(sweeping rate approximately 30 cps), and then applied to the tested network in the form of a signal simultaneously modulated in frequency and swept at a lower frequency rate. The limiter and discriminator circuits remain as described previously. The second harmonic voltage of the demodulated signal is applied to the vertical plates of the cathode-ray oscilloscope. The 10 kc filter should be adjusted to avoid excessive ringing as a consequence of fast changing levels and, at the same time, provide selectivity adequate for suppression of both fundamental and third harmonic voltage.

The attenuation of the filter for 5 kc signals is of the order of 60 db. Oscillographic displays obtained for various adjustments of the intermediate-frequency amplifier under test agree well with curves obtained using step-by-step method. Fig. 7 shows the oscillogram taken for four interstages of \( kQ = 1 \) (slightly misaligned).

In conclusion it can be stated that the method of displaying second harmonic distortion or (in another scale) delay derivative versus frequency, provides more distinctive means of recognizing the conditions of network alignment than the conventional magnitude versus frequency display. The authors feel it may provide a useful means of aligning amplifiers intended for the reproduction of FM or transient signals.

**Excitation Coefficients and Beamwidths of Tschebyscheff Arrays**

ROBERT J. STEGEN†

Summary—In this paper, exact expressions are obtained for the excitation coefficients of a Tschebyscheff array by equating the array space factor to a Fourier series whose coefficients are readily calculated.

A set of curves is presented showing half-power beam width versus antenna length for various side-lobe levels. An approximate though very accurate expression for the half-power beamwidth is derived.

INTRODUCTION

DOLPH† SUGGESTED that a Tschebyscheff polynomial could be made to coincide with the polynomial representing an antenna pattern. He then proved rigorously that the resulting pattern would yield a minimum beamwidth when the side-lobe levels are fixed and a minimum side-lobe level when the beamwidth is specified. He derived the expressions for the currents of the Tschebyscheff arrays as

\[
I_q = \frac{1}{A_{2q}} \left\{ A_{2q} Z_0^{2q-1} - \sum_{k=q+1}^{N} I_k A_{2q-1} Z_0^{2q-1} \right\}
\]

for \( 2N \) elements and

\[
I_q = \frac{1}{A_{2q}} \left\{ A_{2q} Z_0^{2q} - \sum_{k=q+1}^{N} I_k A_{2q-1} Z_0^{2q} \right\}
\]

for \( 2N+1 \) elements where

\[
A_{2m}^{2n} = (-1)^{n-m} \sum_{p=-n-m}^{n} \left( \frac{p}{2p} \right) \left( \frac{2n}{2p} \right),
\]

\[
Z_0 = \frac{1}{2} \{ [r + \sqrt{r^2 - 1}]^{1M} + [r - \sqrt{r^2 - 1}]^{1M} \},
\]

\[
T_M(Z) = \cos (M \arccos Z), \quad |Z| \leq 1,
\]

\[
T_M(Z) = \cosh (M \arccos Z), \quad |Z| \geq 1,
\]

and

\[
T_M(Z) = \frac{1}{2} [(Z + \sqrt{Z^2 - 1})^M + (Z - \sqrt{Z^2 - 1})^M], \quad \text{all } Z.
\]
By clever algebraic manipulation, Barbiere was able to obtain expressions for the currents which were much easier to handle than those of Dolph. The Barbiere current expressions are

\[ I_k = \sum_{q=k}^{N} (-1)^{N-q} Z_{q}^{2q} \frac{(2N - 1)(q + N - 2)!}{(q - k)!(q + k - 1)!(N - q)!} \]  

(9)

for 2N elements and

\[ I_k = \sum_{q=k}^{N} (-1)^{N-q} Z_{q}^{2q} \frac{(2N)(q + N - 1)!}{(q - k)!(q + k)!(N - q)!} \]  

(10)

for 2N+1 elements.

In addition Barbiere also proved that

\[ Z_0 = \cosh \left( \frac{1}{M} \text{arc sinh} r \right) \]  

(11)

which is much easier to evaluate than (4) by either using tables or appropriate series expansions. These expressions of Barbiere are a big improvement over those of Dolph; however, they have one serious drawback. This is the factor \((-1)^{N-q}\), in (9) and (10), which results in the currents being the difference of two large and almost equal numbers. For example, take the calculation of \( I_k \) in a 24 element array designed to have \(-40\) db side-lobe level. \( Z_0 = 1.02665 \) and \( I_k \) then becomes \( I_k = 93,040,583.6338 \). Six significant figures are lost in this calculation. It was necessary to keep \( Z_0 \) to 12 significant figures to obtain this result, i.e., \( Z_0 \) was assumed to be \( Z_0 = 1.0266500000 \). As the number of elements in the array becomes larger, the number of significant figures required increases.

The use of Dolph’s or Barbiere’s expressions for calculating the current distributions of Tschebyscheff arrays requires a tremendous amount of calculations for an array having a large number of elements. It is also possible to determine the excitation coefficients of the Tschebyscheff array by equating the array space factor to a Fourier series whose coefficients are readily calculated. This leads to expressions for the currents which are somewhat more amenable to calculations.

**The Tschebyscheff-Fourier Coefficients**

The space factor of a Tschebyscheff array is given by

\[ T_{2N}(Z) = \sum_{m=0}^{N} I_m \cos \left( 2mu \right) \]  

(12)

where

\[ I_m = \text{the excitation coefficient of the } m\text{th element on each side of the array center-line,} \]

\[ Z = Z_0 \cos u \]  

(14)

\[ u = \frac{\pi d}{\lambda} \]  

(15)

and \( Z_0 \) is defined by (4) or (7).

The excitation coefficients, \( I_m \), are real and symmetrical about the array center because the power pattern is symmetrical and the space factor is real.

Whittaker and Robinson and Sokolnikoff present a method of finding a sum

\[ F(x) = \sum_{m=0}^{N} (a_m \cos mx + b_m \sin mx) \]  

(16)

which furnishes the best possible representation of a function \( u(x) \) when we are given that \( u(x) \) takes the values \( u_0, u_1, u_2, u_3, \cdots u_{n-1} \) when \( x \) takes the values 0, \( 2\pi/n, 4\pi/n, \cdots 2(n-1)\pi/n \), respectively, where \( n \geq 2r + 1 \). The coefficients are evaluated by use of the following equations

\[ a_0 = \frac{1}{n} \sum_{k=0}^{n-1} u_k \]  

(17)

\[ a_m = \frac{2}{n} \sum_{k=0}^{n-1} u_k \cos \left( \frac{2\pi km}{n} \right) \]  

(18)

\[ b_m = \frac{2}{n} \sum_{k=0}^{n-1} u_k \sin \left( \frac{2\pi km}{n} \right) \]  

(19)

The excitation coefficients of the Tschebyscheff array are easily determined by equating the Tschebyscheff polynomial to (16) and solving for the \( I_m \) in terms of the \( a_m \) and \( b_m \) coefficients. The space factor for the Tschebyscheff array of \( 2N+1 \) elements, (12) may be equated to the Fourier series (16) by setting \( 2m=x \) and \( N=r \). Then

\[ I_m = a_m \quad m = 0, 1, 2, \cdots, N, \]  

(20)

\[ b_m = 0, \]  

(21)

\[ n = 2r + 1 = 2N + 1. \]  

(22)

\( Z_0 \) may be calculated from (4) or (11).

The values of \( u(x) \) are obtained from

\[ u(x) = T_{2N}(Z) = T_{2N}(Z_0 \cos u) = T_{2N} \left( \frac{Z_0 \cos x}{2} \right) \]  

(23)

at the \( 2N+1 \) values of \( x \), namely,

\[ x = \frac{2\pi}{2N + 1} \cdot s, \quad s = 0, 1, 2, \cdots, 2N. \]  

(24)

\( T_{2N}(Z) \) may be computed using the appropriate closed forms (6), (7), or (8).

The expressions for the excitation coefficients of the Tschebyscheff linear array then become


The excitation coefficients of a 144-element array having 
-40 db side-lobe level were calculated using (35). The 
values of $I_{71}$ and $I_{71}$ obtained from (35) were exactly 
the same as those calculated from (1) or (9). This was 
used as a check for errors in the calculations of $T_{2N-1}$ 
($Z_0 \cos \pi/2N$).

### The Half-Power Beamwidth

The beamwidth of a Tschebyscheff array may readily 
be calculated using (5) and (7). The maximum amplitude of the main beam is

$$T_M(Z_0) = r = \cosh (M \arccosh Z_0)$$  \hspace{1cm} (36)

or

$$Z_0 = \cosh \left( \frac{1}{M} \arccosh r \right)$$  \hspace{1cm} (37)

At the half-power points

$$T_M(Z_1) = \frac{r}{\sqrt{2}}$$  \hspace{1cm} (38)

or

$$Z_1 = \cosh \left( \frac{1}{M} \arccosh \frac{r}{\sqrt{2}} \right)$$  \hspace{1cm} (39)

where

$$Z_1 = Z_0 \cos u_1$$  \hspace{1cm} (40)

$$u_1 = \frac{\pi d}{\lambda} (\sin \theta_1 - \sin \tilde{\theta})$$  \hspace{1cm} (41)

$\tilde{\theta}$ = the angle of the main beam from broadside

and

$d$ = the element spacing.

#### Half-power beamwidth for broadside beam ($\tilde{\theta} = 0^\circ$)

$$\theta_{HP} = 2\theta_1,$$  \hspace{1cm} (42)

and is shown in Fig. 1 for a spacing

$$d = \frac{\lambda}{2}.$$  \hspace{1cm} (43)

The spacing of the elements has only a minor effect on 
the beamwidth. For other than broadside beams, i.e.,

$$I_{m+1} = \frac{1}{N} \left[ r + 2 \sum_{i=1}^{N-1} T_{2N}(Z_0 \cos \frac{s\pi}{2N+1}) \cos \frac{2s\pi m}{2N+1} \right]$$  \hspace{1cm} (35)

where

$m = 0, 1, 2, \ldots, N - 1.$
\[ \theta_{HP} = \theta_{1^+} - \theta_{1^-}. \] (44)

An approximate expression for the broadside half-power beamwidth may be obtained, which is quite accurate for the smaller beamwidths, by letting

\[ \cos u_1 = 1 - \frac{u_1^2}{2}, \] (45)

cosh \( v = 1 + \frac{v^2}{2} \),

arc cosh \( r = \log 2r - \frac{1}{4r^2}, \) (47)

and

\( l = Md = \) the length of the array.

By direct substitution the final expression becomes

\[ \sin \left( \frac{\theta_{HP}}{2} \right) = \frac{1}{\pi l/\lambda} \sqrt{\frac{3}{4} \log_2 2 + 2 \log_2 2 \log_e r + \frac{\log_e r}{2r^2}}. \] (49)

For

\[ \theta_{HP} < 12^\circ \]

\[ \theta_{HP} = \frac{0.636}{l/\lambda} \sqrt{0.360 + 0.693 \log_2 r + \frac{\log_e r}{2r^2}}, \] (50)

where \( A \) depends on the side-lobe level \( r \). Some values are listed below.

<table>
<thead>
<tr>
<th>( r ) (db)</th>
<th>( A ) (degrees)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-20</td>
<td>51.1</td>
</tr>
<tr>
<td>-25</td>
<td>56.0</td>
</tr>
<tr>
<td>-30</td>
<td>60.6</td>
</tr>
<tr>
<td>-35</td>
<td>65.0</td>
</tr>
<tr>
<td>-40</td>
<td>68.7</td>
</tr>
</tbody>
</table>

These may be compared with the value \( \theta_{HP} = 50.9/l/\lambda \) for a uniform array.

Contributors to Proceedings of the I.R.E.

For a photograph and biography of Marvin Chodorow, see page 163 of the January, 1953 issue of the PROCEEDINGS OF THE I.R.E.

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