2)}$  for  $NPPTS(G)$  under strict majority threshold where  $\Delta$  and  $\delta$  are maximum and minimum degree of vertices respectively.*

The approximation factor of the algorithm follows from previous lemma and the lower bound provided by Theorem 1:

**Corollary 1.** *The Greedy NPPTS algorithm is a  $\frac{\Delta(\Delta+1)(\delta+2)}{8\Delta+2(\Delta+1)(\delta-2)}$  approximation algorithm for NPPTS problem.*

**NP-Hardness.** In this section, we use a reduction from the Minimum Dominating Set problem (MDS) [2] to prove the NP-hardness of computing  $NPPTS(G)$ .

**Theorem 3.** *If there exists a polynomial-time algorithm for computing  $NPPTS(G)$  for a given graph  $G$  under the strict majority threshold, then  $P = NP$ .*

### 3 Non-Progressive Spread of Influence in Power-law graphs

In this section, we investigate the non-progressive spread of influence in power-law graphs, and show that the greedy algorithm presented in the previous section

is indeed a constant-factor approximation algorithm for power-law graphs. For each natural number  $x$ , we assume that the number of vertices with degree  $x$  is proportional to  $x^{-\gamma}$  and use  $\alpha$  as the normalization coefficient. The value of  $\gamma$ , known as power-law coefficient, is known to be between 2 and 3 in real-world social networks. We denote the number of vertices of degree  $x$  by  $P(x) = \alpha x^{-\gamma}$ . Let  $n$  be the number of vertices of graph, so we have:

$$n = \sum_{x=1}^{\infty} \alpha x^{-\gamma} = \alpha \zeta(\gamma) \Rightarrow \alpha = \frac{n}{\zeta(\gamma)},$$

where  $\zeta$  is the Riemann Zeta function [21].

**Lower bound.** Consider a power-law graph  $G$  with a threshold function  $t$  and a perfect target set  $f_0$ . Denoting the set of initially influenced vertices by  $B$  and the rest of the vertices by  $W$ , from the Equation 1, we have:

$$\sum_{v \in W} (2t(v) - d(v)) \leq \sum_{v \in B} (d(v) - t(v)).$$

The maximum cardinality of  $W$  is achieved when the degree of all vertices in  $B$  is greater than or equal to the degree of all vertices in  $W$ . In this case, assume that the minimum degree of vertices in  $B$  is  $k$  and  $0 \leq p \leq 1$  is the proportion of the vertices of degree  $k$  in  $B$ , so under strict majority threshold we have:

$$\begin{aligned} \sum_{x=1}^{k-1} \alpha x^{-\gamma} + (1-p)\alpha k^{-\gamma} &\leq |W| \leq \sum_{v \in W} (2t(v) - d(v)) \\ &\leq \sum_{v \in B} (d(v) - t(v)) \leq \sum_{x=k+1}^{\infty} \alpha x^{-\gamma} \left(\frac{x-1}{2}\right) + p\alpha k^{-\gamma} \frac{k-1}{2} \\ \Rightarrow \sum_{x=1}^{k-1} x^{-\gamma} + (1-p)k^{-\gamma} &\leq \frac{\sum_{x=k+1}^{\infty} (x^{1-\gamma} - x^{-\gamma}) + pk^{-\gamma}(k-1)}{2} \\ \Rightarrow \zeta(\gamma) - \zeta(\gamma, k-1) + (1-p)k^{-\gamma} \\ &\leq \frac{\zeta(\gamma-1, k) - \zeta(\gamma, k) + pk^{-\gamma}(k-1)}{2}. \end{aligned}$$

By estimating the value of Riemann Zeta function, we can estimate the upper bound of  $k$  and lower bound of  $p$  for that  $k$  to provide a lower bound for  $|B|$ . Assuming that we have the maximum possible value of  $k$  and minimum value of  $p$  for that  $k$ , then:

$$|B| \geq \sum_{x=k+1}^{\infty} \alpha x^{-\gamma} + \alpha p k^{-\gamma} = \frac{\zeta(\gamma, k) + p k^{-\gamma}}{\zeta(\gamma)} n.$$

The estimated values of lower bound for  $2 \leq \gamma \leq 2.8$  are shown in Figure 1.

**Upper bound.** Suppose that one has run Greedy NPPTS algorithm under strict majority threshold on a graph with power-law degree distribution. The following theorem shows that unlike general graphs, the Greedy NPPTS algorithm guarantees a constant factor upper bound on power-law graphs.

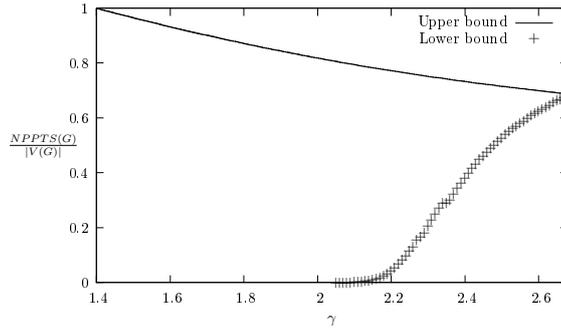
**Theorem 4.** *Algorithm Greedy NPPTS initially influences at most  $(1 + \frac{1}{2^{\gamma+1}} - \frac{1}{2\zeta(\gamma)})n$  vertices under the strict majority threshold on a power-law graphs of order  $n$ .*

*Proof.* We may assume that the input graph is connected. We prove that the number of uninfected vertices of degree 1 are sufficient for this upper bound. Let  $v$  be a vertex of degree more than 1 with  $k$  adjacent vertices of degree 1 say  $u_1, u_2 \dots u_k$ . If  $d(v)$  is odd, it is clear that at least  $\frac{k}{2}$  of the vertices  $u_1, u_2 \dots u_k$  will be uninfected since  $k \leq d(v)$ . Note that according to the greedy algorithm, the value of  $f_0$  for degree 1 vertices are determined before any other vertex. If  $d(v)$  is even, at least  $\frac{k}{2} - 1$  of vertices  $u_1, u_2 \dots u_k$  will be uninfected. Therefore we have:

$$\begin{aligned} NPPTS(G) &\leq n - \frac{1}{2}(P(1) - \sum_{x=1}^{\infty} P(2x)) \\ &\leq n - \frac{1}{2}(\alpha \frac{1}{1^\gamma} - \alpha \sum_{x=1}^{\infty} \frac{1}{(2x)^\gamma}) \\ &= n - \frac{\alpha}{2}(1 - \frac{1}{2^\gamma} \zeta(\gamma)) = n(1 + \frac{1}{2^{\gamma+1}} - \frac{1}{2\zeta(\gamma)}). \end{aligned}$$

□

By previous theorem, we conclude that the Greedy NPPTS algorithm is a constant-factor approximation algorithm on power-law graphs under strict majority threshold. The lower bound and upper bound for different values of  $\gamma$  are shown in Figure 1. As you can see our algorithm acts optimally on social networks with large value of power-law coefficient since upper and lower bound diagram meet each other for these values of the power-law coefficient.



**Fig. 1.** Values of upper bound and lower bound in power-law graphs

## 4 Convergence Issues

Let the state graph  $H$  of a non-progressive spread of influence process for graph  $G$  be as follows: Each node of this graph represents one of possible states of the graph. An edge between two states  $A$  and  $B$  in  $H$  models the fact that applying one step of the influence process on state  $A$  changes the state to state  $B$ . First of all, one can easily see that the non-progressive model may not result to a singleton steady state for any dynamics. To see this, consider the following example: a cycle with  $2k$  vertices  $C = v_1v_2\dots v_{2k}$  and at time 0 infect vertices with odd indices. In this case, the process will oscillate between exactly two states. In fact, one can show a general theorem that any dynamics will converge to either one or two states:

**Theorem 5.** *The non-progressive spread of influence process on a graph reaches a cycle of length of at most two.*

Using this intuition, one can define the convergence time of a non-progressive influence process under the strict majority rule as the time it takes to converge to a cycle of size of two states, i.e., the convergence time is the minimum time  $T$  at which  $f_T(v) = f_{T+2}(v)$  for all vertices  $v \in V(G)$ . For a set  $S$  of initially infected vertices, let  $ct_G(S)$  to be the convergence time of the non-progressive process under the strict majority model( $T$ ). In the following theorem, we formally prove an upper bound of  $O(|E(G)|)$  for this convergence time:

**Theorem 6.** *For a given graph  $G$  and any set  $S \subseteq V(G)$ , we have  $ct_G(S) = O(|E(G)|)$ .*

The above theorem is tight i.e. there exists a set of graphs and initial states with convergence time of  $\Omega(|E(G)|)$ . In power-law graphs since average degree is constant, the number of edges is  $O(|V|)$  and thus the convergence time of these graphs is  $O(|V|)$ .

Finally, we study convergence time of non-progressive dynamics on several real-world graphs, and observe the fast convergence of such dynamics on those graphs.

## 5 Conclusions

In this paper, we study the minimum target set selection problem in the non-progressive influence model under the strict majority rule and provide theoretical and practical results for this model. Our main results include upper bound and lower bounds for these graphs, hardness and an approximation algorithm for this problem. We also apply our techniques on power-law graphs and derive improved constant-factor approximation algorithms for this kind of graphs.

An important follow-up work is to study the minimum perfect set problem for non-progressive models under other influence propagation rules, e.g. the general linear threshold model. It is also interesting to design approximation algorithms

for other special kinds of complex graphs such as small-world graphs. Another interesting research direction is to study maximum active set problem for non-progressive models.

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