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The singular sources method for cracks

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SUMMARY

The singular sources method is given to detect the shape of a thin infinitely cylindrical obstacle from a knowledge of the TM-polarized scattered electromagnetic field in large distance. The basic idea is based on the singular behaviour of the scattered field of the incident point source on the cross-section of the cylinder. We assume that the scatterer is a perfect conductor which is possibly coated by a material and investigate two models with different boundary conditions. Also we give a uniqueness proof for the shape reconstruction. Copyright © 2006 John Wiley & Sons, Ltd.

KEY WORDS: inverse scattering; singular sources method; cylindrical obstacle

1. INTRODUCTION

Inverse scattering problems have been used to determine the scattering obstacle from a knowledge of the scattered wave at large distance. In some problems, the scatterer is a thin infinitely cylindrical conductor. We consider that the incident electric field is polarized in TM mode. This case is modelled by the Helmholtz equation defined in the exterior of an open arc in $\mathbb{R}^2$. We will encounter two model cases with respect to the kind of obstacle. If the obstacle is a perfect conductor then the boundary condition on the open arc is Dirichlet, but if it is coated on one side by some material then it leads to a mixed boundary condition model.

In order to reconstruct the shape of the open arc, Kress used a Newton's method for a perfect conductor by *a priori* information that the scatterer is an open arc [1]. In [2], the authors applied the factorization method to reconstruct the shape without any *a priori* information. Also, the other advantage of this approach over Newton's method is that it is not necessary to solve a forward problem at each step of an iterative process. These two methods are not applicable to cracks with mixed boundary conditions. Cakoni and Colton [3] used the linear sampling method for both the cases, Dirichlet and mixed boundary condition.

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In this paper we develop the singular sources method to reconstruct the shape of the open arc. This method is proposed by Potthast [4], and is used in several models with inverse scattering problem [5, 6]. The singular sources method is able to reconstruct obstacles without any *a priori* information about the boundary condition. The basic idea of the method is based on the singular behaviour of the scattered field of the incident point source on the boundary of the obstacle. This means that if $\Phi^{\mathrm{s}}(.\,,z)$ denote this scattered field and z tends to the boundary then

$$|\Phi^{\mathrm{s}}(z,z)| \longrightarrow \infty$$

This behaviour shows that the boundary of the obstacle is the set of points where the scattered field $\Phi^{\mathrm{s}}(z,z)$ becomes singular. We will show that this approach is applicable for a thin cylindrical obstacle. In order to apply the singular sources method we need to estimate $\Phi^{\mathrm{s}}(z,z)$ from the knowledge of the far-field pattern, $u^{\infty}(\hat{x},d)$, which is derived by the mixed reciprocity relation and backprojection operator.

In Section 2, the direct scattering problem is considered and a theorem related to the mixed reciprocity relation is proved. We show the singular behaviour of the scattered field $\Phi^{\mathrm{s}}(z,z)$ in Section 3. We will prove the uniqueness of the shape reconstruction in this section, too. Finally in Section 3, we will apply the singular sources method to reconstruct the shape of the crack.

2. SCATTERING PROBLEMS

Let $\Gamma \subset \mathbb{R}^2$ be an oriented smooth curve. We denote the right-hand side of Γ with respect to the chosen orientation by Γ^+ and the left-hand side by Γ^-. There is an unit normal vector, v, pointing to the side of Γ^+, and it is defined everywhere except in a finite number of points on Γ.

The scattering of time-harmonic electromagnetic plane waves from a thin infinitely long cylindrical perfect conductor with the electromagnetic field E-polarized leads to the following problem:

$$\begin{aligned} \Delta u + k^2 u = 0 \quad & \text{in } \mathbb{R}^2\backslash\Gamma \\ u_{\pm} = 0 \quad & \text{on } \Gamma^{\pm} \end{aligned} \tag{1}$$

where $u_{\pm}(x) = \lim_{h\to 0^+} u(x \pm hv)$ for $x \in \Gamma$, and $k>0$ is the wave number. The solution, $u \in H^1_{\mathrm{loc}}(\mathbb{R}^2\backslash\overline{\Gamma})$, is the total field and decomposed $u = u^{\mathrm{i}} + u^{\mathrm{s}}$, into the given incident field u^{i}, and the scattered field u^{s}, which is required to satisfy Sommerfield radiation condition

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^{\mathrm{s}}}{\partial r} - \mathrm{i}ku^{\mathrm{s}}\right) = 0 \tag{2}$$

uniformly in all directions $\hat{x} := x/|x|$, with $r = |x|$.

If one side of the thin cylindrical obstacle is coated by a material with surface impedance $\lambda>0$, we obtain the following mixed crack problem for the total field $u = u^{\mathrm{i}} + u^{\mathrm{s}}$:

$$\begin{aligned} \Delta u + k^2 u = 0 \quad & \text{in } \mathbb{R}^2\backslash\Gamma \\ u_- = 0 \quad & \text{on } \Gamma^- \\ \frac{\partial u_+}{\partial v} + \mathrm{i}k\lambda u_+ = 0 \quad & \text{on } \Gamma^+ \end{aligned} \tag{3}$$

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where $(\partial u_+/\partial \nu)(x) = \lim_{h\to 0^+} \nu\cdot\nabla u(x + h\nu)$, and u^s satisfies the Sommerfield radiation condition (2). Here, the surface impedance λ is a smooth real non-negative function and $\lambda \in L^\infty(\Gamma)$.

In [3], it is shown that for every arbitrary incident wave u^i, both problems (1) and (3) with radiation condition (2), have a unique solution. The radiation condition (2) in both the models implies an asymptotic behaviour of the scattered wave

$$u^s(x) = \frac{e^{ikr}}{\sqrt{r}} u^\infty(\hat{x}) + o(r^{-3/2}) \tag{4}$$

uniformly in all direction $\hat{x} = x/|x|$, where $r = |x|$, see [7]. The amplitude factor u^∞ is known as the far-field pattern of the scattered wave, u^s. If we consider a plane incident wave $u^i(x, d) = e^{ikx\cdot d}$, in the direction d, with $|d| = 1$, then its scattered field and far field are denoted by $u^s(x, d)$ and $u^\infty(\hat{x}, d)$, respectively.

The fundamental solution of Helmholtz equation is given by

$$\Phi(x, y) = \frac{i}{4} H_0^{(1)}(k|x - y|)$$

where $H_0^{(1)}$ denotes the Hankel function of the first kind of order zero [7]. Consider the point source $\Phi(.\,, z)$, $z \in \mathbb{R}^2\backslash\Gamma$, its scattered wave is denoted by $\Phi^s(.\,, z)$. This means that $\Phi^s(.\,, z)$ satisfies the Helmholtz equation in $\mathbb{R}^2\backslash\Gamma$ with condition (2). Also, $\Phi^s(.\,, z) + \Phi(.\,, z)$ satisfies the boundary condition in the problem (1) or (3). We also denote the far-field pattern of $\Phi^s(.\,, z)$ by $\Phi^\infty(\hat{x}, z)$.

In order to investigate inverse problem in the next section, we need to recall the definition of the following Sobolev spaces. First, we extend the arc Γ, to an arbitrary piecewise smooth, closed curve ∂D enclosed in a bounded domain D, such that the normal vector ν on Γ coincides with the outward normal vector on ∂D. If $H^1(D)$ denotes the usual Sobolev space and $H^{1/2}(\partial D)$ its usual trace space, then we define

$$\begin{aligned} H^{1/2}(\Gamma) &:= \{u|_\Gamma : u \in H^{1/2}(\partial D)\} \\ \widetilde{H}^{1/2}(\Gamma) &:= \{u \in H^{1/2}(\partial D) : \operatorname{supp} u \subseteq \overline{\Gamma}\} \\ H^{-1/2}(\Gamma) &:= (\widetilde{H}^{1/2}(\Gamma))^* \quad \text{the dual space of } \widetilde{H}^{1/2}(\Gamma) \\ \widetilde{H}^{-1/2}(\Gamma) &:= (H^{1/2}(\Gamma))^* \quad \text{the dual space of } H^{1/2}(\Gamma) \end{aligned}$$

Also, we define $[u] := u_+ - u_-$ and $[\partial u/\partial \nu] := \partial u_+/\partial \nu - \partial u_-/\partial \nu$, the jump of u and $\partial u/\partial \nu$ across Γ, respectively. It is clear that if u is the solution of (1) or (3) then $[u] = [\partial u/\partial \nu] = 0$ on $\partial D\backslash\Gamma$.

In order to determine Γ and λ, we will use the singular sources method. This method is based on the mixed reciprocity relation, relating the scattered field u^s on the exterior of the scatterer to the far-field distribution of the fundamental solution $\Phi(.\,, z)$ due to a point source $z \in \mathbb{R}^2\backslash\Gamma$. In [6], this relation has been proved for hard and soft obstacles. We will prove the mixed reciprocity relation for cracks in the following theorem.

Theorem 1 (*Mixed reciprocity relation*)
For the scattering of the plane waves $u^i(.\,, d)$, $d \in \Omega$ and the point sources $\Phi(.\,, z)$, $z \in \mathbb{R}^2\backslash\Gamma$, from a crack scatterer Γ in both model (1) or (3), we have

$$\Phi^\infty(-d, z) = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} u^s(z, d), \quad z \in \mathbb{R}^2\backslash\Gamma, \quad d \in \Omega$$

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Proof

First of all note that by the Green representation formula, the scattered wave $u^s(z,d)$ is equal to

$$u^s(z,d) = \begin{cases} \int_{\partial D} u^s_+(y,d)\dfrac{\partial\Phi(z,y)}{\partial\nu(y)} - \dfrac{\partial u^s_+(y,d)}{\partial\nu(y)}\Phi(z,y)\,\mathrm{d}s(y), & z\in\mathbb{R}^2\backslash\overline{D} \\ \int_{\partial D} -u^s_-(y,d)\dfrac{\partial\Phi(z,y)}{\partial\nu(y)} + \dfrac{\partial u^s_-(y,d)}{\partial\nu(y)}\Phi(z,y)\,\mathrm{d}s(y), & z\in D \end{cases} \tag{5}$$

where ∂D is an extension of Γ to a closed curve. Since $u^s(.,d)$ and $\Phi(z,.)$ are solutions of Helmholtz equation in D, when $z\in\mathbb{R}^2\backslash\overline{D}$, so

$$\int_{\partial D} u^s_-(y,d)\frac{\partial\Phi(z,y)}{\partial\nu(y)} - \frac{\partial u^s_-(y,d)}{\partial\nu(y)}\Phi(z,y)\,\mathrm{d}s(y) = 0 \tag{6}$$

Also, $u^s(.,d)$ and $\Phi(z,.)$ are radiation solutions of Helmholtz equation in $\mathbb{R}^2\backslash\overline{D}$, when $z\in D$, then

$$\int_{\partial D} u^s_+(y,d)\frac{\partial\Phi(z,y)}{\partial\nu(y)} - \frac{\partial u^s_+(y,d)}{\partial\nu(y)}\Phi(z,y)\,\mathrm{d}s(y) = 0 \tag{7}$$

By considering relations (5)–(7), we conclude that in both the cases $z\in D$ or $z\in\mathbb{R}^2\backslash\overline{D}$ the following relation is true:

$$u^s(z,d) = \int_{\partial D} [u^s(y,d)]\frac{\partial\Phi(z,y)}{\partial\nu(y)} - \left[\frac{\partial u^s(y,d)}{\partial\nu(y)}\right]\Phi(z,y)\,\mathrm{d}s(y)$$

Note that if $u^s(.,d)$ is smooth away from Γ, then $[u^s]=[\partial u^s/\partial\nu]=0$ on $\partial D\backslash\Gamma$, also we have $[u^s]=[u]$ and $[\partial u^s/\partial\nu]=[\partial u/\partial\nu]$, where $u(.,d)$ denotes the total field with respect to the incident plane wave of direction d, therefore

$$u^s(z,d) = \int_{\Gamma} [u(y,d)]\frac{\partial\Phi(z,y)}{\partial\nu(y)} - \left[\frac{\partial u(y,d)}{\partial\nu(y)}\right]\Phi(z,y)\,\mathrm{d}s(y) \tag{8}$$

On the other hand, if we substitute $\Phi^s(x,z)$ instead of $u^s(z,d)$ in (5) and let $|x|\longrightarrow\infty$, then we imply that the far field $\Phi^\infty(\hat{x},z)$ for every $z\in\mathbb{R}^2\backslash\Gamma$ and $\hat{x}\in\Omega$ is equal to

$$\Phi^\infty(\hat{x},z) = \frac{\mathrm{e}^{\mathrm{i}\pi/4}}{\sqrt{8\pi k}}\int_{\partial D} \Phi^s_+(y,z)\frac{\partial \mathrm{e}^{-\mathrm{i}k\hat{x}\cdot y}}{\partial\nu(y)} - \frac{\partial\Phi^s_+(y,z)}{\partial\nu(y)}\mathrm{e}^{-\mathrm{i}k\hat{x}\cdot y}\,\mathrm{d}s(y) \tag{9}$$

Also, we know that $u^s(.,d)$ and $\Phi^s(.,z)$ are radiation solutions of Helmholtz equation in $\mathbb{R}^2\backslash\overline{D}$, then we have

$$\int_{\partial D} \Phi^s_+(y,z)\frac{\partial u^s_+(y,d)}{\partial\nu(y)} - \frac{\partial\Phi^s_+(y,z)}{\partial\nu(y)}u^s_+(y,d)\,\mathrm{d}s(y) = 0 \tag{10}$$

It follows from (9) and (10) with $-d$ replaced by $\hat{x}$, that

$$\Phi^\infty(-d,z) = \frac{\mathrm{e}^{\mathrm{i}\pi/4}}{\sqrt{8\pi k}}\int_{\partial D} \Phi^s_+(y,z)\frac{\partial u_+(y,d)}{\partial\nu(y)} - \frac{\partial\Phi^s_+(y,z)}{\partial\nu(y)}u_+(y,d)\,\mathrm{d}s(y) \tag{11}$$

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Moreover $u(.\,,d)$ and $\Phi^s(.\,,z)$ are solutions of Helmholtz equation in D, then from Green's theorem we conclude that

$$\int_{\partial D} \Phi^s_-(y,z)\frac{\partial u_-(y,d)}{\partial \nu(y)} - \frac{\partial \Phi^s_-(y,z)}{\partial \nu(y)} u_-(y,d)\,\mathrm{d}s(y) = 0 \tag{12}$$

Thus, from (11) and (12), we obtain

$$\Phi^\infty(-d,z) = \frac{\mathrm{e}^{\mathrm{i}\pi/4}}{\sqrt{8\pi k}} \int_{\partial D} \left\{ \Phi^s_+(y,z)\frac{\partial u_+(y,d)}{\partial \nu(y)} - \frac{\partial \Phi^s_+(y,z)}{\partial \nu(y)} u_+(y,d) \right.$$
$$\left. -\Phi^s_-(y,z)\frac{\partial u_-(y,d)}{\partial \nu(y)} + \frac{\partial \Phi^s_-(y,z)}{\partial \nu(y)} u_-(y,d) \right\} \mathrm{d}s(y)$$

Now note that the jump of $u(.\,,d)$, $\Phi^s(.\,,z)$ and their derivatives are zero across $\partial D \backslash \Gamma$, then

$$\Phi^\infty(\hat{x},z) = \frac{\mathrm{e}^{\mathrm{i}\pi/4}}{\sqrt{8\pi k}} \int_{\Gamma} \left\{ \Phi^s_+(y,z)\frac{\partial u_+(y,d)}{\partial \nu(y)} - \frac{\partial \Phi^s_+(y,z)}{\partial \nu(y)} u_+(y,d) \right.$$
$$\left. -\Phi^s_-(y,z)\frac{\partial u_-(y,d)}{\partial \nu(y)} + \frac{\partial \Phi^s_-(y,z)}{\partial \nu(y)} u_-(y,d) \right\} \mathrm{d}s(y)$$

Now consider the boundary conditions in Dirichlet case (1), we have

$$u_+(.\,,d) = u_-(.\,,d) = 0 \qquad \text{on } \Gamma$$
$$\Phi^s_+(.\,,z) = \Phi^s_-(.\,,z) = -\Phi(.\,,z) \quad \text{on } \Gamma$$

Therefore,

$$\Phi^\infty(-d,z) = \frac{\mathrm{e}^{\mathrm{i}\pi/4}}{\sqrt{8\pi k}} \int_{\Gamma} -\Phi(y,z)\left[\frac{\partial u(y,d)}{\partial \nu(y)}\right] \mathrm{d}s(y)$$

According to relation (8) and relation $[u]=0$, on Γ in the Dirichlet case, the theorem will be proved in this case. Similarly in the mixed boundary condition (3), we have

$$u_- = 0, \quad \frac{\partial u_+}{\partial \nu} + \mathrm{i}k\lambda u_+ = 0 \qquad \text{on } \Gamma$$
$$\Phi^s_- = -\Phi, \quad \frac{\partial \Phi^s_+}{\partial \nu} + \mathrm{i}k\lambda \Phi^s_+ = -\frac{\partial \Phi}{\partial \nu} - \mathrm{i}k\lambda\Phi \quad \text{on } \Gamma$$

and these relations imply that

$$\Phi^\infty(-d,z) = \frac{\mathrm{e}^{\mathrm{i}\pi/4}}{\sqrt{8\pi k}} \int_{\Gamma} [u(y,d)]\frac{\partial \Phi(y,z)}{\partial \nu(y)} - \left[\frac{\partial u(y,d)}{\partial \nu(y)}\right]\Phi(y,z)\,\mathrm{d}s(y)$$

which is equal to relation (8). □

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3. INVERSE SCATTERING

The inverse scattering problem in this model is to determine the shape of the arc from the far-field pattern, $u^{\infty}(\hat{x}, d)$ for all observation directions, $\hat{x} \in \Omega$ and all incident directions, $d \in \Omega$. In [3], the linear sampling method is used to recover the arc, Γ. This method is applicable without any *a priori* knowledge of the kind of scattering model, i.e. we do not need to know if the scatterer is an open arc or an non-empty interior obstacle sound-soft or hard-soft.

In this section we develop the singular sources method to reconstruct the shape of the open arc. This method is used in [6] to reconstruct the shape of an non-empty interior scatterer without knowing the boundary condition or physical properties of the scatterer, in the cases of soft obstacle, hard obstacle and inhomogeneous medium scattering. In this method we use the field $\Phi^{s}(z, z)$ to reconstruct the shape of the scattering object. The arc, Γ, is found to be the set of points where $\Phi^{s}(z, z)$ becomes singular. First, we prove the following lemma which is necessary for investigation of singular behaviour of $\Phi^{s}(z, z)$ on Γ.

Lemma 2
If $D \subset \mathbb{R}^2$, is an open set then there exist constants $\tau, c>0$, such that

$$\|\Phi(.\,, z)\|^2_{H^{1/2}(\partial D)} \leqslant c|\ln d(z, D)|$$

$$\left\|\frac{\partial \Phi(.\,, z)}{\partial v}\right\|^2_{H^{-1/2}(\partial D)} \leqslant c|\ln d(z, D)|$$

for every $z \notin D$, which satisfies $0<d(z, D)<\tau$.

Proof
For every $z \in \mathbb{R}^2 \setminus \overline{D}$, $\Phi(.\,, z)$ satisfies the Helmholtz equation in D, then from Theorem 3.37 and Lemma 4.3 in [8], we conclude

$$\|\Phi(.\,, z)\|_{H^{1/2}(\partial D)}, \quad \left\|\frac{\partial \Phi(.\,, z)}{\partial v}\right\|_{H^{-1/2}(\partial D)} \leqslant C\|\Phi(.\,, z)\|_{H^1(D)}$$

On the other hand Lemma 2 in [5] estimates the norm of $\Phi(.\,, z)$ on D for every $z \notin D$, which satisfies $0<d(z, D)<\tau$

$$\|\Phi(.\,, z)\|^2_{H^1(D)} \leqslant c|\ln d(z, D)| \qquad \square$$

Now we investigate the behaviour of $\Phi^{s}(z, z)$ when z tends to the arc, Γ, in the following theorem.

Theorem 3
Let $\Phi^{s}(.\,, z)$ be the scattering field of the point-source $\Phi(.\,, z)$ by an open arc, Γ. In both the models, Dirichlet and mixed boundary condition, we have

$$\lim_{z \to z^*} |\Phi^{s}(z, z)| = \infty$$

for every $z^* \in \Gamma$.

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Proof

We consider the theorem in two cases, Dirichlet and mixed boundary condition.

Case 1—Dirichlet model. Suppose that z approaches to $z^* \in \Gamma$. Let D be a bounded domain such that Γ is a part of ∂D and $z \notin D$. Now consider $u(.\,,z)$ to be the radiating solution of Helmholtz equation

$$\Delta u + k^2 u = 0 \qquad \text{in } \mathbb{R}^2 \backslash \overline{D}$$

$$u = -\Phi(.\,,z) \quad \text{on } \partial D$$

In fact $u(.\,,z)$ is the scattered wave of point source $\Phi(.\,,z)$ with respect to the obstacle D. Then by Theorem 2.1.15 in [6], we conclude that for every z near D

$$|u(z,z)| \geqslant c|\ln d(z,D)| \tag{13}$$

Now let $w(.\,,z) = \Phi^s(.\,,z) - u(.\,,z)$, then $w(.\,,z)$ satisfies the Helmholtz equation in $\mathbb{R}^2 \backslash \overline{D}$, with the boundary condition

$$w(.\,,z) = 0 \quad \text{on } \Gamma$$

We show that for z near D, the rate of growth of $w(z,z)$ is less than the rate of growth of $|\ln d(z,D)|$. That is

$$|w(z,z)| \leqslant c\sqrt{|\ln d(z,D)|}$$

In order to see this estimate, let G_1 and G_2 be two neighbourhoods of z^*, with $\overline{G_1} \Subset G_2$ and $G_2 \cap \partial D \subseteq \Gamma$. Then note that Γ is smooth and $w(.\,,z) = 0 \in H^{3/2}(G_2 \cap \Gamma)$, thus theorem of regularity of the solution up to the boundary in [8] implies that $w(.\,,z) \in H^2(\Omega_1)$, and

$$\|w(.\,,z)\|_{H^2(\Omega_1)} \leqslant C(\|w(.\,,z)\|_{H^1(\Omega_2)} + \|w(.\,,z)\|_{H^{3/2}(G_2\cap\Gamma)}) \leqslant C\|w(.\,,z)\|_{H^1(\Omega_2)}$$

where $\Omega_i = G_i \backslash \overline{D}$. Hence, by the definition of $w(.\,,z)$, we have

$$\|w(.\,,z)\|_{H^2(\Omega_1)} \leqslant C(\|\Phi^s(.\,,z)\|_{H^1(\Omega_2)} + \|u(.\,,z)\|_{H^1(\Omega_2)})$$

Now note that the estimation of the radiation solution of Helmholtz equation from the boundary values for a crack scattering, Theorem 2.4 in [3] implies that

$$\|\Phi^s(.\,,z)\|_{H^1(\Omega_2)} \leqslant C\|\Phi(.\,,z)\|_{H^{1/2}(\Gamma)}$$

Also, if $u(.\,,z)$ is a radiating solution of Helmholtz equation in $\mathbb{R}^2 \backslash D$, then

$$\|u(.\,,z)\|_{H^1(\Omega_2)} \leqslant C\|\Phi(.\,,z)\|_{H^{1/2}(\partial D)}$$

Therefore, according to Lemma 2, we imply that

$$\|w(.\,,z)\|_{H^2(\Omega_1)} \leqslant C\sqrt{|\ln d(z,D)|}$$

for every z near D. On the other hand, the imbedding theorem implies that $w(.\,,z)$ is a Hölder continuous function and

$$|w(x,z)| \leqslant C\|w(.\,,z)\|_{H^2(\Omega_1)}$$

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for every $x \in \Omega_1$. Therefore,

$$|w(z,z)| \leqslant c\sqrt{|\ln d(z,D)|} \tag{14}$$

for every $z \in \Omega_1$. By considering (13) and (14) the proof of theorem is complete in this case.

Case 2—Mixed boundary condition. Let z approach to $z^* \in \Gamma$ and extend Γ to ∂D similar to the previous case. If z approaches from left-hand side of Γ then we consider Helmholtz equation in $\mathbb{R}^2\backslash\overline{D}$ with Dirichlet boundary condition. The proof of the first case works in this situation. Thus, we assume that z approaches to z^* from the right-hand side of Γ. Now let $u(.,z)$ be the radiating solution of Helmholtz equation

$$\Delta u + k^2 u = 0 \qquad \text{in } \mathbb{R}^2\backslash\overline{D}$$

$$\frac{\partial u}{\partial \nu} + \mathrm{i}k\lambda u = -\frac{\partial \Phi(.,z)}{\partial \nu} - \mathrm{i}k\lambda\Phi(.,z) \quad \text{on } \partial D$$

Similar to the first case, consider $w(.,z) = \Phi^{\mathrm{s}}(.,z) - u(.,z)$, then $w(.,z)$ satisfies the Helmholtz equation in $\mathbb{R}^2\backslash\overline{D}$, with the boundary condition

$$\frac{\partial w}{\partial \nu} + \mathrm{i}k\lambda w = 0 \quad \text{on } \Gamma$$

We claim that u and w satisfy the following estimations:

$$|w(z,z)| \leqslant c\sqrt{|\ln d(z,D)|}$$

$$|u(z,z)| \geqslant c|\ln d(z,D)|$$

In order to see them, let v be the radiating solution of Helmholtz equation with the following boundary condition:

$$\frac{\partial v}{\partial \nu} = \mathrm{i}k\lambda \quad \text{on } \partial D$$

Then $w_1 = \mathrm{e}^v w$ satisfies

$$\Delta w_1 + k^2 w_1 = f \quad \text{in } \mathbb{R}^2\backslash\overline{D}$$

$$\frac{\partial w_1}{\partial \nu} = 0 \quad \text{on } \Gamma$$

where $f = \mathrm{e}^v(w\nabla v\cdot\nabla v + 2\nabla v\cdot\nabla w - k^2 v w)$. Now similar to the first case, by theorem of regularity of the solution upto the boundary in the neighbourhood of z^*, we conclude that $w_1 \in H^2(\Omega_1)$, and we have

$$\|w_1\|_{H^2(\Omega_1)} \leqslant C\left(\|w_1\|_{H^1(\Omega_2)} + \left\|\frac{\partial w_1}{\partial \nu}\right\|_{H^{1/2}(G_2\cap\Gamma)} + \|f\|_{L^2(\Omega_2)}\right)$$

Now since v is bounded on Ω_2, then

$$\|f\|_{L^2(\Omega_2)} \leqslant C\|w\|_{H^1(\Omega_2)}$$

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and

$$\|w_1\|_{H^1(\Omega_2)} \leqslant C\|w\|_{H^1(\Omega_2)}$$

Thus,

$$\|w_1\|_{H^2(\Omega_1)} \leqslant C\|w\|_{H^1(\Omega_2)}$$

Also, we have $w = \mathrm{e}^{-v} w_1$, then

$$\|w\|_{H^2(\Omega_1)} \leqslant C\|w_1\|_{H^2(\Omega_1)} \leqslant C\|w\|_{H^1(\Omega_2)}$$

and so

$$\|w\|_{H^2(\Omega_1)} \leqslant C(\|\Phi^{\mathrm{s}}(.\,,z)\|_{H^1(\Omega_2)} + \|u(.\,,z)\|_{H^1(\Omega_2)})$$

According to Theorem 2.5 in [3], we have

$$\begin{aligned}\|\Phi^{\mathrm{s}}(.\,,z)\|_{H^1(\Omega_2)} &\leqslant C\left(\|\Phi(.\,,z)\|_{H^{1/2}(\Gamma)} + \left\|\frac{\partial\Phi(.\,,z)}{\partial v} + \mathrm{i}k\lambda\Phi(.\,,z)\right\|_{H^{-1/2}(\Gamma)}\right)\\ &\leqslant C\left(\|\Phi(.\,,z)\|_{H^{1/2}(\Gamma)} + \left\|\frac{\partial\Phi(.\,,z)}{\partial v}\right\|_{H^{-1/2}(\Gamma)}\right)\end{aligned}$$

Also $u(.\,,z)$ satisfies the following estimate:

$$\begin{aligned}\|u(.\,,z)\|_{H^1(\Omega_2)} &\leqslant C\left\|\frac{\partial\Phi(.\,,z)}{\partial v} + \mathrm{i}k\lambda\Phi(.\,,z)\right\|_{H^{-1/2}(\partial D)}\\ &\leqslant C\left(\|\Phi(.\,,z)\|_{H^{1/2}(\partial D)} + \left\|\frac{\partial\Phi(.\,,z)}{\partial v}\right\|_{H^{-1/2}(\partial D)}\right)\end{aligned}$$

Therefore by Lemma 2, we have

$$\|w\|_{H^2(\Omega_1)} \leqslant c\sqrt{|\ln d(z,D)|}$$

and by imbedding theorem we conclude that

$$|w(z,z)| \leqslant c\sqrt{|\ln d(z,D)|}$$

for every z near Γ. It remains to show that

$$|u(z,z)| \geqslant c|\ln d(z,D)|$$

for every z near to Γ. Similar to the above, we change u to $\tilde{u} = \mathrm{e}^{v} u$. Now we can write $\tilde{u} = u_1 + u_2$, where u_1 is the radiating solution of

$$\begin{aligned}\Delta u_1 + k^2 u_1 &= 0 &&\text{in } \mathbb{R}^2\backslash\overline{D}\\ \frac{\partial u_1}{\partial v} &= -\frac{\partial(\mathrm{e}^{v}\Phi(.\,,z))}{\partial v} &&\text{on } \partial D\end{aligned}$$

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and u_2 satisfies in the following equation:

$$\Delta u_2 + k^2 u_2 = f \quad \text{in } \mathbb{R}^2 \backslash \overline{D}$$

$$\frac{\partial u_2}{\partial v} = 0 \quad \text{on } \partial D$$

Similar to the above, we can get

$$\|u_2\|_{H^2(\Omega_1)} \leqslant C\|f\|_{L^2(\Omega_2)} \leqslant C\|u\|_{H^1(\Omega_2)} \leqslant c\sqrt{|\ln d(z, D)|}$$

Thus,

$$|u_2(z, z)| \leqslant \frac{c}{d(z, D)^{1/2}}$$

By considering the boundedness of e^v in a neighbourhood of D, the proof will be complete if we show that

$$|u_1(z, z)| \geqslant c|\ln d(z, D)|$$

This estimate follows from Lemma 4 in [5]. □

Before we begin to consider the shape reconstruction problem, we investigate the uniqueness of the reconstruction. In the following theorem we show the uniqueness of the reconstruction of the arc, Γ, which has used the singular behaviour of $\Phi^{\mathrm{s}}(z, z)$.

Theorem 4
Assume that Γ_1 and Γ_2 are two open arc obstacles such that their far-field patterns $u_1^{\infty}(\hat{x}, d)$ and $u_2^{\infty}(\hat{x}, d)$ coincide for all $\hat{x}, d \in \Omega$. Then $\Gamma_1 = \Gamma_2$ and their boundary conditions are similar. Also, if Γ_1 and Γ_2 satisfy in the mixed boundary condition model then $\lambda_1 = \lambda_2$.

Proof
Let G be the unbounded component of $\mathbb{R}^2 \backslash (\Gamma_1 \cup \Gamma_2)$. From

$$u_1^{\infty}(\hat{x}, d) = u_2^{\infty}(\hat{x}, d) \quad \text{for all } \hat{x},\ d \in \Omega$$

and Rellich Lemma in [7], we obtain

$$u_1^{\mathrm{s}}(x, d) = u_2^{\mathrm{s}}(x, d) \quad \text{for all } x \in G,\ d \in \Omega \tag{15}$$

Thus, by Theorem 1 we have

$$\Phi_1^{\infty}(\hat{x}, z) = \Phi_2^{\infty}(\hat{x}, z) \quad \text{for all } \hat{x} \in \Omega,\ \ z \in G$$

Again, we can use Rellich Lemma to achieve the following relation:

$$\Phi_1^{\mathrm{s}}(x, z) = \Phi_2^{\mathrm{s}}(x, z) \quad \text{for all } x, z \in G \tag{16}$$

If $\Gamma_1 \neq \Gamma_2$, then without loss of generality we can assume that there exists $z_0 \in \Gamma_1 \backslash \Gamma_2$. Thus, Theorem 3 implies that

$$\infty > \Phi_2^{\mathrm{s}}(z_0, z_0) = \lim_{z \to z_0, z \in G} \Phi_2^{\mathrm{s}}(z, z) = \lim_{z \to z_0, z \in G} \Phi_1^{\mathrm{s}}(z, z) = \infty$$

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This contradiction shows that $\Gamma_1 = \Gamma_2$. From relation (15), we conclude that traces of $u_1^s(x, d)$ and $u_2^s(x, d)$ are equal on $\Gamma_1 = \Gamma_2$. Thus, the boundary conditions of the obstacles Γ_1 and Γ_2 are similar. Also, by equality of the scattered waves $u_1^s(x, d)$ and $u_2^s(x, d)$, we conclude $\lambda_1 = \lambda_2$ in the mixed boundary condition model. □

Now we apply the singular sources method to reconstruct the shape of the obstacle. According to Theorem 3 the arc, Γ, is the set of points where $\Phi^s(z, z)$ is large. In order to determine this set we should calculate $\Phi^s(z, z)$ from the far-field pattern $u^\infty(\hat{x}, d)$.

Suppose we know by *a priori* information that $\Gamma \subset B$, where B is a bounded domain. For every $z \in B$, let $G = G(z)$ be a smooth region which does not have Dirichlet eigenvalue $-k^2$, $z \notin G$, $\Gamma \subseteq G \subseteq B$ and the exterior of G is connected, where k is the wave number. For every ε, there is $g \in L^2(\Omega)$ such that

$$\|\Phi(.\,, z) - v_g\|_{L^2(\partial G)} < \varepsilon$$

where $v_g(x) := \int_\Omega g(d) e^{ikx\cdot d}\, ds(d)$ is a Herglotz wave (see Lemma 3.1.2 in [6]). Now note that the functions v_g and $\Phi(.\,, z)$ are the solutions of Helmholtz equation in G, hence

$$\|\Phi(.\,, z) - v_g\|_{H^1(G)} \leqslant c_1 \varepsilon$$

Therefore, for every z and τ, there exists $g_\tau(z, .) \in L^2(\Omega)$ such that

$$\|\Phi(.\,, z) - v_{g_\tau}\|_{H^1(G)} \leqslant \tau$$

If we consider the trace of $\Phi(.\,, z)$ and v_{g_τ} on Γ, from Theorem 3.37 in [8], we see that

$$\|\Phi(.\,, z) - v_{g_\tau}\|_{H^{1/2}(\Gamma)} \leqslant c_2 \|\Phi(.\,, z) - v_{g_\tau}\|_{H^1(G)} \leqslant c_2 \tau$$

where c_2 is a constant which depends only on Γ. Also from Lemma 4.3 in [8], we have

$$\left\| \frac{\partial}{\partial \nu}\Phi(.\,, z) - \frac{\partial}{\partial \nu} v_{g_\tau} \right\|_{H^{-1/2}(\Gamma)} \leqslant c_3 \tau$$

Therefore, for every $\tau > 0$ and $z \in B \backslash \Gamma$ in both models with Dirichlet and mixed boundary condition, there is $g_\tau(z, .) \in L^2(\Omega)$ such that

$$\|\Phi^s(.\,, z) - v_{g_\tau}^s\|_{H^1(B\backslash\Gamma)} \leqslant C\tau \tag{17}$$

and

$$\|\Phi^\infty(.\,, z) - v_{g_\tau}^\infty\|_{L^2(\Omega)} \leqslant C\tau \tag{18}$$

where C depends only on Γ and B, while $v_{g_\tau}^s$ and $v_{g_\tau}^\infty$ are the scattered field and the far field with respect to the Herglotz wave v_{g_τ}.

Let $\Gamma_\rho = \{z \in \mathbb{R}^m \mid d(z, \Gamma) \leqslant \rho\}$. Then according to the regularity of the solution of elliptic equation and relation (17) we conclude that for every $x \in B \backslash \Gamma_\rho$

$$\|\Phi^s(.\,, z) - v_{g_\tau}^s\|_{H^2(B_\rho(x))} \leqslant C_1 \|\Phi^s(.\,, z) - v_{g_\tau}^s\|_{H^1(B_\rho(x))} \leqslant C_1 C\tau$$

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where B_ρ is a ball with centre in x and radius ρ, moreover C_1 depends on ρ. Also the imbedding theorem and the above result imply that $\Phi^{\mathrm{s}}(.\,,z)-v^{\mathrm{s}}_{g_\tau}$ is a Hölder continuous function on B_ρ and we have

$$|\Phi^{\mathrm{s}}(x,z)-v^{\mathrm{s}}_{g_\tau}(x)|\leqslant C_2\|\Phi^{\mathrm{s}}(.\,,z)-v^{\mathrm{s}}_{g_\tau}\|_{H^2(B_\rho(x))}\leqslant C_2C_1C\tau=C_\rho\tau \tag{19}$$

where C_ρ depends only on ρ, B and Γ.

On the other hand, we know that

$$v^{\mathrm{s}}_{g_\tau}(x)=\int_\Omega g_\tau(z,d)u^{\mathrm{s}}(x,d)\,\mathrm{d}s(d) \tag{20}$$

for every $x\in\mathbb{R}^m\backslash\Gamma$. Moreover,

$$v^{\infty}_{g_\tau}(\hat{x})=\int_\Omega g_\tau(z,d)u^{\infty}(\hat{x},d)\,\mathrm{d}s(d) \tag{21}$$

for every $\hat{x}\in\Omega$. Thus, from (19), (20) and Theorem 1 we have

$$\left|\Phi^{\mathrm{s}}(x,z)-\frac{1}{\gamma_m}\int_\Omega g_\tau(z,d)\Phi^{\infty}(-d,x)\,\mathrm{d}s(d)\right|\leqslant C_\rho\tau$$

Now from (18) and (21) we conclude that there is $g_\eta(x,.)$ such that

$$\left\|\Phi^{\infty}(.\,,x)-\int_\Omega g_\eta(x,\tilde{d})u^{\infty}(.\,,\tilde{d})\,\mathrm{d}s(\tilde{d})\right\|_{L^2(\Omega)}\leqslant C\eta$$

Thus,

$$\left|\int_\Omega g_\tau(z,d)\left\{\Phi^{\infty}(-d,x)-\int_\Omega g_\eta(x,\tilde{d})u^{\infty}(-d,\tilde{d})\,\mathrm{d}s(\tilde{d})\right\}\mathrm{d}s(d)\right|$$

$$\leqslant\|g_\tau(z,.)\|_{L^2(\Omega)}.\|\Phi^{\infty}(.\,,x)-v^{\infty}_{g_\eta}\|_{L^2(\Omega)}\leqslant C\eta\|g_\tau(z,.)\|_{L^2(\Omega)}$$

Therefore, for every $x,z\in B\backslash\Gamma_\rho$, we have

$$\left|\Phi^{\mathrm{s}}(x,z)-\frac{1}{\gamma_m}\int_\Omega\int_\Omega g_\eta(x,\tilde{d})g_\tau(z,d)u^{\infty}(-d,\tilde{d})\,\mathrm{d}s(\tilde{d})\,\mathrm{d}s(d)\right|\leqslant C_\rho\tau+\frac{C\eta}{\gamma_m}\|g_\tau(z,.)\|_{L^2(\Omega)}$$

Now we formulate the previous calculations and results in the following theorem. In order to do this we define the backprojection operator, Q as

$$(Qw)(x,z):=\frac{1}{\gamma_m}\int_\Omega\int_\Omega g_\eta(x,\tilde{d})g_\tau(z,d)u^{\infty}(-d,\tilde{d})\,\mathrm{d}s(\tilde{d})\,\mathrm{d}s(d)$$

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Theorem 5
Let Γ be a crack obstacle then there are kernels g_τ and g_η for every τ and η such that

$$|\Phi^s(z,z) - (Qu^\infty)(z,z)| \leqslant C_\rho \tau + \frac{C\eta}{\gamma_m} \|g_\tau(z,.)\|_{L^2(\Omega)}$$

for every $z \in \mathbb{R}^2 \backslash \Gamma_\rho$, moreover C and C_ρ are constants.

Remark 1
For an appropriate choice of τ and η the above error can be made arbitrarily small. In fact for given $\tau > 0$, we can choose η such that $\eta \|g_\tau(z,.)\|_{L^2(\Omega)}$ becomes sufficiently small. Now if $\tau \to 0$, and $\eta(\tau) \to 0$, then the error tends to zero.

Remark 2
In order to apply this method, we need to choose the region $G(z)$ with the property $\Gamma \subset G(z)$, but this seems impossible when Γ is unknown. In order to take care of this trouble, we start with a number of fixed directions $p_1, \ldots, p_8$ which divided the plane into eight symmetric regions. For every direction p_i, we choose a special region $G_i(z)$ and compute $a_i^{(1)}(z)$ as an approximation $\Phi^s(z,z)$ using the operator Q, where Q is depending on $G_i(z)$. We can obtain a first approximation Γ_1 as the set

$$\Gamma_1 := \{z \in B : |a_i^{(1)}(z)| > C \text{ for } i = 1, \ldots, 8\}$$

In each further step, we adapt the choice $G(z)$ according to the reconstruction Γ_n of the nth step, $\overline{\Gamma_n} \subset G(z)$, and repeat the procedure to obtain the $(n+1)$th approximation Γ_{n+1}.

Remark 3
We estimate $\Phi^s(z,z)$ from the far field, u^∞ by the operator Q, and if the far field is measured with some noise and there is u_δ^∞ an approximation of u^∞ such that

$$\|u^\infty - u_\delta^\infty\|_{L^2(\Omega\times\Omega)} \leqslant \delta$$

then the error for the approximation of $\Phi^s(z,z)$ by $(Qu_\delta^\infty)(z,z)$ is estimated by

$$\begin{aligned} |\Phi^s(z,z) - (Qu_\delta^\infty)(z,z)| &\leqslant |\Phi^s(z,z) - (Qu^\infty)(z,z)| + |Q(u^\infty - u_\delta^\infty)(z,z)| \\ &\leqslant C_\rho \tau + \frac{C\eta}{\gamma_m} \|g_\tau(z,.)\|_{L^2(\Omega)} + \frac{\delta}{\gamma_m} \|g_\tau(z,.)\|_{L^2(\Omega)} \|g_\eta(z,.)\|_{L^2(\Omega)} \end{aligned}$$

Therefore, the ill-posedness of the reconstruction of Γ is mainly influenced by the norm of the densities g_τ and g_η.

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