Extrapolation Methods for Sommerfeld Integral Tails

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***Abstract*—A review is presented of the extrapolation methods for accelerating the convergence of Sommerfeld-type integrals (i.e., semi-infinite range integrals with Bessel function kernels), which arise in problems involving antennas or scatterers embedded in planar multilayered media. Attention is limited to partition-extrapolation procedures in which the Sommerfeld integral is evaluated as a sum of a series of partial integrals over finite subintervals and is accelerated by an extrapolation method applied over the real-axis tail segment (a, ∞) of the integration path, where $a > 0$ is selected to ensure that the integrand is well behaved. An analytical form of the asymptotic truncation error (or the remainder), which characterizes the convergence properties of the sequence of partial sums and serves as a basis for some of the most efficient extrapolation methods, is derived. Several extrapolation algorithms deemed to be the most suitable for the Sommerfeld integrals are described and their performance is compared. It is demonstrated that the performance of these methods is strongly affected by the horizontal displacement of the source and field points ρ and by the choice of the subinterval break points. Furthermore, it is found that some well-known extrapolation techniques may fail for a number of values of ρ and ways to remedy this are suggested. Finally, the most effective extrapolation methods for accelerating Sommerfeld integral tails are recommended.**

***Index Terms*—Electromagnetic radiation, electromagnetic scattering, nonhomogeneous media, numerical analysis.**

I. Introduction

GREEN'S functions and integral equations that arise in layered media problems comprise Sommerfeld-type integrals of the form [1]

$$I = \int_0^\infty \underbrace{G(z, z'; \xi) J_\nu(\xi\rho)\xi}_{f(\xi)}\, d\xi \qquad \nu = 0, 1, 2 \tag{1}$$

where G is a spectral domain Green's function (or integral equation kernel) of the layered medium, J_ν is the Bessel function of order ν, ρ is the horizontal distance between the field and source points, and z and z' are the vertical coordinates of those points. These integrals are difficult to evaluate because their integrands possess singularities near the integration path and are, in general, oscillatory and slowly convergent. To facilitate the integration, the semi-infinite range in (1) is usually split into two parts—$(0, a)$ and (a, ∞)—and the first path segment is deformed into the first quadrant of the complex plane to avoid the guided-wave poles and branch points of the integrand, as illustrated in Fig. 1. The value of a is selected to ensure that on the remaining tail segment the integrand is free of singularities. Since a significant fraction of the overall computational effort is typically spent on the *tail integral*, it is essential that this integration be done as efficiently as possible. The proven and most popular approach is the *integration then summation* procedure [2]–[4] in which the integral is evaluated as a sum of a series of partial integrals over finite subintervals as follows:

$$S = \int_a^\infty f(\xi)\, d\xi = \sum_{i=0}^{\infty} u_i \tag{2}$$

with

$$u_i = \int_{\xi_{i-1}}^{\xi_i} f(\xi)\, d\xi \tag{3}$$

where $\xi_{-1} < \xi_0 < \xi_1 < \cdots$ with $\xi_{-1} = a$ and $\lim_{n\to\infty} \xi_n = \infty$ is a sequence of suitably selected interpolation points. These *break points* may be selected based on the asymptotic behavior of the integrand $f(\xi)$. Namely, the spectral function G arising in layered media problems has the asymptotic form

$$G(z, z'; \xi) \sim \frac{e^{-\zeta\xi}}{\xi^\mu}\left[C + \mathcal{O}(\xi^{-1})\right] \tag{4}$$

where C is a constant and where ζ, which is related to z and z', and μ are easily determined [1]. It is also well known that the Bessel function behaves for large arguments as [5, p. 364]

$$J_\nu(\xi\rho) \sim \sqrt{\frac{2}{\pi\xi\rho}}\, \cos(\xi\rho - \nu\pi/2 - \pi/4). \tag{5}$$

Hence, the simplest choice of break points are the equidistant points [6], [7]

$$\xi_n = b + nq, \qquad n \geq 0 \tag{6}$$

where $q = \pi/\rho$ is the asymptotic half-period of the Bessel function and b denotes the first break point greater than a. The value of b may be adjusted, for example, to coincide with the first zero of the Bessel function exceeding a. Unless otherwise stated, here we simply set $b = a + q$. Other possible choices of break points include the (exact) zero crossings and extremum points of the Bessel function [4], [8] in which case the subinterval length $q_n = \xi_{n+1} - \xi_n$ varies with n. In the special case of $\rho = 0$, the subinterval length must be chosen based on the exponential behavior of G, rather than the

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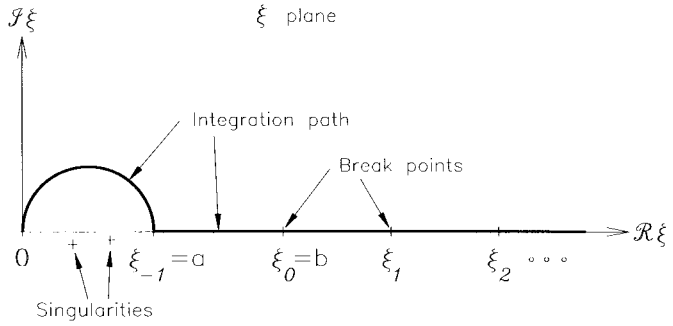

Fig. 1. Integration path in the complex ξ plane. Between zero and a the path is deformed from the real axis to avoid the integrand singularities. The partition-extrapolation method is applied over the real-axis tail segment (a, ∞).

oscillations of the Bessel function. Here, we choose $q = \pi/\zeta$, but other values may also work well.

The computation of the tail integral in (2) has thus been reduced to finding the limit of a sequence of the partial sums

$$S_n = \sum_{i=0}^{n} u_i \tag{7}$$

as $n \to \infty$. However, this sequence usually approaches S slowly, i.e., the *remainders*

$$r_n = S_n - S = -\int_{\xi_n}^{\infty} f(\xi)\, d\xi, \qquad n \geq 0 \tag{8}$$

do not decay rapidly with increasing n. One, therefore, seeks a transformation T of the sequence $\{S_n\}$ into another sequence $\{S'_n\}$ with remainders $\{r'_n\}$, possessing the property that as $n \to \infty$

$$\frac{|r'_n|}{|r_n|} = \mathcal{O}(\xi_n^{-p}), \qquad p > 0. \tag{9}$$

It is then said that the transformation T, which may be linear or nonlinear, *accelerates the convergence* of the sequence $\{S_n\}$. Any sequence transformation may be viewed as an extrapolation method [9] since the underlying principle is to obtain an improved estimate from the sequence of approximate values. The integration then summation procedure, followed by extrapolation, is referred to as the *partition-extrapolation method* [10].

For Sommerfeld-type integrals, the remainders (8) will be shown to possess the asymptotic expansion

$$r_n \sim \omega_n \sum_{i=0}^{\infty} a_i \xi_n^{-i}, \qquad n \to \infty \tag{10}$$

where ω_n are referred to as *remainder estimates*. These remainder estimates play an important role in sequence acceleration since they provide structural information about the behavior of the dominant term of the remainder for large n [11]. As will become clear later, the most effective extrapolation methods explicitly utilize the information contained in the remainder estimates.

The convergence of the sequence of partial sums (7) may be characterized based on the behavior of the ratio

$$\lambda_n = \frac{\Delta S_n}{\Delta S_{n-1}} = \frac{u_{n+1}}{u_n} \tag{11}$$

where Δ is the forward difference operator defined by $\Delta S_n = S_{n+1} - S_n$. If $\lambda_n \to \lambda$ as $n \to \infty$, then the sequence is said to converge linearly if $|\lambda| < 1$ and logarithmically if $|\lambda| = 1$. If $|\lambda| > 1$, the sequence is divergent. Furthermore, the sequence is asymptotically monotone if $\lambda > 0$ and alternating if $\lambda < 0$. Alternating sequences are generally the easiest to accelerate, while logarithmic monotone convergence is the most challenging. Fortunately, as will be shown later, the convergence of the sequences that arise in the present case is either linear or logarithmic alternating.

We now give a brief history of previous work relevant to the partition-extrapolation method for Sommerfeld-type integrals. This approach was originated by Longman [2], who used it in conjunction with the Euler transformation [12, p. 62]. The earliest applications of this technique to the Sommerfeld half-space problem were by Frischknecht [13] and by Siegel and King [14]. Ryu *et al.* [15] used the partition-extrapolation method in conjunction with the G transform of Gray and Atchison [16], which was especially designed for infinite integrals. Higher order version of this transform was later developed by Gray *et al.* [17]. Cornille [18] used the first-order G transform, as well as the Euler method, but his comparisons appear nonconclusive. Chisholm *et al.* [19] used Wynn's ϵ algorithm [20], which is an efficient recursive implementation of the Shanks transformation [21] and is closely related to the Padé summation method [22]. Levin [23] compared the Shanks transformation to his t transformation and found the latter to be superior. Iterations of the Aitken Δ^2 process, which is just the first-order Shanks transformation, were used by Alaylioglu *et al.* [24]. Squire [10] also used this method as well as the ϵ algorithm and suggested the Levin transformation [23] as a promising alternative. Blakemore *et al.* [3] have found Levin's v transformation to be more efficient than the ϵ algorithm, albeit only marginally so. Bubenik [25] employed the ϵ algorithm. Hillion and Nurdin [26] used the Euler method, the ϵ algorithm, and the iterated Aitken process and found the latter to be superior. The iterated Aitken process was also used by Burke *et al.* [27], who applied the partition-extrapolation method along various paths in the complex plane. Lin and Mei [28] used the Shanks transformation. Sidi [29] developed a new extrapolation method, especially for infinite oscillatory integrals, known as the W transformation. This method utilizes the analytic form of the remainder estimates and, thus, requires the knowledge of the asymptotic behavior of the integrand, which, however, is usually available for Sommerfeld-type integrals. Mosig and Gardiol [30], [31] employed the Euler method and developed its much more efficient version, which became known as the weighted-averages method. The latter also depends on the knowledge of the asymptotic properties of the integrand. Chave [32] used the Padé summation method implemented via the continued fraction expansion algorithm [33]—a procedure equivalent to the ϵ algorithm. The ϵ algorithm was also implemented by Piessens *et al.* [34] in the QUADPACK library. Lyness [8] (see also [35]) revisited the Euler method, which he felt to be unjustly neglected and suggested using break points corresponding to the arithmetic average of two consecutive zero crossings of the Bessel function. Hasegawa and Tori [36] applied the W

transformation in conjunction with a quadrature rule-based Chebyshev series expansion, which is particularly effective when the nonoscillatory part of the integrand is smooth. Sidi [37] demonstrated the application of his W transformation to divergent oscillatory integrals and developed its modified version [38], [39] (later named the mW transformation [40]), which does not require any information on the asymptotic behavior of the integrand. Mosig [6] derived a refined version of the weighted-averages method [30], [31] and found it superior to the Shanks and W transformations. Ehrenmark [41] developed an improved method for choosing the break points in Sidi's W transformation. Espelid and Overholt [7] applied a modified Euler transformation and some variations of the Overholt method [42], which require the knowledge of the asymptotic behavior of the integrand. They found Overholt's P-order-two method to be at least as efficient as Levin's u transform. Lucas and Stone [4] compared the Euler method, the ϵ algorithm, and the mW transformation, and concluded that the latter, with the Lyness break points, was the best accelerator available. Kinayman and Aksun [43] compared the Euler transformation, the iterated Aitken process, the ϵ algorithm, the θ algorithm [44], the Chebyshev–Toeplitz algorithm [45], and the method of averages [31] and found the latter to be superior to the other methods. Hasegawa and Sidi [40] used the quadrature rule of Hasegawa and Tori [36] in conjunction with the mW transformation. Finally, we mention that the partition-extrapolation method is also treated in the books by Brezinski and Redivo Zaglia [9], Wimp [46], Davis and Rabinowitz [47], and Evans [48].

Based on this review of previous work, we conclude that the Shanks transformation (usually implemented via the ϵ algorithm) has been the most popular extrapolation method for accelerating Sommerfeld-type integrals, but there are indications that the less frequently used iterated Aitken Δ^2 process and some of the more recently developed accelerators, such as the Levin transformations, the weighted-averages method, and the W and mW transformations are more efficient and as easy to implement. However, these most promising methods have not yet been subjected to the same suite of realistic test cases. Furthermore, with the single exception of Mosig's work [6], all tests to date were done for a single value of the parameter ρ and for $a = 0$. This is inadequate, because in many applications the integral (2) must be evaluated for a wide range of values of ρ and a finite fixed value of a. In addition, as Lucas and Stone [4] have shown, the performance of the extrapolation methods can be strongly affected by the choice of break points, which should, therefore, be carefully examined. Finally, since the asymptotic behavior of the integrand in (1) is typically known, it should be possible to characterize the convergence properties of the sequence of partial sums (7) and to predict which of the acceleration methods are most likely to be effective for the problem in hand.

It is our purpose here to address all those issues and to recommend the most effective acceleration methods for the Sommerfeld integral tails (2). This paper is organized as follows. In Section II, we derive an explicit analytical expression of the remainder estimates ω_n, which serves as a basis for some of the most efficient extrapolation techniques and allows us to characterizes the convergence of the sequence of partial sums (7). In Section III, we develop in detail the extrapolation algorithms that we consider to be the most promising for the problem in hand. In particular, we discuss the weighted averages and W algorithms, which may be used to efficiently implement various sequence transformations resulting from different choices of the remainder estimates. In Section IV, we present numerical tests and comparison of methods. Finally, in Section V, we draw conclusions and offer recommendations.

II. Estimating the Remainders

The remainder estimates ω_n appearing in (10) play an important role in the development of extrapolation algorithms, as will become clear in Section III. In this section, we derive an explicit analytical form of ω_n, based on the asymptotic behavior of G, which is usually known in Sommerfeld-type problems. Furthermore, we obtain (as a byproduct) the asymptotic form of λ_n, which will allow us to characterize the convergence of the sequence of partial sums (7). Our approach is similar to that of Espelid and Overholt [7] (see also [6] and [49]).

In view of (4) and (5), let us assume that for $\xi \geq b$, $f(\xi)$ may written as

$$f(\xi) = g(\xi)p(\xi) \tag{12}$$

where $g(\xi)$ has the asymptotic expansion

$$g(\xi) \sim \frac{e^{-\zeta\xi}}{\xi^{\alpha}} \sum_{i=0}^{\infty} \frac{c_i}{\xi^i} \tag{13}$$

and $p(\xi)$ is periodic with period $2q$ such that

$$p(\xi + q) = -p(\xi). \tag{14}$$

Observe that the integrand of (2) is asymptotically of this form with $q = \pi/\rho$ and $\alpha = \mu - 1/2$. Next, substituting (13) into (8) gives

$$r_n \sim -\sum_{i=0}^{\infty} c_i \int_{\xi_n}^{\infty} \frac{p(\xi)e^{-\zeta\xi}}{\xi^{\alpha+i}}\, d\xi \tag{15}$$

which upon changing the variable of integration and using (6) and (14) becomes

$$r_n \sim (-1)^{n+1} e^{-nq\zeta} \sum_{i=0}^{\infty} c_i \int_{b}^{\infty} \frac{p(t)\, e^{-\zeta t}}{(t+nq)^{\alpha+i}}\, dt. \tag{16}$$

Finally, upon expanding the denominator in a Taylor series about $t = b$ and formally integrating term by term, one obtains (10) with the remainder estimates

$$\omega_n = (-1)^{n+1} \frac{e^{-nq\zeta}}{\xi_n^{\alpha}} \tag{17}$$

and the leading coefficient

$$a_0 = c_0 \int_{b}^{\infty} p(t) e^{-\zeta t}\, dt. \tag{18}$$

The higher index coefficients a_i, which are similar in form but progressively more complicated, will not explicitly be used

and are omitted for brevity. Observe that a_0 may vanish if a proper value of b is chosen, resulting in faster convergence [7]. It is readily shown that for $\zeta = 0$, the optimal b results in break points (6) corresponding to the extrema of the periodic function $p(\xi)$.

The above procedure may also be used to demonstrate that u_{n+1} has the same asymptotic form as (16), except that the integrals defining the coefficients a_i are now over the finite range $(b, b+q)$. It then follows from (11) that

$$\lambda_n \sim -e^{-q\zeta}\left[1 - \frac{\alpha}{\beta+n} + \mathcal{O}(n^{-2})\right] \underset{n\to\infty}{\longrightarrow} -e^{-q\zeta} \tag{19}$$

where $\beta = b/q$. Hence, we conclude that the series of partial integrals (2) is asymptotically alternating. Furthermore, the convergence is linear for $\zeta > 0$ and logarithmic for $\zeta = 0$.

We have tacitly assumed in the above that $\rho > 0$. However, the special case where $\rho = 0$ with $\nu = 0$ and $\zeta > 0$ is also of practical interest. Clearly, in this case $p(\xi) = 1$ and (14) no longer holds. The latter contributes the $(-1)^n$ factors in (16) and (17) and causes the minus sign in (19), which must now be removed. Hence, in this case, the convergence is linear monotone.

Clearly, the analytical expression for the remainder estimates given in (17) provides much insight into the convergence properties of the sequence of partial sums (7). As will become evident later, these analytical remainder estimates also serve as a basis for some of the most powerful extrapolation methods currently known. In cases where the information on the asymptotic behavior of the integrand is not available or difficult to extract, numerically derived remainder estimates—based on the actual numerical values of one or a few consecutive terms of the series (2)—have been suggested in the literature. For later reference, we list below the alternative choices of ω_n, which have been found to be particularly effective [23], [39], [50]:

$$\omega_n = \Delta S_n = u_{n+1} \tag{20}$$

$$\omega_n = \Delta S_{n-1} = u_n \tag{21}$$

$$\omega_n = \xi_n \Delta S_{n-1} = \xi_n u_n \tag{22}$$

$$\omega_n = -\frac{\Delta S_{n-1}\Delta S_n}{\Delta^2 S_{n-1}} = \frac{u_n u_{n+1}}{u_n - u_{n+1}}. \tag{23}$$

Heuristic arguments supported by numerical evidence indicate that (20) and (21) are most suitable for alternating convergence, while (22) and (23) are best choices for logarithmic monotone and linear convergence.

III. Accelerating Convergence by Extrapolation

Series acceleration methods are based on the idea that the information contained in the sequence of partial sums $S_0, \cdots, S_n$ should be extracted and utilized in a way that is more efficient than the conventional process of adding up one term after the other [11]. Below, we discuss the most important series extrapolation techniques currently known, with emphasis on those methods that appear to be most suitable for the acceleration of the Sommerfeld integral tails.

A. Euler Transformation

As a simple introduction to the extrapolation methods for sequence acceleration, we first consider the Euler transformation [8], [12, p. 62]. One step of this method is the simple averaging procedure

$$S'_n = S_n + \tfrac{1}{2}\Delta S_n = \tfrac{1}{2}(S_n + S_{n+1}) \tag{24}$$

which clearly is a linear transformation. If we define $S_n^{(0)} = S_n$, the formula (24) may be applied repeatedly, resulting in the triangular scheme

$$S_n^{(k+1)} = \tfrac{1}{2}[S_n^{(k)} + S_{n+1}^{(k)}], \qquad n, k \geq 0. \tag{25}$$

If $S_0, \cdots, S_k$ are known, $S_0^{(k)}$ is the best approximation of S. We recognize (25) as a recursive implementation of the method of averages [30], [31]. It will be shown later that this method is effective for logarithmic alternating convergence.

B. Iterated Aitken Transformation

To derive the Aitken Δ^2 process [47, p. 43], we postulate that

$$S \approx S_n + c\Delta S_n \tag{26}$$

and determine the coefficient c by applying the operator Δ to both sides of the equation. Upon substituting this value back into (26), we obtain

$$S'_n = S_n - \frac{(\Delta S_n)^2}{\Delta^2 S_n} = S_{n+1} + \frac{1}{\Delta(1/\Delta S_n)} \tag{27}$$

where S'_n defines the transformed sequence. This procedure, which clearly is a nonlinear transformation, may be repeated, thus leading to the iterated Aitken algorithm

$$S_n^{(k+1)} = S_n^{(k)} - \frac{[\Delta S_n^{(k)}]^2}{\Delta^2 S_n^{(k)}}, \qquad n, k \geq 0 \tag{28}$$

where we have used the first form given in (27). If $S_0, \cdots, S_{2k}$ are known, $S_0^{(2k)}$ is the best approximation of S, whereas if $S_0, \cdots, S_{2k+1}$ are known, one should use $S_1^{(2k)}$. Observe that the Aitken process improves upon the Euler transformation by allowing the coefficient c to be adaptively computed at each iteration, rather than being set to a fixed value of 1/2. Numerical tests indicate that the iterated Aitken method is effective for alternating and linear monotone convergence [51].

C. Weighted-Averages Method

The weighted-averages method is a more sophisticated version of the Euler transformation, which uses *weighted* means of consecutive partial sums, with weights selected based on the remainder estimates ω_n. To derive this algorithm, we denote the weight associated with S_n by W_n and generalize (24) as [52]

$$S'_n = \frac{W_n S_n + W_{n+1} S_{n+1}}{W_n + W_{n+1}} \tag{29}$$

which, upon using (8), may be written as

$$S_n' = S + \underbrace{\frac{W_n r_n + W_{n+1} r_{n+1}}{W_n + W_{n+1}}}_{r_n'}. \tag{30}$$

It is now clear that the remainders r_n' of the transformed sequence will be annihilated if the weights are in the ratio

$$\eta \equiv \frac{W_{n+1}}{W_n} = -\frac{r_n}{r_{n+1}}. \tag{31}$$

The problem, of course, is that the remainders r_n are unknown. However, if r_n possess the asymptotic expansion (10), we readily show that

$$\frac{r_n}{r_{n+1}} = \frac{\omega_n}{\omega_{n+1}}[1 + \mathcal{O}(\xi_n^{-2})], \qquad n \to \infty \tag{32}$$

which suggests that we choose

$$\eta = -\frac{\omega_n}{\omega_{n+1}}. \tag{33}$$

Then, if

$$1 + \eta = \mathcal{O}(\xi_n^{-\sigma}) \tag{34}$$

the condition (9) is satisfied with $p = 2 - \sigma$. Furthermore, with this choice, the remainders r_n' have the same form as (10) except that the remainder estimates are now scaled by ξ_n^{-p}. Consequently, the transformation (29) may be applied repeatedly, leading to the recursive scheme

$$S_n^{(k+1)} = \frac{S_n^{(k)} + \eta_n^{(k)} S_{n+1}^{(k)}}{1 + \eta_n^{(k)}}, \qquad n, k \geq 0 \tag{35}$$

with

$$\eta_n^{(k)} = -\frac{\omega_n}{\omega_{n+1}} \left(\frac{\xi_{n+1}}{\xi_n} \right)^{pk} \tag{36}$$

$$\approx -\frac{\omega_n}{\omega_{n+1}} \left(1 + pk \frac{q_n}{\xi_n} \right) \tag{37}$$

where the second form is valid for large n. Given the partial sums $S_0, \cdots, S_k$, this algorithm, which will be referred to as the generalized weighted-averages method, produces $S_0^{(k)}$ as the best approximation of S.

Using the analytical remainder estimates (17) in (34), we find that $\sigma = 0$ (i.e., $p = 2$) for both $\rho > 0$ (alternating convergence) and $\rho = 0$ (linear monotone convergence). Furthermore, replacing ω_n by (17) in (36) and (37), we obtain

$$\eta_n^{(k)} = \pm e^{q\zeta} \left(\frac{\xi_{n+1}}{\xi_n} \right)^{\alpha + pk} \tag{38}$$

$$\approx \pm e^{q\zeta} \left(1 + \frac{\alpha + pk}{\beta + n} \right), \qquad p = 2 \tag{39}$$

where the plus and minus signs apply to the alternating and linear monotone cases, respectively. Note that the equidistant break points (6) are used in (39). The recursion (35) with the coefficients $\eta_n^{(k)}$ given by (38) and (39) is a linear transformation known as the weighted-averages method.

When the asymptotic coefficients (39) are employed for the alternating convergence case, the weighted-averages method can be shown to be equivalent to the P-order-two transformation of Espelid and Overholt [7], if the latter is extended to accommodate the ζ dependence of the integrand.

Note that (24) is a special case of (29) when $\eta = 1$. Hence, from (31) and (32) it follows that the Euler transformation is only effective if $\omega_n/\omega_{n+1} = -1 + \mathcal{O}(\xi_n^{-p})$. Using the analytical remainder estimates (17), we find that this condition is only satisfied if $\zeta = 0$ and $\rho > 0$ (i.e., for logarithmic alternating convergence), in which case $p = 1$. Therefore, as expected, the weighted-averages method is superior to the Euler transformation.

The weighted-averages method presented above was first developed by Mosig [6], using an approach that can be traced back to Hart *et al.* [53, p. 39]. However, Mosig used a different asymptotic expansion of the remainders than (10) and, consequently, the coefficients $\eta_n^{(k)}$ he obtained differ from, but are for $n \gg 1$ equivalent to, those given by (38) (with the plus sign since he limited attention to the alternating case). In view of the fact that the derivation was based on the asymptotic behavior of the integrand, Mosig remarked that using $p = 2$ could be too optimistic and recommended using $p = 1$ instead. This conservative choice was also employed in the earlier version of the weighted-averages algorithm [30], [31], [54], which does not take into account the dependence of the integrand on ζ.

Finally, we note that the generalized weighted-averages algorithm (35)–(37) can also be used in conjunction with the numerical remainder estimates (20) and (21). In particular, we find that if (20) is used in (36), the first iteration of (35) yields the Aitken Δ^2 process. This combination of the weighted-averages algorithm and the remainder estimates (20) gives rise to a new, nonlinear transformation, which will be referred to as the M transformation.

D. E Transformation

A common feature of the extrapolation algorithms discussed thus far is that each was obtained by an iterative application of some very simple sequence transformation. However, many of the existing sequence acceleration methods are based on a different approach, in which one constructs an algorithm that is exact for a certain *model sequence*. The most general extrapolation method was introduced by Schneider [55], Håvie [56], and Brezinski [57], who independently postulated the model sequence

$$S_n = S + \sum_{i=0}^{k-1} a_i \psi_i(n), \qquad n \geq 0,\ k \geq 1 \tag{40}$$

where $\psi_i(n)$ are known but otherwise arbitrary functions. Writing (40) for the indexes $n, \cdots, n + k$ leads to a linear equation system, which may be solved for S. However, unless the sequence $\{S_n\}$ exactly satisfies (40), the value of S so obtained will depend on the indexes n and k. Hence, if we denote this solution by $E_k(S_n)$ and use Cramer's rule, we obtain

$$E_k(S_n) = \frac{D_k[S_n;\, \psi_i(n)]}{D_k[1;\, \psi_i(n)]} \tag{41}$$

where

$$D_k[S_n;\,\psi_i(n)] = \begin{vmatrix} S_n & \cdots & S_{n+k} \\ \psi_0(n) & \cdots & \psi_0(n+k) \\ \vdots & \ddots & \vdots \\ \psi_{k-1}(n) & \cdots & \psi_{k-1}(n+k) \end{vmatrix} \tag{42}$$

and where $D_k[1;\,\psi_i(n)]$ is the same determinant with the elements of the first row replaced by one. The ratio of determinants (41) may be computed by the recursive procedure known as the Brezinski–Håvie protocol [46, p. 186] or the E algorithm [57], [58] or by its more efficient implementation due to Ford and Sidi [59]. The E transformation (41) contains some of the most effective sequence transformations currently known as special cases corresponding to different choices of the functions $\psi_i(n)$ [9, p. 57]. In those special cases, however, the particular structure of the determinants can usually be exploited to devise more efficient procedures than the general E algorithm. The transformation (41) is linear if the functions $\psi_i(n)$ do not depend on S_n and it is nonlinear if $\psi_i(n)$ involve S_n. Below we discuss two choices of $\psi_i(n)$, which lead to the Shanks and generalized Levin transformations.

E. Shanks Transformation

The Shanks transformation [21], which is nonlinear, arises from the choice $\psi_i(n) = \Delta^{i+1}S_n = \Delta S_{n+i}$ in the model sequence (40). Hence, in this transformation, the limit S is approximated by the nth partial sum plus a weighted sum of the next k terms of the series. From (41), we find that the determinantal representation of the Shanks transformation is

$$e_k(S_n) = \frac{D_k(S_n;\,\Delta S_{n+i})}{D_k(1;\,\Delta S_{n+i})}. \tag{43}$$

The matrices that arise in (43) are of the Hankel type for which efficient algorithms exist [46, p. 198], [9, p. 33]. However, the most convenient way to compute the Shanks transformation is the ϵ algorithm, which Wynn [20] developed from (43) using determinantal identities (see also [46, p. 244]). Rather than repeating Wynn's rigorous arguments, here we give a simple, heuristic derivation of this algorithm [60].

First, we note that the Shanks transformation may be considered a higher order generalization of the Aitken Δ^2 process since for $k = 1$ (43) reduces to (27). Hence, with this in mind, we define

$$\epsilon_{-1}^{(n)} = 0, \qquad \epsilon_0^{(n)} = S_n \tag{44}$$

and

$$\epsilon_1^{(n)} = 1/\Delta S_n = \epsilon_{-1}^{(n+1)} + 1/\Delta\epsilon_0^{(n)} \tag{45}$$

which allows us to write the second form given in (27) as

$$\epsilon_2^{(n)} = \epsilon_0^{(n+1)} + 1/\Delta\epsilon_1^{(n)}. \tag{46}$$

Generalizing the above, we obtain

$$\epsilon_{k+1}^{(n)} = \epsilon_{k-1}^{(n+1)} + 1/\Delta\epsilon_k^{(n)}, \qquad n,\ k \geq 0 \tag{47}$$

which, together with (44), is a recursive version of the Shanks transformation. Namely, it can be shown that $\epsilon_{2k}^{(n)} = e_k(S_n)$, whereas the elements with odd subscripts are auxiliary quantities [20]. Thus, if $S_0, \cdots, S_{2k}$ are known, $\epsilon_{2k}^{(0)}$ is the best approximation of S, while if $S_0, \cdots, S_{2k+1}$ are known, one should use $\epsilon_{2k}^{(1)}$. Numerical tests indicate that the ϵ algorithm is effective for alternating and linear monotone convergence [51].

When the Shanks transformation or the ϵ algorithm are applied to a power series with coefficients u_i, then it can be shown that $\epsilon_{2k}^{(0)} = e_k(S_0)$ yield the diagonal Padé approximants [21], [22], [61]. Furthermore, the successive convergents of the *corresponding* continued fractions lie on a staircase line of the ϵ array [62]. Consequently, series acceleration using continued fractions and the ϵ algorithm are equivalent, but it transpires that the implementation of the latter is more economical.

F. Generalized Levin Transformation

The generalized Levin transformation [23] arises by choosing $\psi_i(n) = \omega_n\xi_n^{-i}$ in the model sequence (40). Hence, in this transformation the limit S is approximated by the nth partial sum plus the nth remainder estimate multiplied by a correction function (a polynomial of order $k-1$ in ξ_n^{-1}), which approaches a constant as $n \to \infty$. From (41), we find that the determinantal representation of the generalized Levin transformation is

$$L_k(S_n) = \frac{D_k(S_n/\omega_n;\,\xi_n^{-i})}{D_k(1/\omega_n;\,\xi_n^{-i})}. \tag{48}$$

When the determinants in (48) are expanded by the first row, the minors that arise are of the Vandermonde type, for which explicit expansions are available [63, p. 9]. As a result, one readily obtains an explicit representation of $L_k(S_n)$ [46, p. 189]. For the equidistant break points (6), this expression reduces to a simpler form given by Weniger [11], which, for $\beta = 1$, further specializes to the formula originally derived by Levin [23]. A more efficient, recursive form of this transformation was developed by Fessler *et al.* [50] and was later extended for arbitrary $\beta > 0$ by Weniger [11]. In the general case, however, the transformation (48) is most efficiently computed by the W algorithm of Sidi [29], [37], [49], which may be derived as follows.

With $\psi_i(n) = \omega_n\xi_n^{-i}$ in the model sequence (40), we rewrite the latter as

$$(S_n - S)/\omega_n = \sum_{i=0}^{k-1} a_i\xi_n^{-i}, \qquad n \geq 0,\ k \geq 1 \tag{49}$$

and observe that the right-hand member in the above is a polynomial of degree $k-1$ in the variable $\xi_n^{-1} \equiv x_n$. Hence, if we consider S_n and ω_n as functions of the continuous variable x evaluated at $x = x_n$, and apply the *divided difference* [63] of order k to both sides of (49), we obtain

$$\delta^k[(S_n - S)/\omega_n] = 0 \tag{50}$$

because δ^k annihilates polynomials of degree smaller than k. Here, the divided difference operator δ^k is defined recursively

as [11, p. 245], [9, p. 116]

$$\delta^{k+1}(u_n) = \frac{\delta^k(u_{n+1}) - \delta^k(u_n)}{x_{n+k+1} - x_n}, \qquad n,\ k \geq 0 \tag{51}$$

with $\delta^0(u_n) = u_n$. Since δ^k is a linear operator, we readily find from (50) that S may be estimated as

$$L_k(S_n) = \frac{\delta^k(S_n/\omega_n)}{\delta^k(1/\omega_n)} \tag{52}$$

which is an equivalent form of (48). If now we denote the numerator and the denominator of (52) by $M_n^{(k)}$ and $N_n^{(k)}$, respectively, it follows from (51) that both obey the same three-term recurrence formula

$$R_n^{(k+1)} = \frac{R_{n+1}^{(k)} - R_n^{(k)}}{\xi_{n+k+1}^{-1} - \xi_n^{-1}}, \qquad n,\ k \geq 0 \tag{53}$$

with the starting values $M_n^{(0)} = S_n/\omega_n$ and $N_n^{(0)} = 1/\omega_n$. If $S_0, \cdots, S_k$ are known, $L_k(S_0) = M_0^{(k)}/N_0^{(k)}$ is the best approximation of S.

Different choices of ω_n in the generalized Levin transformation (52) result in extrapolation methods with different acceleration properties. The numerical remainder estimates (21)–(23) yield, respectively, the t, u, and v transformations of Levin [23], while (20) gives rise to the modified t transformation of Smith and Ford [64], which will be referred to as the t' transformation. All four transformations, which are nonlinear, may efficiently be computed by the W algorithm using the appropriate remainder estimates ω_n.

It is readily shown that the first-order t' transformation is identical with the Aitken Δ^2 process (27). This is not surprising if one notes that the latter is based on (26), which is a simple one-term model sequence with $\omega_n = \Delta S_n$. Hence, the iterated Aitken method may also be viewed as a technique utilizing numerical remainder estimates.

The Levin u transformation is considered to be among the most versatile and powerful convergence accelerators currently known [64]. It appears that this transformation was already known to Bickley and Miller [65], who also appear to have originated the modern approach of constructing sequence transformations via annihilating difference operators [11].

With the explicit, analytical remainder estimates (17), the generalized Levin transformation reduces to the W transformation of Sidi [29], [37], which was developed especially for infinite oscillatory integrals such as (1). It follows from (52) that the W transformation is linear.

It was also suggested by Sidi [38], [39] that the numerical remainder estimates (20) be used in his W transformation if the information on the asymptotic behavior of the integrand is not available. The so-obtained mW transformation [4], [40] is thus equivalent to the t' transformation.

G. Other Methods

There exist other well-known extrapolation techniques, which are either equivalent to the methods discussed thus far in this paper, or have been shown to be less effective (or not applicable at all) for the class of problems considered here. Below, we give a brief review of some of the most important methods in this category.

The $\mathcal{J}$ transformation of Homeier [66], [67] is an iterative method, which makes explicit use of the remainder estimates and also depends on a family of auxiliary sequences. Many of the existing extrapolation methods—including the weighted-averages method and the generalized Levin transformation—can be obtained as variants of this transformation by properly choosing the auxiliary sequences. In this sense, the $\mathcal{J}$ transformation is very general and it provides a powerful theoretical framework for the development of new convergence accelerators.

The Chebyshev–Toeplitz and Legendre–Toeplitz algorithms of Wimp [45] are linear rhombus rules with weights related to the coefficients of the respective orthogonal polynomials. Numerical experiments indicate that the former is effective for alternating and the latter for logarithmic monotone convergence [64].

The G transformation of Gray *et al.* [17], which was especially developed for infinite integrals, can be obtained as a special case of the E transformation (41) when $\psi_i(n) = f(\xi_{n+i})$. Clearly, in this transformation, the limit S is approximated by the nth partial sum plus a weighted sum of the integrand values at the next k break points. Therefore, the G transformation is closely related to the Shanks transformation and may be expected to perform similarly as the latter. A recursive computational scheme for the G transformation was derived by Pye and Atchison [68].

The ρ algorithm of Wynn [69] (whose rules are quite similar to those of the ϵ algorithm) can also be obtained from (41) by properly choosing the auxiliary functions $\psi_i(n)$ [9, p. 101]. This algorithm, which is very powerful for logarithmic monotone sequences, but fails to accelerate alternating convergence, is based on Thiele's continued-fraction interpolation [70, ch. 3].

The θ algorithm of Brezinski [44], which was derived by modifying the ϵ algorithm, accelerates both linear and logarithmic convergence. The second-order θ algorithm $\theta_2^{(n)}$ is identical with the Lubkin W transform [71] (not to be confused with Sidi's W transformation discussed above), which may also be considered as a generalization of the Aitken Δ^2 process. An iterative algorithm based on $\theta_2^{(n)}$ was developed by Weniger [11], [72].

IV. Numerical Tests and Comparison of Methods

We now present numerical test results for the extrapolation methods discussed in Section III. Since the Euler transformation was shown to be a less effective variant of the weighted-averages method, we have dropped it from consideration in favor of the latter. All computations were done in double precision (15–16 decimal digits).

As the first test case, we consider the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}} = 0.604898643421630 \tag{54}$$

which was used by Levin [23], among others. This series exhibits logarithmic alternating convergence and has asymptotic

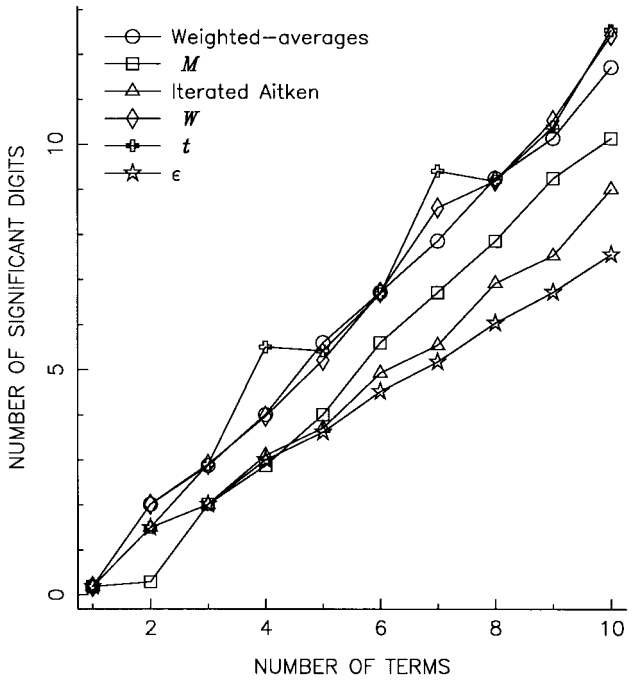

Fig. 2. Performance of various extrapolation algorithms for the series (54).

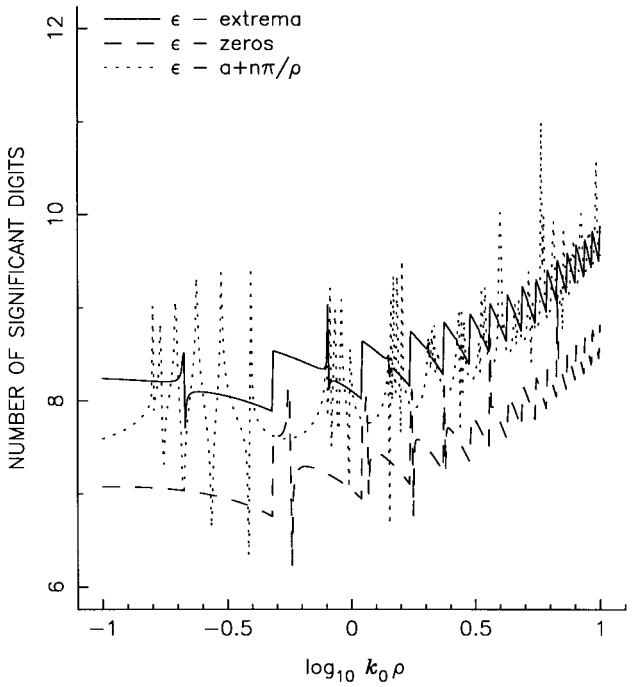

Fig. 3. Performance of the ϵ algorithm for the integral (55) with three choices of break points: the equidistant points (6) with $q = \pi/\rho$, the extremum points, and zeros of the Bessel function. The number of significant digits obtained after ten integration subintervals is plotted versus $k_0\rho$ for $z = 0$.

behavior similar to that of common Sommerfeld integrands. In Fig. 2, we plot the number of significant digits—computed as $-\log_{10}|\text{relative error}|$—versus the number of terms, obtained by the major extrapolation methods. Since the performance of the t', t, u, and v transformations was very similar, we have only included the results for the t transformation. For the weighted-averages method and the W transformation, which require explicit analytical remainder estimates, we set ω_n to the $n+1$st term of the series. We have found that for reasons unknown, the asymptotic form of the coefficients $\eta_n^{(k)}$ given by (37) results in a faster convergence than the exact form (36). Consequently, the former was used (with $p = 2$) to generate the results presented in Fig. 2.

As the next case, we consider the integral

$$\int_a^\infty \frac{e^{-jk_z|z|}}{jk_z} J_0(\xi\rho)\xi\, d\xi = \frac{e^{-jkr}}{r} - \int_0^a \frac{e^{-jk_z|z|}}{jk_z} J_0(\xi\rho)\xi\, d\xi \tag{55}$$

which is based on the Sommerfeld identity [73, p. 242]. Here, $r = \sqrt{\rho^2+z^2}$, $k_z = \sqrt{k^2-\xi^2}$, and $k = k_0\sqrt{\epsilon}$, where k_0 is the free-space wavenumber and ϵ is the (possibly complex) dielectric constant of the medium. The square roots defining k_z and k are to be taken with negative imaginary parts. Observe that the left-side integral in (55) is of the form (2), with $\zeta = |z|$ and $\alpha = 1/2$. To avoid the branch point singularity of the integrand at $\xi = k_0\sqrt{\epsilon}$, the value of a (see Fig. 1) was chosen as $a/k_0 = \sqrt{\Re\{\epsilon\}} + 1$ and the integral on the right side of (55) was evaluated on a semi-elliptical path in the complex ξ plane, as suggested by Gay-Balmaz and Mosig [74]. This integral and the partial integrals (3) were computed to machine accuracy by an adaptive quadrature based on the Patterson's formulas [75], [76]. The numerical results presented here are for $\epsilon = 16 - j0.1$. With one exception, we use $z = 0$, which is the most challenging case, resulting in logarithmic alternating convergence.

We have tested the partition-extrapolation method for three choices of break points: the equidistant points (6) with $q = \pi/\rho$, the zero crossings, and extremum points of the Bessel function. For the first five zeros the tabulated values were used and the higher order zeros were computed from the McMahon's asymptotic expansions [5, p. 371]. The Newton–Raphson iteration [4] was also tried, but for the low orders of Bessel functions considered here it resulted in only insignificant improvement of the extrapolation results. The extremum points were approximated by taking an average value of two consecutive Bessel function zeros, as suggested by Lyness [8].

Unless otherwise stated, the acceleration is applied from the first integration subinterval possible, which in most cases means extrapolation beginning with S_0. As will be seen below, some methods do not work well unless the extrapolation is delayed by one subinterval. Such delay is automatically built-in for some transformations—as, for example, the t' transformation, which uses the numerical remainder estimates (20) and can thus be first applied only after S_1 becomes available.

We first apply to (55) the ϵ algorithm, which is known for its robustness and is thus an appropriate benchmark for other techniques. In Fig. 3, we plot the number of significant digits obtained after ten integration subintervals versus the normalized horizontal distance between the source and observation points, $k_0\rho$. The number of integration subintervals required to achieve a certain accuracy is a good estimate of the computational cost involved and thus a good indicator of the efficiency of the method. The three curves in Fig. 3 correspond

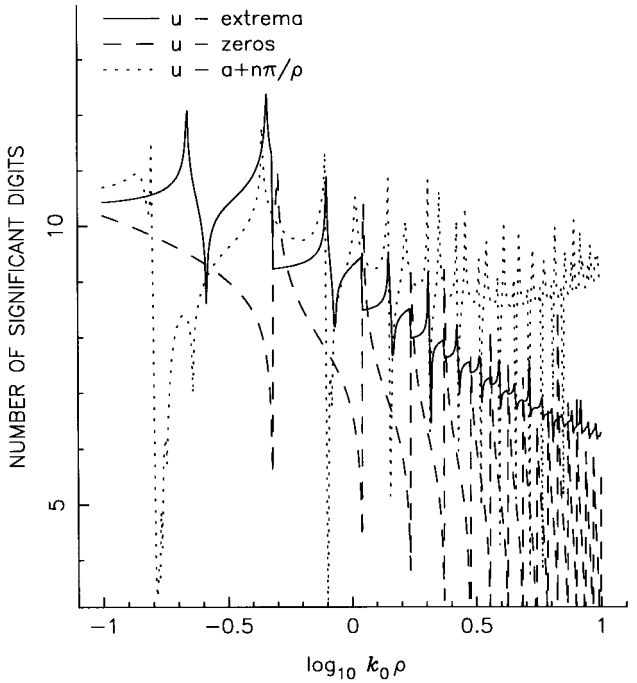

Fig. 4. Performance of the u transformation for the integral (55) with three choices of break points: the equidistant points (6) with $q = \pi/\rho$, the extremum points, and zeros of the Bessel function. The number of significant digits obtained after ten integration subintervals is plotted versus $k_0\rho$ for $z = 0$.

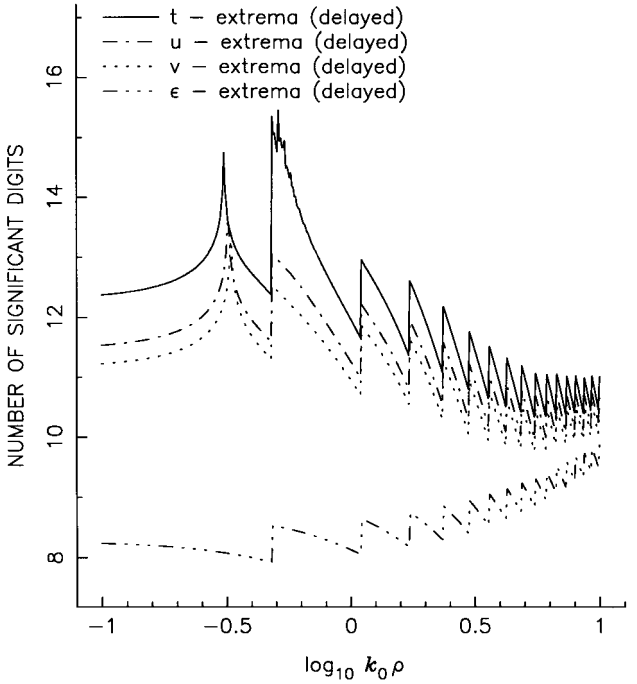

Fig. 5. Performance of the t, u, v transformations, and the ϵ algorithm for the integral (55) when Bessel function extremum points are used as break points and the extrapolation is delayed by one subinterval. The number of significant digits obtained after ten integration subintervals is plotted versus $k_0\rho$ for $z = 0$.

to the three different choices of break points discussed above. It is noted that for the equidistant break points the curve exhibits an erratic behavior—almost certainly caused by the break points not being related to the actual location of the zeros (or extrema) of the Bessel function (except that the spacing of those points is equal to the asymptotic half-period of the latter). As the parameter ρ varies while the lower integration limit a remains fixed, the break points and the Bessel function zeros continuously change their relative positions, thus affecting the convergence properties of the sequence of the partial sums (7). Apparently, this causes near breakdowns of the ϵ algorithm for a number of values of ρ.

The curves in Fig. 3 corresponding to Bessel function zeros and extremum points as break points exhibit a staircase behavior (previously observed by Mosig [6]) with some "glitches" superposed on it. The staircase steps can be correlated with the values of ρ for which the consecutive break points pass through the fixed value a, thus causing the length of the first integration interval to vanish. The glitches are apparently also related to this phenomenon, since (as is shown later in Fig. 5) they disappear when the extrapolation is delayed by one interval. We note that, as expected, the convergence is faster when the break points are based on the Bessel function extrema rather than zero crossings.

In Fig. 4, we show similar results for the u transformation. In this case, the breakdowns are much more pronounced than those observed in Fig. 3. The t and v transformations exhibit a similar behavior (not shown, for brevity). The performance of these transformations with break points corresponding to the extrema of the Bessel function can be significantly improved by delaying the extrapolation by one subinterval. This is illustrated in Fig. 5, where the corresponding results for the ϵ algorithm are also included for comparison. Clearly, all three Levin transformations are superior to the ϵ algorithm, but the performance margin narrows considerably for higher values of $k_0\rho$.

In Figs. 6–8, plots similar to those in Figs. 3 and 4 are shown for the t', iterated Aitken, and M transformations, respectively, the latter using the asymptotic coefficients (37) with $p = 2$. We note the absence of glitches on the staircase curves corresponding to break points based on the Bessel function extrema and zeros, which may be attributed to the fact that all three transformations have a built-in delay in the extrapolation. However, with the equidistant break points, these transformations fail for a number of discrete values of $k_0\rho$, similarly to the t, u, and v transformations. This is further evident in Fig. 9, where the number of significant digits achieved by the u transformation is plotted versus the number of integration subintervals for $k_0\rho = 0.164$, which corresponds to the first dip of the dotted lines in Figs. 4 and 6–8. As explained in connection with Fig. 3, these dips may be attributed to the continuously changing convergence properties of the sequence of partial sums (7) as the break points and the zero crossings of the Bessel function change their relative positions with varying ρ. This conjecture is supported by Fig. 10, where the sequences of ten initial partial sums computed with the equidistant break points (6) and with the break points based on the Bessel function extrema are plotted for the critical value of $k_0\rho = 0.164$. (Only the real parts of S_n are shown, because the imaginary parts are negligible in this case.) Observe that for this value of $k_0\rho$, the sequence of partial sums computed with the equidistant break

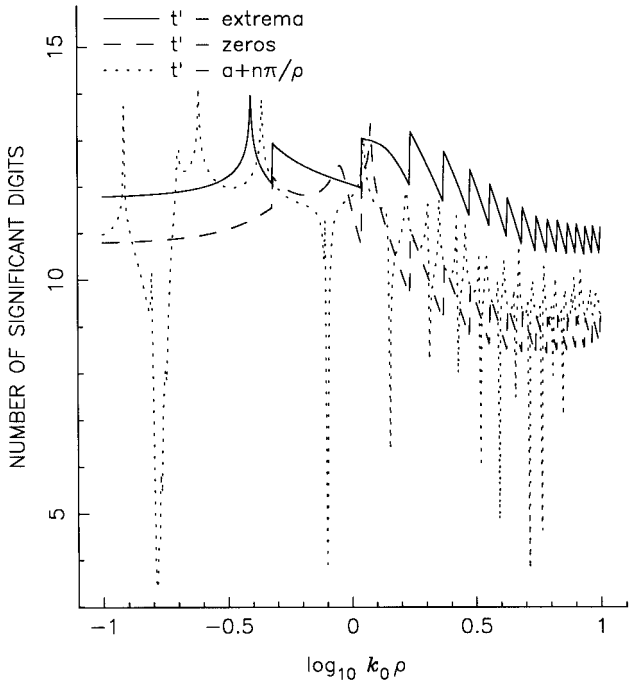

Fig. 6. Performance of the t' transformation for the integral (55) with three choices of break points: the equidistant points (6) with $q = \pi/\rho$, the extremum points, and zeros of the Bessel function. The number of significant digits obtained after ten integration subintervals is plotted versus $k_0\rho$ for $z = 0$.

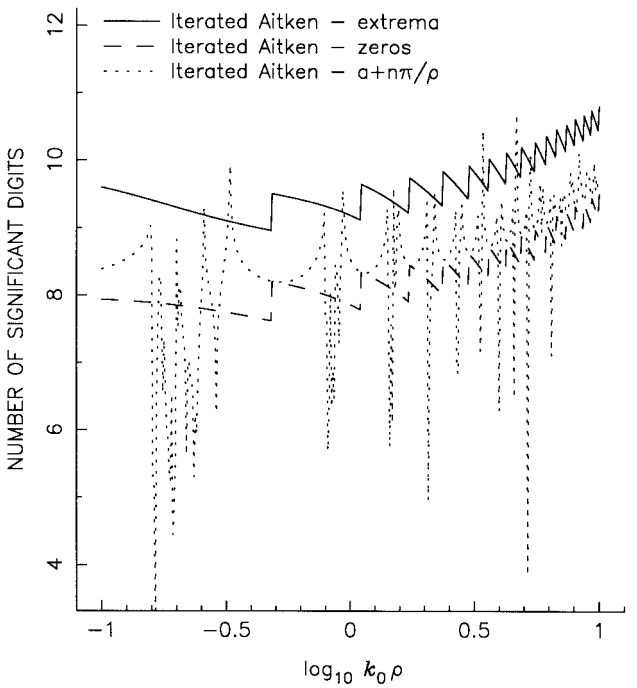

Fig. 7. Performance of the iterated Aitken transformation for the integral (55) with three choices of break points: the equidistant points (6) with $q = \pi/\rho$, the extremum points, and zeros of the Bessel function. The number of significant digits obtained after ten integration subintervals is plotted versus $k_0\rho$ for $z = 0$.

points ceases to be strictly alternating, since there is at least one instance where two consecutive partial sums are greater than the limit of the sequence—a phenomenon previously observed by Lucas and Stone [4].

The results in Fig. 11 illustrate the performance of the W transformation and the weighted-averages method, which both utilize the analytical remainder estimates (17) and the equidistant break points (6). (The break points based on the Bessel function extrema were also tried, but this choice resulted in a slight deterioration of the performance.) In the case of the weighted-averages method, results are presented for the exact and asymptotic forms of the coefficients $\eta_n^{(k)}$, given by (38) and (39), respectively (with the plus sign). Observe that the asymptotic form appears to have an advantage for the smaller values of $k_0\rho$. (Following Mosig's [6] recommendation, we have also tried using $p = 1$ in (38), but this change did not improve the performance of the method.) The fact that neither the W transformation nor the weighted-averages method suffer any breakdowns with the equidistant break points may be attributed to the fact that the analytical remainder estimates (17) depend on the fixed asymptotic length of the integration subinterval q. As a result, these methods are less affected by changes in the character of the sequence of partial sums caused by the varying position of the break points relative to the Bessel function zeros.

As mentioned in Section III, the commonly used earlier version of the weighted-averages method [30], [31], [54] does not take into account the dependence of the integrand on ζ, i.e., the exponential factor in (38) is not included, even if $\zeta \neq 0$. However, this approach may result in a significant performance penalty, as the results in Fig. 12 illustrate.

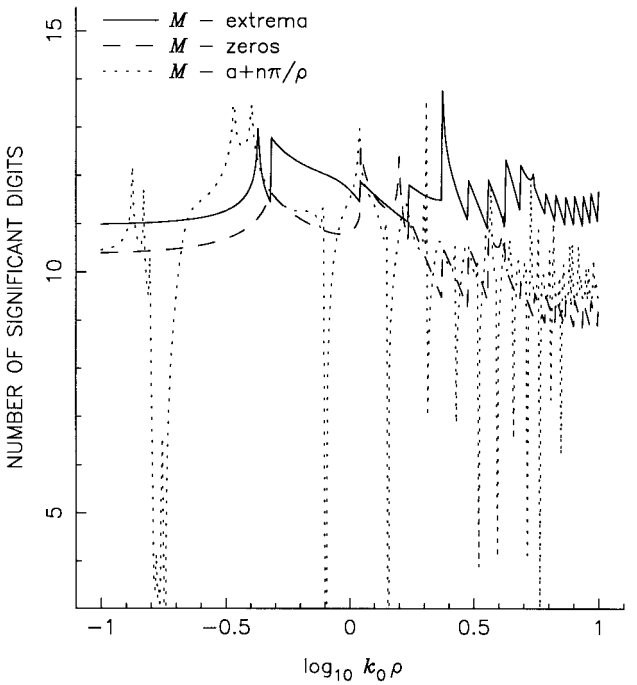

Fig. 8. Performance of the M-transformation for the integral (55) with three choices of break points: the equidistant points (6) with $q = \pi/\rho$, the extremum points, and zeros of the Bessel function. The number of significant digits obtained after ten integration subintervals is plotted versus $k_0\rho$ for $z = 0$.

As mentioned in Section I, for $\rho = 0$ we use equidistant break points with $q = \pi/\zeta$, where we assume that $\zeta > 0$. In this case, because of the exponential decay of the integrands, the integrals are rapidly convergent and acceleration is not

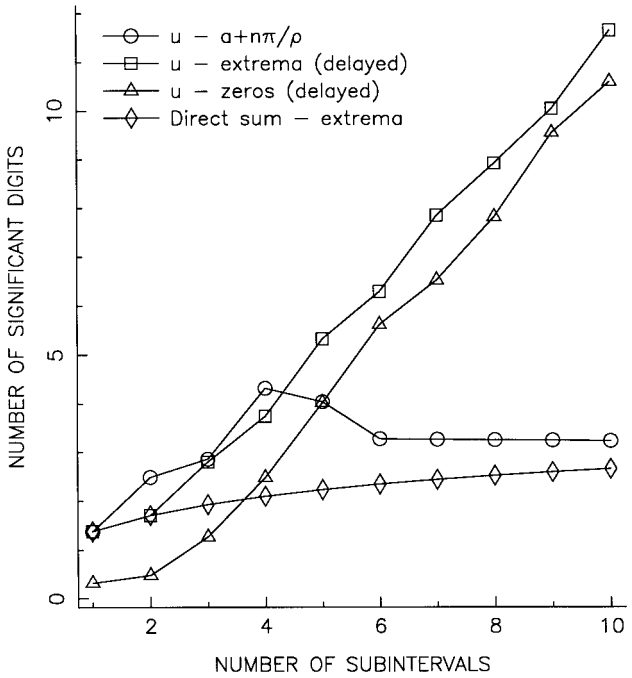

Fig. 9. Performance of the u-transformation for the integral (55) using three choices of break points: the equidistant points (6) with $q = \pi/\rho$, the extremum points, and zeros of the Bessel function, at $z = 0$ and $k_0\rho = 0.164$ (which corresponds to the first dip of the dotted lines in Figs. 4 and 6–8). The number of significant digits obtained is plotted versus the number of integration subintervals. Plotted is also the nonaccelerated direct sum using the extremum points as break points.

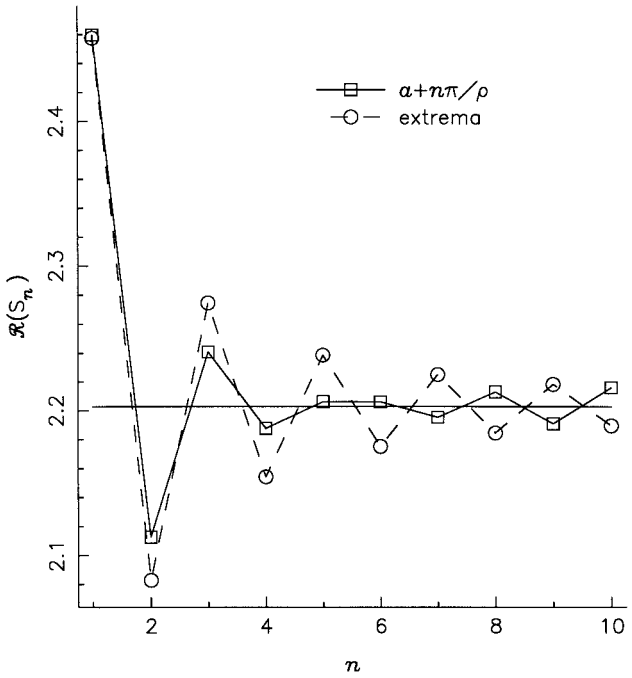

Fig. 10. Real part of the sequence of ten initial partial sums for the integral (55) evaluated at $z = 0$ and $k_0\rho = 0.164$ (which corresponds to the first dip of the dotted lines in Figs. 4 and 6–8), using the equidistant break points (6) with $q = \pi/\rho$ and the break points corresponding to the Bessel function extrema. The value of the integral to which the sequences are converging is shown as the horizontal line.

essential. Nevertheless, as the results in Fig. 13 indicate, even in this case the convergence can be improved by extrapolation.

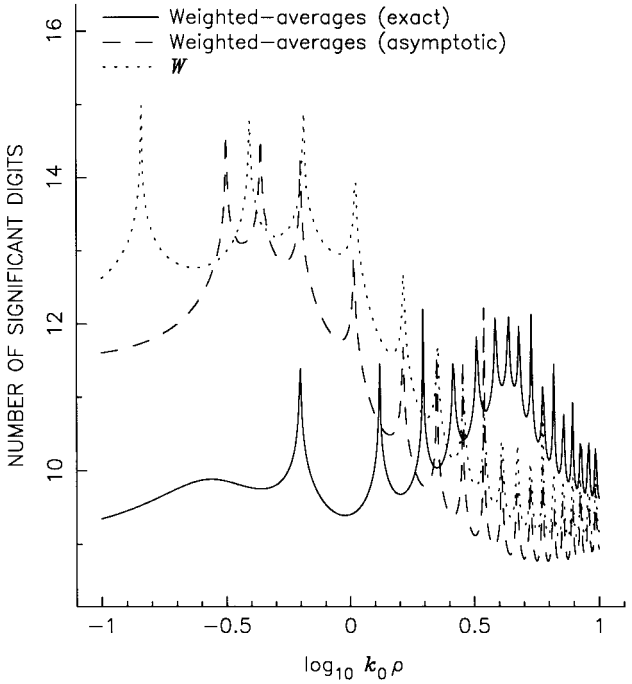

Fig. 11. Performance of the weighted-averages method and the W transformation for the integral (55). The number of significant digits obtained after ten integration subintervals is plotted versus $k_{0\rho}$ for $z = 0$.

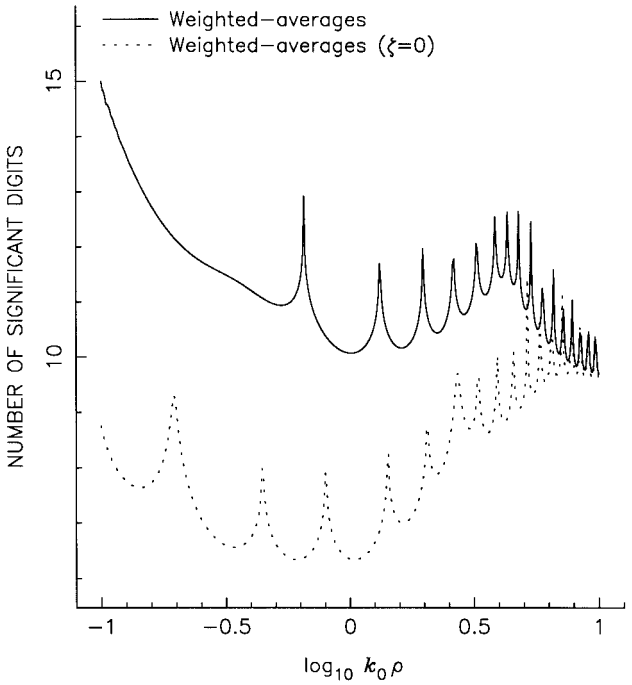

Fig. 12. Performance of the weighted-averages method for the integral (55). The number of significant digits obtained after ten integration subintervals is plotted versus $k_0\rho$ for $k_0 z = 0.05$. The solid and dotted lines are obtained using (38) with $\zeta = z$ and $\zeta = 0$, respectively.

As the last test case, we consider the integral

$$\int_a^\infty \frac{e^{-jk_z|z|}}{jk_z} J_1(\xi\rho)\xi^2\, d\xi = (1+jkr)\frac{\rho e^{-jkr}}{r^3} - \int_0^a \frac{e^{-jk_z|z|}}{jk_z} J_1(\xi\rho)\xi^2\, d\xi \quad (56)$$

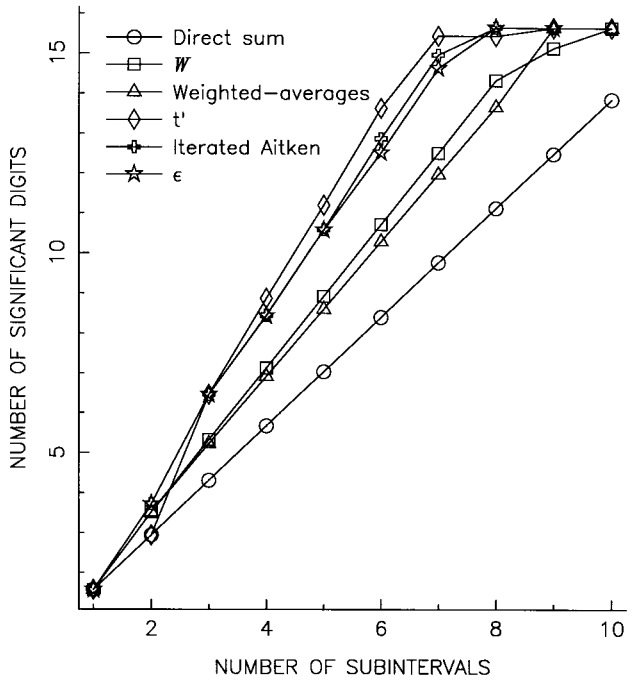

Fig. 13. Performance of various extrapolation algorithms for the integral (55) with $\rho = 0$ and $k_0 z = 0.1$. The equidistant break points (6) with $q = \pi/\zeta$ were employed.

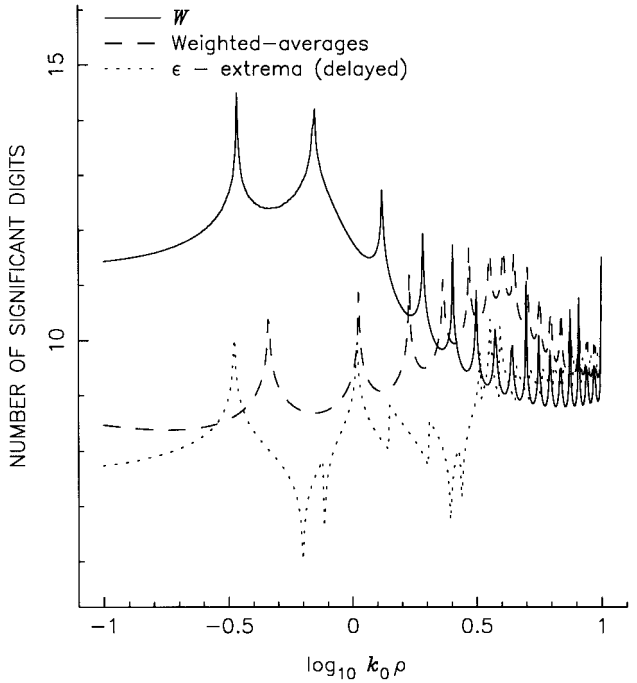

Fig. 14. Performance of the weighted-averages method, the W transformation and the ϵ algorithm for the integral (56). The number of significant digits obtained after ten integration subintervals is plotted versus $k_0\rho$ for $z = 0$.

obtained by differentiating (55). Observe that the left-side integral in (56) has the form of (2), with $\zeta = |z|$ and $\alpha = -1/2$. When $z = 0$, this integral has a diverging integrand and is defined in the sense of Abel summability [37]. For the last case, in Fig. 14, we compare the performance of the weighted-averages method, the W transformation and the ϵ algorithm (the latter using break points based on the Bessel function extrema, with extrapolation delayed by one interval). Similar plots for the iterated Aitken method and the Levin-type transformations, which utilize numerical remainder estimates, indicate that these methods perform poorly for divergent integrands, barely achieving three significant digits of accuracy in ten integration subintervals (not shown, for brevity). We should stress, however, that integrals of this type do not arise when the mixed-potential integral equation (MPIE) formulation is employed [1].

V. Conclusion

We have presented a review of the most promising extrapolation methods currently known for the acceleration of Sommerfeld-type integral tails, with particular emphasis on those algorithms, which explicitly utilize remainder estimates since they tend to be particularly effective. The remainder estimates can be numerically derived or analytical. In the latter case, the knowledge of the asymptotic behavior of the integrand is required. We have found that the performance of the extrapolation methods depends strongly on ρ and also on the choice of subinterval break points. The equidistant break points based on the asymptotic half-period of the Bessel function can only be used with those techniques that utilize analytical remainder estimates, i.e., the W transformation and the weighted-averages method. Methods based on numerical remainder estimates, i.e., the generalized Levin transformations t, u, and v, the t' and M transformations, the iterated Aitken method, as well as (to a lesser degree) the ϵ algorithm (i.e., the Shanks transformation), fail for a number of values of ρ, when the equidistant break points are employed. These failures can be remedied by choosing the break points based on the Bessel function extrema. Even with this choice, however, the generalized Levin transformations and (to a lesser degree) the ϵ algorithm perform poorly unless the extrapolation is delayed by at least one integration subinterval. The t' and M transformations and the iterated Aitken method have a built-in delay, hence, no special action is required in their case.

We have conducted tests for convergent integrals and for divergent integrals defined in the sense of Abel summability. In the former case, the t' and M transformations are, by a narrow margin, the most efficient convergence accelerators when considered over a wide range of ρ. They are closely followed by the generalized Levin transformations—particularly the t transformation and by the W transformation, the iterated Aitken method, and the weighted-averages method with the ϵ algorithm trailing farther behind.

For the divergent integrals, only the W transformation, the weighted-averages method—which both utilize analytical remainder estimates—and the ϵ algorithm are effective, with the W transformation being the most efficient of the three techniques. The other methods, which are based on numerical remainder estimates, perform poorly in this case.

Hence, if the information on the asymptotic behavior of the integrand is available, which is usually the case for Sommerfeld-type integrals in multilayered media, the W transformation and the weighted-averages method emerge as the most versatile and efficient currently known convergence accelerators for Sommerfeld integral tails.

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