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On the Scalar Potential of a Point Charge Associated with a Time-Harmonic Dipole in a Layered Medium

KRZYSZTOF A. MICHALSKI, MEMBER, IEEE

Abstract—**It is demonstrated that one can choose the form of the magnetic vector potential to render the scalar potential of a single point charge associated with a horizontal, time-harmonic dipole in a layered medium identical to that associated with a vertical dipole, provided that the source and observation points are within the same layer. This proves the existence of the so-called mixed-potential electric field integral equation for objects of arbitrary shape in layered media.**

I. INTRODUCTION

In solving radiation and scattering problems of electromagnetics, it is often useful to introduce the notion of a scalar potential due to a single point charge associated with a time-harmonic Hertzian dipole [1]–[4]. It is well known that in a homogeneous space this potential does not depend on the orientation of the dipole [1], [2]. In a layered medium, however, the scalar potential depends on the chosen form of the magnetic vector potential, which is not unique [5]. Hence, the scalar potential of a point charge associated with a horizontal dipole is, in general, different from that associated with a vertical dipole, when the medium is stratified [3]. The purpose of this communication is to demonstrate that one can choose the form of the magnetic vector potential in a layered medium such that those scalar potentials are identical, provided that the source and observation points are within the same layer. This has important implications relative to the existence of the so-called mixed-potential electric field integral equation in layered medium [6]. Our development is limited, for the sake of simplicity, to the case of a medium consisting of two contiguous half-spaces. The conclusions, however, are also valid for a dielectric medium comprising any number of planar layers.

II. STATEMENT OF THE PROBLEM

Consider a time-harmonic Hertzian dipole (the $e^{j\omega t}$ time dependence is assumed and suppressed) residing above an interface between two dielectric half-spaces, which is taken to be the xy-plane of a Cartesian coordinate system (x, y, z) with unit vectors $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})$. The medium of the upper $(z > 0)$ half-space has permittivity ϵ_1, permeability μ_1, and wavenumber k_1, and the corresponding parameters of the lower $(z < 0)$ half-space are ϵ_2, μ_2, and k_2. The dipole, whose orientation is arbitrary and is defined by a unit vector $\hat{\mathbf{l}}'$, is of infinitesimal length dl' and has a current moment $I d\mathbf{l}'$, where $d\mathbf{l}' = \hat{\mathbf{l}}' dl'$. In accord with the equation of continuity, associated with this dipole are two point charges $\pm Q$ where $\hat{\mathbf{l}}'$ points from $-Q$ to $+Q$ and $I = j\omega Q$. The magnetic vector potential due to the dipole is given by

$$\mathbf{A}(\mathbf{r}) = \bar{\mathbf{G}}_A(\mathbf{r}|\mathbf{r}') \cdot I d\mathbf{l}' \tag{1}$$

where $\mathbf{r}$ and $\mathbf{r}'$ are the position vectors of the observation and source points, respectively, defined with respect to the global coordinate origin, and $\bar{\mathbf{G}}_A$ is the dyadic Green's function, which can be expressed as [6]

$$\bar{\mathbf{G}}_A = (\hat{\mathbf{x}}\hat{\mathbf{x}} + \hat{\mathbf{y}}\hat{\mathbf{y}}) G_{xx} + \hat{\mathbf{z}}\hat{\mathbf{x}} G_{zx} + \hat{\mathbf{z}}\hat{\mathbf{y}} G_{zy} + \hat{\mathbf{z}}\hat{\mathbf{z}} G_{zz}. \tag{2}$$

This form of the Green's function results from the traditional approach [7], which postulates that a horizontal, say, x-directed dipole, generates the x- and z-components of the vector potential. However, one may as well take the y-component of the vector potential to accompany the primary x-component [5]. This strategy leads to a different form of the dyadic Green's function:

$$\bar{\mathbf{G}}_A = \hat{\mathbf{x}}\hat{\mathbf{x}} G'_{xx} + \hat{\mathbf{y}}\hat{\mathbf{y}} G_{yy} + (\hat{\mathbf{x}}\hat{\mathbf{y}} + \hat{\mathbf{y}}\hat{\mathbf{x}}) G_{xy} + \hat{\mathbf{z}}\hat{\mathbf{z}} G_{zz}. \tag{3}$$

When a cylindrical coordinate system (ρ, ϕ, z) is inscribed in the Cartesian system, the elements of the dyadics (2) and (3) can be expressed in terms of Sommerfeld-type integrals [7]. These expressions are listed for easy reference in the Appendix for the case where $\mathbf{r}$ and $\mathbf{r}'$ are in the upper half-space (i.e., $z > 0$ and $z' > 0$).

In the upper half-space, the scalar potential Φ_d of the dipole is given by the Lorentz gauge as [3]

$$\Phi_d(\mathbf{r}) = \frac{j\omega}{k_1^2} \nabla \cdot \mathbf{A}(\mathbf{r}). \tag{4}$$

By analogy to electrostatics [8], we can associate with Φ_d a scalar potential Φ of a single, time-harmonic point charge Q, as

$$\Phi_d(\mathbf{r}) = \frac{\partial}{\partial l'} \Phi(\mathbf{r}) \equiv \hat{\mathbf{l}}' \cdot \nabla' \Phi(\mathbf{r}) \tag{5}$$

where the primed operator nabla acts on source coordinates, which are implicit in $\Phi(\mathbf{r})$. Our objective is to find Φ, given the vector potential Green's function $\bar{\mathbf{G}}_A$. To this end, let us suppose that a scalar function K_Φ exists, such that

$$\frac{j\omega}{k_1^2} \nabla \cdot \bar{\mathbf{G}}_A(\mathbf{r}|\mathbf{r}') = \frac{1}{j\omega} \nabla' K_\Phi(\mathbf{r}|\mathbf{r}'). \tag{6}$$

Using this and (1) in (4) allows us to express the latter as

$$\Phi_d(\mathbf{r}) = \hat{\mathbf{l}}' \cdot \nabla' K_\Phi(\mathbf{r}|\mathbf{r}') Q dl'. \tag{7}$$

Comparing this with (5), we finally conclude that

$$\Phi(\mathbf{r}) = K_\Phi(\mathbf{r}|\mathbf{r}') Q dl'. \tag{8}$$

To recapitulate, if a function K_Φ exists such that (6) is satisfied, then a scalar potential of a single point charge Q associated with a time-harmonic Hertzian dipole $I d\mathbf{l}'$ can be defined and is given by (8). If, for simplicity, the dipole moment is taken to be unity $(Q dl' = 1)$, then, obviously, $\Phi(\mathbf{r}) = K_\Phi(\mathbf{r}|\mathbf{r}')$, and K_Φ is the sought-after scalar potential. In Section III, we demonstrate that in a layered medium K_Φ satisfying (6) does not, in general, exist if the traditional form (2) of the vector potential Green's function $\bar{\mathbf{G}}_A$ is employed. In Section IV, we show that K_Φ *does* exist, if the alternative form (3) of $\bar{\mathbf{G}}_A$ is used.

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The author is with the Department of Electrical Engineering, Texas A&M University, College Station, TX 77843.

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III. Traditional Formulation

When the traditional form (2) of $\bar{\mathbf{G}}_A$ is employed and use is made of the explicit expressions for its elements given in the (21)–(24), the x-, y-, and z-components of the left-hand side of (6) can be expressed as

$$\frac{j\omega}{k_1^2}\left\{\frac{\partial}{\partial x}G_{xx}+\frac{\partial}{\partial z}G_{zx}\right\}=\frac{1}{j\omega}\frac{\partial}{\partial x'}K_\Phi^H \tag{9}$$

$$\frac{j\omega}{k_1^2}\left\{\frac{\partial}{\partial y}G_{xx}+\frac{\partial}{\partial z}G_{zy}\right\}=\frac{1}{j\omega}\frac{\partial}{\partial y'}K_\Phi^H \tag{10}$$

and

$$\frac{j\omega}{k_1^2}\frac{\partial}{\partial z}G_{zz}=\frac{1}{j\omega}\frac{\partial}{\partial z'}K_\Phi^V \tag{11}$$

respectively, where (in the notation of the Appendix)

$$K_\Phi^H(\mathbf{r}|\mathbf{r}')=\frac{1}{4\pi\epsilon_1}\left\{K_0(\mathbf{r}|\mathbf{r}')+S_0\left(\frac{k_1^2}{j\beta_1\lambda^2}\Gamma^h+\frac{j\beta_1}{\lambda^2}\Gamma^e\right)\right\} \tag{12}$$

and

$$K_\Phi^V(\mathbf{r}|\mathbf{r}')=\frac{1}{4\pi\epsilon_1}\left\{K_0(\mathbf{r}|\mathbf{r}')+S_0\left(\frac{\Gamma^e}{j\beta_1}\right)\right\}. \tag{13}$$

Hence, keeping the right-hand side of (6) in mind, we conclude that in this case the function K_Φ does not exist for an arbitrarily oriented dipole. However, we can interpret K_Φ^H and K_Φ^V as the scalar potentials of point charges associated, respectively, with a horizontal and a vertical dipole. Obviously, these potentials are not identical.

When $\mu_1 = \mu_2$, the potentials K_Φ^H and K_Φ^V given in (12) and (13), respectively, reduce to those previously used in the analyses of wire antennas above a dielectric half-space [3], [9].

IV. Alternative Formulation

When the alternative form (3) of $\bar{\mathbf{G}}_A$ is employed and use is made of (22)–(27), the x- and y-components of the left-hand side of (6) can be expressed as

$$\frac{j\omega}{k_1^2}\left\{\frac{\partial}{\partial x}G'_{xx}+\frac{\partial}{\partial y}G_{xy}\right\}=\frac{1}{j\omega}\frac{\partial}{\partial x'}K_\Phi^V \tag{14}$$

and

$$\frac{j\omega}{k_1^2}\left\{\frac{\partial}{\partial y}G_{yy}+\frac{\partial}{\partial x}G_{xy}\right\}=\frac{1}{j\omega}\frac{\partial}{\partial y'}K_\Phi^V \tag{15}$$

respectively, with the z-component still given by (11). Hence, in this case a function K_Φ satisfying (6) *does* exist and is given by (13), i.e., $K_\Phi(\mathbf{r}|\mathbf{r}') \equiv K_\Phi^V(\mathbf{r}|\mathbf{r}')$.

V. Conclusion

We have demonstrated that when the alternative form (3) of the vector potential Green's function is employed, the scalar potentials of point charges associated with the horizontal and vertical dipoles in a layered medium are identical, provided that the source and observation points are within the same layer. Consequently, it is possible in this case to define a scalar potential of a single point charge associated with an arbitrarily oriented, time-harmonic dipole. This is tantamount to saying that the mixed potential electric field integral equation [6] does exist, provided that the scatterer or antenna is restricted to a single dielectric layer.

Appendix

Components of the Vector Potential Green's Functions

In this Appendix, we give the explicit expressions for the elements of the dyadic $\bar{\mathbf{G}}_A$, in both the conventional form (2) and in the alternative form (3), for the case where the source and the observation points are in the upper half-space ($z > 0$, $z' > 0$). To make the formulas more compact, we first introduce the notation:

$$K_0(\mathbf{r}|\mathbf{r}')=\frac{e^{-jk_1|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \tag{16}$$

$$S_n(f)=\int_0^\infty f(\lambda)e^{-j\beta_1(z+z')}J_n(\lambda\xi)\lambda^{n+1}\,d\lambda \tag{17}$$

$$\xi=|\boldsymbol{\rho}-\boldsymbol{\rho}'|,\quad \zeta=\arctan\left\{\frac{y-y'}{x-x'}\right\} \tag{18}$$

$$\Gamma^e=-\frac{\epsilon_2\beta_1-\epsilon_1\beta_2}{\epsilon_2\beta_1+\epsilon_1\beta_2},\quad \Gamma^h=\frac{\mu_2\beta_1-\mu_1\beta_2}{\mu_2\beta_1+\mu_1\beta_2} \tag{19}$$

$$\beta_i^2=k_i^2-\lambda^2,\quad \text{Im}\,(\beta_i)\le 0 \tag{20}$$

where $k_i^2 = \omega^2\mu_i\epsilon_i$, $i = 1, 2$. With these definitions, we can express the components of the dyadic (2) as listed below.

$$G_{xx}(\mathbf{r}|\mathbf{r}')=\frac{\mu_1}{4\pi}\left\{K_0(\mathbf{r}|\mathbf{r}')+S_0\left(\frac{\Gamma^h}{j\beta_1}\right)\right\} \tag{21}$$

$$G_{zx}(\mathbf{r}|\mathbf{r}')=-\frac{\mu_1}{4\pi}\frac{\partial}{\partial x}S_0\left(\frac{\Gamma^e-\Gamma^h}{\lambda^2}\right) \tag{22a}$$

or, equivalently,

$$G_{zx}(\mathbf{r}|\mathbf{r}')=\cos\zeta\,\frac{\mu_1}{4\pi}S_1\left(\frac{\Gamma^e-\Gamma^h}{\lambda^2}\right) \tag{22b}$$

$$G_{zy}(\mathbf{r}|\mathbf{r}')=-\frac{\mu_1}{4\pi}\frac{\partial}{\partial y}S_0\left(\frac{\Gamma^e-\Gamma^h}{\lambda^2}\right) \tag{23a}$$

with the equivalent form

$$G_{zy}(\mathbf{r}|\mathbf{r}')=\sin\zeta\,\frac{\mu_1}{4\pi}S_1\left(\frac{\Gamma^e-\Gamma^h}{\lambda^2}\right). \tag{23b}$$

Finally,

$$G_{zz}(\mathbf{r}|\mathbf{r}')=\frac{\mu_1}{4\pi}\left\{K_0(\mathbf{r}|\mathbf{r}')-S_0\left(\frac{\Gamma^e}{j\beta_1}\right)\right\}. \tag{24}$$

Similarly, the elements of the dyadic (3) can be expressed as listed below.

$$G'_{xx}(\mathbf{r}|\mathbf{r}')=\frac{\mu_1}{4\pi}\left\{K_0(\mathbf{r}|\mathbf{r}')-\frac{\partial^2}{\partial x^2}S_0\left(\frac{\Gamma^e}{j\beta_1\lambda^2}\right)\right.$$
$$\left.-\frac{\partial^2}{\partial y^2}S_0\left(\frac{\Gamma^h}{j\beta_1\lambda^2}\right)\right\} \tag{25a}$$

or, equivalently,

$$G'_{xx}(\mathbf{r}|\mathbf{r}')=\frac{\mu_1}{4\pi}\left\{K_0(\mathbf{r}|\mathbf{r}')+S_0\left(\frac{\Gamma^e+\Gamma^h}{j2\beta_1}\right)\right.$$
$$\left.-\cos 2\zeta S_2\left(\frac{\Gamma^e-\Gamma^h}{j2\beta_1\lambda^2}\right)\right\} \tag{25b}$$

$$G_{yy}(\mathbf{r}|\mathbf{r}')=\frac{\mu_1}{4\pi}\left\{K_0(\mathbf{r}|\mathbf{r}')-\frac{\partial^2}{\partial x^2}S_0\left(\frac{\Gamma^h}{j\beta_1\lambda^2}\right)-\frac{\partial^2}{\partial y^2}S_0\left(\frac{\Gamma^e}{j\beta_1\lambda^2}\right)\right\} \quad (26a)$$

or

$$G_{yy}(\mathbf{r}|\mathbf{r}')=\frac{\mu_1}{4\pi}\left\{K_0(\mathbf{r}|\mathbf{r}')+S_0\left(\frac{\Gamma^e+\Gamma^h}{j2\beta_1}\right)+\cos 2\zeta S_2\left(\frac{\Gamma^e-\Gamma^h}{j2\beta_1\lambda^2}\right)\right\} \quad (26b)$$

$$G_{xy}(\mathbf{r}|\mathbf{r}')=-\frac{\mu_1}{4\pi}\frac{\partial^2}{\partial x\partial y}S_0\left(\frac{\Gamma^e-\Gamma^h}{j\beta_1\lambda^2}\right) \quad (27a)$$

or

$$G_{xy}(\mathbf{r}|\mathbf{r}')=-\sin 2\zeta\frac{\mu_1}{4\pi}S_2\left(\frac{\Gamma^e-\Gamma^h}{j2\beta_1\lambda^2}\right). \quad (27b)$$

The element G_{zz} in (3) is, of course, identical to the corresponding element in (2), and is given by (24).

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Field Representation in Gyrotropic Media by One Scalar Superpotential

WERNER WEIGLHOFER

Abstract—**The electromagnetic field of an arbitrary current density distribution parallel to the distinguished axis of a gyrotropic medium is represented in terms of one scalar superpotential. The fourth-order partial differential equation for the superpotential is solved for some special forms of the impressed current density distribution.**

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The author is with the Department of Physics, The University of Adelaide, GPO Box 498, Adelaide, South Australia 5001, on leave from the Institute for Theoretical Physics, Technical University of Graz, Petersgasse 16, A-8010 Graz, Austria.

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INTRODUCTION

In recent papers [1], [2], the method of scalar Hertz potentials for gyrotropic media—which was originally developed for source-free regions, [3], [4]—was generalized to gyrotropic media including electric and magnetic current density distributions of arbitrary form.

Here we restrict our attention to longitudinal current density distributions, i.e., sources parallel to the distinguished axis of the gyrotropic medium (This axis is chosen to be the z-axis, with unit vector $\mathbf{e}^z$.) So the electric and magnetic current density distribution is given by

$$\mathbf{J}(\mathbf{r})=J_z(\mathbf{r})\mathbf{e}^z,\ \mathbf{M}(\mathbf{r})=M_z(\mathbf{r})\mathbf{e}^z. \quad (1)$$

(Vectors are bold, tensors are bold with an overbar.) The tensors of permittivity and permeability of the gyrotropic medium have the form

$$\bar{\epsilon}=\begin{pmatrix}\epsilon_1 & i\epsilon_2 & 0\\ -i\epsilon_2 & \epsilon_1 & 0\\ 0 & 0 & \epsilon\end{pmatrix},\quad \bar{\mu}=\begin{pmatrix}\mu_1 & i\mu_2 & 0\\ -i\mu_2 & \mu_1 & 0\\ 0 & 0 & \mu\end{pmatrix}, \quad (2)$$

wherein the matrix elements shall be constant numbers.

The electromagnetic field $\{\mathbf{E}, \mathbf{H}\}$ satisfying Maxwell's equations

$$i\omega\bar{\epsilon}\cdot\mathbf{E}-\nabla\times\mathbf{H}=-\mathbf{J},$$
$$\nabla\times\mathbf{E}+i\omega\bar{\mu}\cdot\mathbf{H}=-\mathbf{M}, \quad (3)$$

can then be represented by two scalar Hertz potentials $u(\mathbf{r})$ and $v(\mathbf{r})$ in the form

$$\mathbf{E}=\bar{\epsilon}^{-1}\cdot(\nabla\times\bar{\epsilon})\cdot(\nabla\times u\mathbf{e}^z)-(i\omega\mu_1/\epsilon_1)\bar{\epsilon}^T\cdot(\nabla\times v\mathbf{e}^z)-(1/i\omega\epsilon)J_z\mathbf{e}^z,$$
$$\mathbf{H}=\bar{\mu}^{-1}\cdot(\nabla\times\bar{\mu})\cdot(\nabla\times v\mathbf{e}^z)+(i\omega\epsilon_1/\mu_1)\bar{\mu}^T\cdot(\nabla\times u\mathbf{e}^z)-(1/i\omega\mu)M_z\mathbf{e}^z, \quad (4)$$

(the superscript T denotes the transposed tensor) if the scalar Hertz potentials u and v are solutions of

$$(\epsilon_1/\epsilon)\hat{H}_e u+\omega\mu_1\tau(\partial v/\partial z)=-J_z/(i\omega\epsilon),$$
$$-\omega\epsilon_1\tau(\partial u/\partial z)+(\mu_1/\mu)\hat{H}_m v=-M_z/(i\omega\mu). \quad (5)$$

The abbreviations used above are

$$\hat{H}_e=\nabla_t^2+(\epsilon/\epsilon_1)(\partial^2/\partial z^2)+k_e^2,\quad \hat{H}_m=\nabla_t^2+(\mu/\mu_1)(\partial^2/\partial z^2)+k_m^2, \quad (6)$$

$$\nabla^2=\nabla\cdot\nabla=\nabla_t^2+(\partial^2/\partial z^2),\quad \nabla=\nabla^t+\mathbf{e}^z(\partial/\partial z), \quad (7)$$

$$k_e^2=k^2(\mu_1^2-\mu_2^2)/\mu\mu_1,\quad k_m^2=k^2(\epsilon_1^2-\epsilon_2^2)/\epsilon\epsilon_1,$$
$$k^2=\omega^2\epsilon\mu,\quad \tau=\epsilon_2/\epsilon_1+\mu_2/\mu_1. \quad (8)$$

The electromagnetic field representation (4) was derived under the assumption of purely longitudinal current density distributions, i.e., the impressed sources are parallel to the distinguished axis of the gyrotropic medium. In the presence of current density distributions which are transversely oriented with respect to the distinguished axis the introduction of scalar Hertz potentials becomes more compli-