Proof of Theorem 1 of "Two-snapshot DOA Estimation via Hankel-structured Matrix Completion "

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Here, we prove Theorem 1 of the paper "Two-snapshot DOA Estimation via Hankel-structured Matrix Completion ". For the sake of completeness. Let $\mathbf{y} \in \mathbb{C}^n$ be the ground-truth noiseless measurement on a ULA array. We also define the Hankel operator $\mathscr{H}(\cdot) : \mathbb{R}^n \mapsto \mathbb{R}^{(d)\times(n-d+1)}$ as

$$\mathscr{H}(\mathbf{x}) := \begin{bmatrix} x_1 & x_2 & \dots & x_{n-d+1} \\ x_2 & x_3 & \dots & x_{n-d+2} \\ \vdots & \vdots & \vdots & \vdots \\ x_d & x_{d+1} & \dots & x_n \end{bmatrix}. \quad (1)$$

where d is called the pencil parameter of the Hankel operator. The goal here is to recover $\mathbf{y}$ from a subset of its elements by exploiting the low-rank structure of $\mathscr{H}(\mathbf{y})$. In particular, the estimated $\mathbf{y}$ denoted by $\widehat{\mathbf{y}}$ is found via

$$\begin{aligned} \widehat{\mathbf{y}} = \underset{\mathbf{g}\in\mathbb{C}^n}{\operatorname{argmin}} \quad & \|\mathscr{H}(\mathbf{g})\|_* \\ \text{s.t.} \quad & \mathcal{P}_\Omega(\mathbf{g}) = \mathbf{y}_o, \end{aligned} \quad (2)$$

where $\Omega \subset \{1,\dots,n\} = [n]$ represents the index of available samples (the location of antennas), $\mathbf{y}_o = \mathcal{P}_\Omega(\mathbf{y})$, and $\mathcal{P}_\Omega$ is the projection operator to the observed samples space.

Theorem 1. *Let $\mathbf{y} \in \mathbb{C}^n$ be the vector of true samples of the ULA for an r-sources. $\mathbf{y}$ can be recovered with probability no less than $1 - n^{-10}$ from the measurements on the SLA $\mathcal{P}_{\widetilde{\Omega}}(\mathbf{y})$ where Ω i.e. index set of the location of the antenna elements in the SLA are randomly chosen by uniformly drawing the indices from $[n]$ by solving the optimization in (2) if*

$$p_k \geq \min\Big\{1\ ,\ \frac{\max\big(c\mu_k r^2 \log^3(n)\ ,\ 1\big)}{n}\Big\}, \quad (3)$$

and $\frac{1}{8\log(n)} \leq \min\{\|\mathbf{U}\mathbf{U}^{\mathsf{H}}\mathbf{e}_1^d\|_{\mathrm{F}}^2, \|\mathbf{e}_n^{n-d+1}\mathbf{V}\mathbf{V}^{\mathsf{H}}\|_{\mathrm{F}}^2\}$, where d is the pencil parameter used in the Hankel operator and $\mathbf{U},\mathbf{V}$ are the unitary matrices of SVD of $\mathscr{H}(\mathbf{y})$. Also μ_k is the leverage score of Definition 1 of the paper and $c > 0$ a scalar.

I. Proof of Theorem 1

W prove by constructing an appropriate dual certificate; the existence of this certificate guarantees that the solution to the problem in (2) is unique. This is a standard approach in the compressed sensing literature (see for instance [1]).

A. Projection

Using the well-studied golfing scheme first used in [2], we show the uniqueness of the solution of the problem. As the first step, we need to define the sampling operator $\mathcal{A}_k$ for any matrix $\mathbf{M} \in \mathbb{C}^{d\times(n-d+1)}$ as $\mathrm{tr}\big(\mathbf{M}^{\mathrm{T}}\mathbf{A}_k\big)\mathbf{A}_k$.. Let Ω be a random subset of $[n]$ such that the element $1 \leq k \leq n$ appears in Ω with probability p_k independent of other elements. We further define the self-adjoint projection operator onto Ω as $\mathcal{A}_\Omega = \sum_{k=1}^n \frac{\delta_k}{p_k}\mathcal{A}_k$. where δ_k is equal to 1 for $k \in \Omega$ and zero elsewhere and p_k is sampling probability of k-th element. It is easy to can check that $\mathbb{E}[\mathcal{A}_\Omega] = \mathcal{A}$, where $\mathcal{A}$ stands for $\sum_{k=1}^n \mathcal{A}_k$. It is also simple to verify that

$$\|\mathcal{A}_\Omega\| = \|\sum_{k=1}^n \frac{\delta_k}{p_k}\mathcal{A}_k\| \leq \frac{\|\sum_{k=1}^n \mathcal{A}_k\|}{\min_k p_k} = \frac{\|\mathcal{A}\|}{\min_k p_k} \leq \frac{1}{\min_k p_k}. \quad (4)$$

We also define the orthogonal operator as $\mathcal{A}^\perp = \mathcal{I} - \mathcal{A}$ where $\mathcal{I}$ is the identity operator. Then the tangent space T with respect to $\mathscr{H}(\mathbf{M}) = \mathbf{U\Sigma V}$ is defined as

$$T := \{\mathbf{U}\mathbf{Y}_1^{\mathrm{H}} + \mathbf{Y}_2\mathbf{V}^{\mathrm{H}} : \mathbf{Y}_1 \in \mathbb{C}^{(n-d+1)\times r}, \mathbf{Y}_2 \in \mathbb{C}^{d\times r}\}. \quad (5)$$

We can now reformulate (2) in form of the following unstructured matrix completion problem:

$$\begin{aligned} \widehat{\mathbf{M}} = \underset{\mathbf{M}\in\mathbb{C}^{d\times(n-d+1)}}{\operatorname{argmin}} \quad & \|\mathbf{M}\|_* \\ \text{s.t.} \quad & \mathcal{Q}_\Omega(\mathbf{M}) = \mathcal{Q}_\Omega(\mathscr{H}(\mathbf{y})), \end{aligned} \quad (6)$$

where $\mathcal{Q}_\Omega$ is $\mathcal{A}_\Omega + \mathcal{A}^\perp$. Using (4), one can see $\|\mathcal{Q}_\Omega\| \leq \frac{1}{\min_k p_k} + 1$ as We further have $\mathbb{E}[\mathcal{Q}_\Omega] = \mathbb{E}[\mathcal{A}_\Omega] + \mathcal{A}^\perp = \mathcal{A} + \mathcal{A}^\perp = \mathcal{I}$.

To prove the exact recovery of the convex optimization, it suffices to produce an appropriate dual certificate, as stated in the following lemma.

Lemma 1. *For a given Ω, let the sampling operator $\mathcal{Q}_\Omega$ fulfill*

$$\|\mathcal{P}_T - \mathcal{P}_T\mathcal{Q}_\Omega\mathcal{P}_T\| \leq \frac{1}{2}, \quad (7)$$

where $\|\cdot\|$ stands for the operator norm and $\mathcal{P}_T$ is the projection to the Tangent space defined in (5). If there exists a matrix $\mathbf{G}$ satisfying

$$\mathcal{Q}_\Omega^\perp(\mathbf{G}) = 0, \quad (8)$$

$$\|\mathcal{P}_T(\mathbf{G} - \mathbf{U}\mathbf{V}^{\mathrm{H}})\|_{\mathrm{F}} \leq \frac{1}{5\|\mathcal{Q}_\Omega\|}, \quad (9)$$

and

$$\|\mathcal{P}_{T^{\perp}}(\mathbf{G})\| \le \frac{1}{2}, \tag{10}$$

then, $\mathbf{M}$ *is the unique solution to* (6).

Proof. This lemma is a standard lemma in golfing scheme, so you can find the proof in [2]. □

We first analyze the condition (7) to construct the proof using Lemma 1. Then, we build up the dual certificate $\mathbf{G}$, and at the end, we validate the dual certificate.

B. Bounding $\|\mathcal{P}_T - \mathcal{P}_T\mathcal{Q}_\Omega\mathcal{P}_T\|$

We first bound the term $\|\mathcal{P}_T(\mathbf{A}_k)\|_{\mathrm{F}}^2$ as

$$\|\mathcal{P}_T(\mathbf{A}_k)\|_{\mathrm{F}}^2 \le \|\mathcal{P}_U(\mathbf{A}_k)\|_{\mathrm{F}}^2 + \|\mathcal{P}_V(\mathbf{A}_k)\|_{\mathrm{F}}^2 \le \frac{2\mu_k r}{n}, \tag{11}$$

where $\mathcal{P}_U(\mathbf{A}_k) = \mathbf{U}\mathbf{U}^{\mathrm{H}}\mathbf{A}_k$ and$\mathcal{P}_V(\mathbf{A}_k) = \mathbf{A}_k\mathbf{V}\mathbf{V}^{\mathrm{H}}$ .Let us define the following family of operators $\mathcal{Z}_k : \mathbb{C}^{d\times(n-d+1)} \mapsto \mathbb{C}^{d\times(n-d+1)}$ as

$$\mathcal{Z}_k := \left(\tfrac{\delta_k}{p_k} - 1\right)\mathcal{P}_T\mathcal{A}_k\mathcal{P}_T \quad \forall\ k \in [n].$$

We can check that for any $\mathbf{M} \in \mathbb{C}^{d\times(n-d+1)}$ we have

$$\begin{aligned}\|\mathcal{Z}_k(\mathbf{M})\|_{\mathrm{F}} &= \|\left(\tfrac{\delta_k}{p_k} - 1\right)\underbrace{\langle \mathbf{A}_k, \mathcal{P}_T(\mathbf{M})\rangle}_{=\langle \mathcal{P}_T(\mathbf{A}_k), \mathbf{M}\rangle}\mathcal{P}_T(\mathbf{A}_k)\|_{\mathrm{F}} \\ &\le \tfrac{1}{p_k}\|\mathcal{P}_T(\mathbf{A}_k)\|_{\mathrm{F}}^2\|\mathbf{M}\|_{\mathrm{F}}.\end{aligned} \tag{12}$$

Therefore, the operator norm $\|\mathcal{Z}_k\|$ is upper-bounded as

$$\|\mathcal{Z}_k\| \le \frac{1}{p_k}\|\mathcal{P}_T(\mathbf{A}_k)\|_{\mathrm{F}}^2 \le \frac{2}{p_k}\frac{\mu_k r}{n} \le \frac{2}{c_0\log(n)}, \tag{13}$$

where we used $p_k \ge c_0\frac{\mu_k r\log(n)}{n}$ in the last inequality.

It is not difficult to see that $\sum_{k=1}^{n}\mathcal{Z}_k = \mathcal{P}_T\mathcal{Q}_\Omega\mathcal{P}_T - \mathcal{P}_T$. Since $\mathbb{E}[\mathcal{Q}_\Omega] = \mathcal{I}$, the latter result shows that $\mathbb{E}[\mathcal{Z}_k] = \mathbf{0}$. Besides, for any $\mathbf{M} \in \mathbb{C}^{d\times(n-d+1)}$, if $\mathcal{Z}_k^2(\mathbf{M})$ represents $\mathcal{Z}_k^*\big(\mathcal{Z}_k(\mathbf{M})\big)$, then, we have

$$\begin{aligned}&\Big\|\sum_k \mathbb{E}[\mathcal{Z}_k^2(\mathbf{M})]\Big\|_{\mathrm{F}} = \|\sum_k \mathbb{E}\big[(\tfrac{\delta_k}{p_k}-1)^2\big]\langle \mathbf{A}_k, \mathcal{P}_T(\mathbf{M})\rangle \\ &\times \langle \mathbf{A}_k, \mathcal{P}_T(\mathbf{A}_k)\rangle\mathcal{P}_T(\mathbf{A}_k)\|_{\mathrm{F}} \\ &\le \max_k \frac{1-p_k}{p_k}\|\mathcal{P}_T(\mathbf{A}_k)\|_{\mathrm{F}}^2\|\sum_k \langle \mathbf{A}_k, \mathcal{P}_T(\mathbf{M})\rangle\mathcal{P}_T(\mathbf{A}_k)\|_{\mathrm{F}} \\ &\le \max_k \frac{1}{p_k}\|\mathcal{P}_T(\mathbf{A}_k)\|_{\mathrm{F}}^2\|\mathbf{M}\|_{\mathrm{F}}.\end{aligned} \tag{14}$$

Therefore, similar to (14) the operator norm can be bounded as $\Big\|\sum_k \mathbb{E}[\mathcal{Z}_k^2]\Big\| \le \frac{2}{c_0\log(n)}$. Then, by the matrix Bernstein inequality, for $c_0 \ge \frac{56}{3}$, we know the existence of some constant $0 < \epsilon \le \frac{1}{2}$ such that

$$\|\sum_k \mathcal{Z}_k\| = \|\mathcal{P}_T - \mathcal{P}_T\mathcal{Q}_\Omega\mathcal{P}_T\| \le \epsilon, \tag{15}$$

with a probability exceeding $1 - n^{-8}$.

C. Dual Certificates construction

We build the dual certificate by using the golfing scheme introduced in [2]. For a small constant $\epsilon < \frac{1}{e}$, let us form $L := \log_{\frac{1}{\epsilon}}(n^2\|\mathcal{Q}_\Omega\|)$ independent subsets $\{\Omega_\ell\}_{\ell=1}^{L}$ of $[n]$ by choosing the elements $1 \le k \le n$ with probability $q_k := 1 - (1-p_k)^{\frac{1}{L}}$ independent of each other. Furthermore, let $\overline{\Omega} = \Omega_1 \cup \cdots \cup \Omega_L$. Next, we construct the dual certificate matrix $\mathbf{G}$ as

$$\mathbf{G} := \sum_{\ell=1}^{L}\mathcal{Q}_{\Omega_\ell}(\mathbf{F}_\ell), \tag{16}$$

where $\mathbf{F}_\ell = \mathcal{P}_T(\mathcal{I} - \mathcal{Q}_{\Omega_\ell})\mathcal{P}_T(\mathbf{F}_{\ell-1})$ and $\mathbf{F}_0 = \mathbf{U}\mathbf{V}^{\mathrm{H}}$. Since $\mathbf{F}_\ell \in \overline{\Omega}$, we can see that $\mathcal{Q}_{\overline{\Omega}}^{\perp}(\mathbf{G}) = 0$; i.e., $\mathbf{G}$ satisfies the first condition of Lemma 1 for $\overline{\Omega}$. Besides, we have that

$$\mathcal{P}_T(\mathbf{F}_\ell) = \mathbf{F}_\ell = \Big(\mathcal{P}_T - \mathcal{P}_T\mathcal{Q}_{\Omega_\ell}\mathcal{P}_T\Big)(\mathbf{F}_{\ell-1}). \tag{17}$$

In addition, from (15), we already know that

$$\Big\|\mathcal{P}_T - \mathcal{P}_T\mathcal{Q}_{\Omega_\ell}\mathcal{P}_T\Big\| \le \epsilon < \frac{1}{2}, \tag{18}$$

with a probability no less than $1 - n^{-8}$.

To bound $\big\|\mathcal{P}_T\big(\mathbf{G} - \mathbf{F}_0\big)\big\|_{\mathrm{F}}$, we follow a similar technique as in [3] to obtain $\mathcal{P}_T(\mathbf{G} - \mathbf{F}_0) = -\mathcal{P}_T(\mathbf{F}_L)$. The latter holds due to $q_\ell \ge \frac{p_\ell}{L} \ge c_0\frac{\mu_\ell r^2\log^2(n)}{n}$. Then, we can bound the term as follows

$$\begin{aligned}\|\mathcal{P}_T\big(\mathbf{G} - \mathbf{F}_0\big)\|_{\mathrm{F}} &= \|\mathcal{P}_T\big(\mathbf{F}_L\big)\|_{\mathrm{F}} \le \epsilon^L\|\mathcal{P}_T(\mathbf{F}_0)\|_{\mathrm{F}} \\ &\le \epsilon^L\|\mathbf{U}\mathbf{V}^{\mathrm{H}}\|_{\mathrm{F}} \le \epsilon^L\sqrt{r} < \frac{1}{5\|\mathcal{Q}_\Omega\|},\end{aligned} \tag{19}$$

with a probability no less than $1 - Ln^{-8}$. This shows that $\mathbf{G}$ satisfies condition (9) of Lemma 1 with high probability. The only condition of Lemma 1 that requires to be satisfied is $\|\mathcal{P}_{T^\perp}(\mathbf{G})\| \le \frac{1}{2}$ which we show it in the next subsection.

D. An Upper Bound on $\|\mathcal{P}_{T^\perp}(\mathbf{G})\|$

We first define the following two useful norms for arbitrary $\mathbf{M} \in \mathbb{C}^{d\times(n-d+1)}$:

$$\|\mathbf{M}\|_{\mathcal{A},\infty} := \max_{k\in[n]}\left|\frac{n\langle \mathbf{A}_k, \mathbf{M}\rangle}{r\mu_k\sqrt{\omega_k}}\right|, \tag{20}$$

$$\|\mathbf{M}\|_{\mathcal{A},2} := \sqrt{\sum_{k\in[n]}\frac{|n\langle \mathbf{A}_k, \mathbf{M}\rangle|^2}{r\mu_k\omega_k}}. \tag{21}$$

Now, we state 3 inequalities regarding the defined norms in form of Lemmas 2-4. In what follows, we provides a set of probabilistic upper bounds in form of three lemmas. All the lemmas can be obtained only by applying matrix Bernstein inequality for the corresponding terms.

Lemma 2. *Suppose* $\mathbf{M}$ *is a complex-valued* $d\times(n-d+1)$ *matrix. If* $p_k \ge c_0\frac{\mu_k r^2\log^2(n)}{n}$ *for all* $k \in [n]$*, then*

$$\begin{aligned}\|\big(\mathcal{Q}_\Omega - \mathcal{I}\big)(\mathbf{M})\| \le &\sqrt{\tfrac{22}{c_0 r\log(n)}}\|\mathbf{M}\|_{\mathcal{A},2} \\ &+ \tfrac{22}{3c_0 r\log(n)}\|\mathbf{M}\|_{\mathcal{A},\infty}\end{aligned} \tag{22}$$

holds with a probability at least $1 - n^{-10}$*, where* $c_0 \ge 22$.

Lemma 3. *For $c_0 \geq 54$ and arbitrary $\mathbf{M} \in \mathbb{C}^{d\times(n-d+1)}$, we have*

$$\left\|\left(\mathcal{P}_T\mathcal{Q}_\Omega - \mathcal{P}_T\right)(\mathbf{M})\right\|_{\mathcal{A},2} \leq \sqrt{8}\Big(\sqrt{\tfrac{20}{c_0}}\,\|\mathbf{M}\|_{\mathcal{A},2} + \tfrac{20}{3c_0}\,\|\mathbf{M}\|_{\mathcal{A},\infty}\Big) \tag{23}$$

with a probability no less than $1-n^{-9}$, given that $p_k \geq c_0\frac{\mu_k r^2}{n}\log^2(n)$ for $k\in[n]$.

Lemma 4. *Suppose we have that*

$$\frac{1}{8\log^2(n)} \leq \min\{\|\mathcal{P}_U(\mathbf{e}_1^d)\|_{\mathrm{F}}^2, \|\mathcal{P}_V(\mathbf{e}_n^{n-d+1})\|_{\mathrm{F}}^2\}. \tag{24}$$

Then, for $c_0 \geq 144$ and arbitrary $\mathbf{M}\in T$, we have

$$\left\|\Big(\mathcal{P}_T\mathcal{Q}_\Omega - \mathcal{P}_T\Big)(\mathbf{M})\right\|_{\mathcal{A},\infty} \leq \sqrt{72}\Big(\sqrt{\tfrac{32}{c_0}}\,\|\mathbf{M}\|_{\mathcal{A},2} + \tfrac{32}{3c_0}\,\|\mathbf{M}\|_{\mathcal{A},\infty}\Big) \tag{25}$$

with probability at least $1-n^{-14}$, given that $p_k \geq c_0\frac{\mu_k r^2}{n}\log^2(n)$ for $k\in[n]$.

Now, recalling (16), we can write that

$$\|\mathcal{P}_{T^\perp}(\mathbf{G})\| \leq \sum_{\ell=1}^{L}\|\mathcal{P}_{T^\perp}\mathcal{Q}_{\Omega_\ell}\mathcal{P}_T(\mathbf{F}_{\ell-1})\|. \tag{26}$$

Next, we bound each term in the right hand summation of (26):

$$\begin{aligned}\|\mathcal{P}_{T^\perp}\mathcal{Q}_{\Omega_\ell}\mathcal{P}_T(\mathbf{F}_{\ell-1})\| &= \left\|\left(\mathcal{P}_{T^\perp}(\mathcal{Q}_{\Omega_\ell}-\mathcal{I})\mathcal{P}_T\right)(\mathbf{F}_{\ell-1})\right\| \\ &\leq \left\|\left((\mathcal{Q}_{\Omega_\ell}-\mathcal{I})\mathcal{P}_T\right)(\mathbf{F}_{\ell-1})\right\| = \|(\mathcal{Q}_{\Omega_\ell}-\mathcal{I})\,(\mathbf{F}_{\ell-1})\| \\ &\overset{\text{Lemma 2}}{\leq} \sqrt{\tfrac{18}{c_0 r\log(n)}}\|\mathbf{F}_{\ell-1}\|_{\mathcal{A},2} + \tfrac{18}{3c_0 r\log(n)}\|\mathbf{F}_{\ell-1}\|_{\mathcal{A},\infty} \\ &\leq \frac{\|\mathbf{F}_{\ell-1}\|_{\mathcal{A},2}+\|\mathbf{F}_{\ell-1}\|_{\mathcal{A},\infty}}{c_1\sqrt{r\log(n)}},\end{aligned} \tag{27}$$

Thus, for a proper c_1, we have

$$\|\mathcal{P}_{T^\perp}(\mathbf{G})\| \leq \tfrac{1}{c_1\sqrt{r\log(n)}}\sum_{\ell=1}^{L}\left(\|\mathbf{F}_{\ell-1}\|_{\mathcal{A},2}+\|\mathbf{F}_{\ell-1}\|_{\mathcal{A},\infty}\right) \tag{28}$$

holds with high probability.

Because of $\mathbf{F}_\ell = \Big(\mathcal{P}_T - \mathcal{P}_T\mathcal{Q}_{\Omega_\ell}\Big)(\mathbf{F}_{\ell-1})$, and b using Lemmas 3 and 4 we can recursively bound $\|\mathcal{P}_{T^\perp}\mathcal{Q}_{\Omega_\ell}\mathcal{P}_T(\mathbf{F}_{\ell-1})\|$:

$$\begin{aligned}\|\mathbf{F}_\ell\|_{\mathcal{A},2}+\|\mathbf{F}_\ell\|_{\mathcal{A},\infty} &\leq \Big(\sqrt{\tfrac{32}{c_0}}+\sqrt{\tfrac{144}{c_0}}\Big)\,\|\mathbf{F}_{\ell-1}\|_{\mathcal{A},2} \\ &\quad + \Big(\tfrac{\sqrt{8}32}{3c_0}+\tfrac{\sqrt{72}16}{3c_0}\Big)\,\|\mathbf{F}_{\ell-1}\|_{\mathcal{A},\infty} \\ &\leq \frac{\|\mathbf{F}_{\ell-1}\|_{\mathcal{A},2}+\|\mathbf{F}_{\ell-1}\|_{\mathcal{A},\infty}}{c_2}\end{aligned} \tag{29}$$

For a suitable choice of $c_2 > 0$. By applying (29) multiple times, we conclude that

$$\|\mathcal{P}_{T^\perp}(\mathbf{G})\| \leq \frac{\|\mathbf{F}_0\|_{\mathcal{A},2}+\|\mathbf{F}_0\|_{\mathcal{A},\infty}}{c_1\sqrt{r\log(n)}}\sum_{\ell=1}^{L}c_2^{1-\ell} \tag{30}$$

with high probability. We further bound $\|\mathbf{F}_0\|_{\mathcal{A},\infty}$ and $\|\mathbf{F}_0\|_{\mathcal{A},2}$ to simplify (30). Also, it is easy to see $\|\mathbf{F}_0\|_{\mathcal{A},\infty}\leq 1$. Then, we only need to bound $\|\mathbf{F}_0\|_{\mathcal{A},2}$. Hence, we use

$$\begin{aligned}\|\mathbf{F}_0\|_{\mathcal{A},2}^2 &= \sum_{k\in[n]}\tfrac{n\,|\langle\mathbf{A}_k,\mathbf{F}_0\rangle|^2}{\omega_k\mu_k r} = \sum_{k\in[n]}\tfrac{\mu_k r}{n}\Big(\tfrac{n\,|\langle\mathbf{A}_k,\mathbf{F}_0\rangle|}{\sqrt{\omega_k\mu_k r}}\Big)^2 \\ &\leq \sum_{k\in n}\frac{\mu_k r}{n} \leq \sum_{k\in n}\|\mathcal{P}_U(\mathbf{A}_k)\|_{\mathrm{F}}^2+\|\mathcal{P}_V(\mathbf{A}_k)\|_{\mathrm{F}}^2\end{aligned} \tag{31}$$

With simple calculation, one can see $\sum_{k\in n}\|\mathcal{P}_U(\mathbf{A}_k)\|_{\mathrm{F}}^2 \leq r\log(n)$ and similarly $\sum_{k\in n}\|\mathcal{P}_V(\mathbf{A}_k)\|_{\mathrm{F}}^2 \leq r\log(n)$. Hence, the direct consequence would lead to

$$\|\mathcal{P}_{T^\perp}(\mathbf{G})\| \leq \frac{\sqrt{2r\log(n)}+1}{c_1\sqrt{r\log(n)}}\sum_{\ell=1}^{L}c_2^{1-\ell} \leq \frac{2\sqrt{2}}{c_1}\sum_{\ell=1}^{L}c_2^{1-\ell} \tag{32}$$

for $q_k \geq c_0\frac{\mu_k}{n}r^2\log^2(n)$, or equivalently $p_k \geq c_0\frac{\mu_k}{n}r^2\log^3(n)$. For $c_2\geq 2$ and $c_1\geq 12$, we can conclude that

$$\|\mathcal{P}_{T^\perp}(\mathbf{G})\| \leq \tfrac{2\sqrt{2}}{c_1}\Big(1+\sum_{\ell=1}^{\infty}(\tfrac{1}{2})^\ell\Big) \leq \tfrac{4\sqrt{2}}{c_1} \leq \frac{1}{2}, \tag{33}$$

with high probability. Therefore, if $p_k \geq c_0\frac{\mu_k}{n}r^2\log^3(n)$ for $k\in[n]$, with probability no less than $1-n^{-10}$, matrix $\mathbf{G}$ is a valid dual certificate. Accordingly, from Lemma 1, the solution of (2) is exact and unique (with high probability).

References

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